Word complexity cutoffs for mixing properties of subshifts

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Abstract Investigating word complexity cutoffs for partial and light mixing in subshifts, we construct partially mixing rank-one subshifts whose word complexity functions $p(q)$ satisfy $\liminf_{q\to\infty} {}^{p(q)}/_{q} < 2+\epsilon$ for any prescribed $\epsilon > 0$ and, conversely, we prove that every lightly mixing rank-one subshift satisfies $\liminf_{q\to\infty} {}^{p(q)}/_{q} > 2$, establishing 2 as the optimal limiting threshold in the rank-one setting. More quantitatively, we show that every α-partially mixing rank-one subshift satisfies $\liminf_{q\to\infty} {}^{p(q)}/_{q} \geq 2+{}^{\alpha}/_{(1+\alpha)}$. Our constructions are a combination of staircase and Chacon-like rank-one stages; we introduce the independently interesting notion of mixing spans and meshes to establish partial mixing while retaining low word complexity.

Introduction

Subshifts are a natural and well-studied class of transformations: transformations on closed subsets $X \subseteq \mathcal{A}^{\mathbb{Z}}$ which are invariant under the shift map (here $\mathcal{A}$, the alphabet, is an arbitrary finite set). A key aspect of subshifts is that there is a finer notion of complexity than for general transformations, namely we can define the **word complexity** function $p(q) = |\{w \text{ is a subword of some } x \in X \,:\, |w| = q\}|$ where $|w|$ is the length. Entropy, the standard measure of complexity for general transformations, cannot distinguish qualitative behavior, such as various forms of measure-theoretic mixing, among zero entropy transformations; however, word complexity can. Questions of what the minimal possible word complexity for subshifts having mixing properties date back to Ferenczi in the 1990s, e.g. [Fer96],[Fer95] and are an active area of current research.

A **word complexity cutoff** is a quantitative statement about the asymptotic nature of $p(q)$ for which some specified qualitative property of the subshift can occur. For example, in [Cre26], the author showed that the word complexity cutoff for measure-theoretically strong mixing is precisely that $p(q)$ is superlinear: any strongly mixing subshift has $\lim p(q)/q = \infty$ and for any $f(q) \to \infty$ there exists a strongly mixing subshift for which $p(q) < qf(q)$ (for sufficiently large q). For another example, the related question for weak mixing was resolved by the author and R. Pavlov [CP24] who showed that $\limsup p(q)/q < 1.5$ implies discrete spectrum, hence precludes weak mixing, and [CP23] where the existence of a weakly mixing subshift with $\limsup {}^{p(q)}/_{q} = 1.5$ and $\liminf {}^{p(q)}/_{q} = 1$ was established.

A striking feature of recent work on word complexity is the appearance of a sequence of thresholds of the form $(k+1)/k$, in several different senses, at which increasingly restrictive structural and dynamical properties cease to be forced. In particular, $3/2$ is the sharp threshold for the occurrence of weak mixing in terms of $\limsup p(q)/q$ [CP24], while $4/3$ marks a transition in the substitutive structures available to low-complexity subshifts [CP23]. Further restrictions on substitution incidence matrices arise below $5/4$, see [CP23], specifically the discussion following Proposition 3.1. At the other end of this hierarchy, strong mixing requires superlinear word complexity, i.e. "1/0". The results of this paper identify $2 = 2/1$ as the corresponding limiting threshold for partial mixing within the rank-one class and provide evidence for the analogous threshold for light mixing in general.

A transformation T is **α-partially mixing** for $0 < \alpha < 1$ if for all measurable sets A and B,

$$\liminf \mu(T^n A \cap B) \geq \alpha\mu(A)\mu(B)$$

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A transformation T is **lightly mixing** if for all positive measure sets A and B, $\liminf \mu(T^n A \cap B) > 0$. Strong mixing is defined as $\lim \mu(T^n A \cap B) = \mu(A)\mu(B)$ and weak mixing as the existence of a sequence of times along which that limit holds; as is well-known the implications

$$\text{strong mixing} \quad \Rightarrow \quad \text{partial mixing} \quad \Rightarrow \quad \text{light mixing} \quad \Rightarrow \quad \text{weak mixing}$$

hold and none of the reverse implications hold.

Main results

Here we establish a quantitative, in terms of word complexity, result regarding partial mixing as it fits into the general word complexity cutoff framework:

Theorem 1. *For every $\epsilon > 0$, there exists a partially mixing subshift such that* $\liminf {}^{p(q)}/_{q} < 2 + \epsilon$.

While weak mixing is possible with $\liminf p(q)/q = 1$ and strong mixing forces $\liminf p(q)/q = \infty$, our work here shows that light mixing can occur with $\liminf p(q)/q < 2 + \epsilon$. Building on this, we establish that, in the context of rank-one subshifts, 2 is the optimal bound (and in fact cannot be attained by a rank-one subshift though we suspect it can be attained by a non-rank-one lightly mixing subshift):

Theorem 2. *For any lightly mixing rank-one subshift,* $\liminf {}^{p(q)}/_{q} > 2$.

We also establish lower bounds for word complexity of arbitrary α-partially mixing rank-one subshifts demonstrating that stronger partial mixing forces increasing lower bounds on word complexity.

Theorem 3. *For any rank-one subshift which is α-partially mixing,* $\liminf {}^{p(q)}/_{q} \geq 2 + {}^{\alpha}/_{(\alpha+1)}$.

Further directions

Motivated by these results, we propose the following:

Conjecture 4. *For any subshift which is α-partially mixing,* $\liminf {}^{p(q)}/_{q} \geq 2 + {}^{\alpha}/_{(\alpha+1)}$.

Conjecture 5. *For any lightly mixing subshift,* $\liminf {}^{p(q)}/_{q} \geq 2$.

The apparent discrepancy with the above and our result is intentional, and in fact we suspect that in the rank-two setting, equality can be attained just as equality with 1.5 can be attained in the weakly mixing rank-two setting despite the fact that weakly mixing rank-one subshifts cannot attain the 1.5 bound.

Conjecture 6. *There exists a lightly mixing subshift with* $\liminf {}^{p(q)}/_{q} = 2$.

Our lightly mixing examples with $\liminf p(q)/q < 2 + \epsilon$ necessarily have $\limsup p(q)/q = \infty$ due to the nature of their construction; we conjecture that such is unavoidable in general when constructing lightly mixing subshifts:

Conjecture 7. *For any lightly mixing subshift,* $\limsup {}^{p(q)}/_{q} = \infty$.

Mixing spans and meshes

A key new ingredient in the proof of Theorem 1 is the notion of mixing spans and meshes, which is of independent interest and worth highlighting. A mesh is, in essence, a way of taking specific subsets of Rokhlin towers which themselves are organized into levels and therefore retain much of the usefulness of towers when estimating mixing-type asymptotics while being significantly more general.

Our examples essentially depend on meshes in that they are built as a combination of staircase-type constructions and Chacon-type constructions; the staircase stages introduce some amount of mixing and meshes allow us to show that said mixing survives in a suitable sense through the Chacon-like stages

of the construction. In terms of word complexity, the staircase stages, as used in our constructions, unavoidably increase the complexity in a superlinear fashion and the Chacon-like stages are required to then reduce the word complexity arbitrarily close to the prescribed bounds; as such the notion of meshes is crucial to the argument.

We introduce the notion of (uniform) mixing stages of rank-one transformations which involves spans of times during which a fixed set is mixed uniformly into every mesh laid over every sufficiently large column. These new ingredients, combined with staircase mixing techniques going back to [AF92] and used, for example, in [Ada98], [CS04], [CS10], [Cre21], [Cre26], form the bulk of the proof of partial mixing (the other ingredient being the nonrigidity of our constructions).

Organization of the paper

Section 1 establishes precise definitions and recalls the standard theorems on subshifts and rank-one constructions, Section 2 defines the class of examples we will use to establish Theorem 1 and Section 3 provides the word complexity estimates for these examples. Sections 4 and 5 establish partial mixing, with Section 4 being largely devoted to developing the theory of meshes as they relate to mixing estimates and Section 5 being more focused on our specific class of examples. Section 6 concretely exhibits the examples proving Theorem 1 and the paper concludes with Section 7 where the lower bounds in Theorems 2 and 3 are demonstrated.

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1 Definitions and preliminaries

Definition 1.1. A **subshift** on the finite set $\mathcal{A}$ is any subset $X \subset \mathcal{A}^{\mathbb{Z}}$ which is closed in the product topology and shift-invariant: for all $x = (x_n)_{n\in\mathbb{Z}} \in X$ and $k \in \mathbb{Z}$, the translate $(x_{n+k})_{n\in\mathbb{Z}}$ of x by k is also in X.

Definition 1.2. A **word** is any element of $\mathcal{A}^\ell$ for some ℓ, the **length** of w, written $|w|$. A word w is a **subword** of a word or bi-infinite sequence x if there exists k so that $w_i = x_{i+k}$ for all $1 \le i \le |w|$. A word u is a **prefix** of w when $u_i = w_i$ for $1 \le i \le |u|$ and a word v is a suffix of w when $v_i = w_{i+|w|-|v|}$ for $1 \le i \le |v|$.

For words v, w, we denote by vw their concatenation–the word obtained by following v immediately by w. We write such concatenations with product or exponential notation, e.g. $\prod_i w_i$ or 0^n.

Definition 1.3. The **language** of a subshift X is $\mathcal{L}(X) = \{w : w \text{ is a subword of some } x \in X\}$.

Definition 1.4. The **word complexity function** of a subshift X over $\mathcal{A}$ is the function $p_X : \mathbb{N} \to \mathbb{N}$ defined by $p_X(q) = |\mathcal{L}(X) \cap \mathcal{A}^q|$, the number of words of length q in the language of X.

When X is clear from context, we suppress the subscript and just write $p(n)$.

For subshifts on the alphabet $\{0, 1\}$, we consider:

Definition 1.5. The set of **right-special** words is $\mathcal{L}^{RS}(X) = \{w \in \mathcal{L}(X) : w0, w1 \in \mathcal{L}(X)\}$.

Cassaigne [Cas97] showed the well-known: $p(q) = p(m) + \sum_{\ell=m}^{q-1} |\{w \in \mathcal{L}^{RS} : |w| = \ell\}|$ for $m < q$.

Definition 1.6. A **transformation** T is a measurable map on a standard Borel or Lebesgue measure space $(Y, \mathcal{B}, \mu)$ that is measure-preserving: $\mu(T^{-1}B) = \mu(B)$ for all $B \in \mathcal{B}$.

Definition 1.7. Two transformations T on $(Y, \mathcal{B}, \mu)$ and T' on $(Y', \mathcal{B}', \mu')$ are **measure-theoretically isomorphic** when there exists a bijective map ϕ between full measure subsets $Y_0 \subset Y$ and $Y_0' \subset Y'$ where $\mu(\phi^{-1}A) = \mu'(A)$ for all measurable $A \subset Y_0'$ and $(\phi \circ T)(y) = (T' \circ \phi)(y)$ for all $y \in Y_0$.

Definition 1.8. A transformation T on a finite measure space is **lightly mixing** if for all positive measure sets A and B,

$$\liminf \mu(T^n A \cap B) > 0$$

Definition 1.9. A transformation T on a finite measure space is **α-partially mixing** for $0 < \alpha < 1$ if for all measurable sets A and B,

$$\liminf \mu(T^n A \cap B) \geq \alpha \mu(A)\mu(B)$$

A **rank-one transformation** is a transformation T constructed by "cutting and stacking". Here Y represents a (possibly infinite) interval, $\mathcal{B}$ is the induced σ-algebra from $\mathbb{R}$, and μ is Lebesgue measure. We give a brief description, referring the reader to [FGH+23] or [Sil08] for more details.

The transformation is defined inductively on larger and larger portions of the space through Rohlin towers or **columns**, denoted C_n. Each column C_n consists of **levels** $I_{n,j}$ where $0 \leq j < h_n$ is the height of the level within the column. All levels $I_{n,j}$ in C_n are intervals with the same length, $\mu(I_n)$, and the total number of levels in a column is the **height** of the column, denoted by h_n. The transformation T is defined on all levels $I_{n,j}$ except the top one I_{n,h_n-1} by sending each $I_{n,j}$ to $I_{n,j+1}$ using the unique order-preserving affine map.

Start with $C_1 = [0, 1)$ with height $h_1 = 1$. To obtain C_{n+1} from C_n, we require a **cut sequence**, $\{r_n\}$ such that $r_n \geq 1$ for all n. Make r_n vertical cuts of C_n to create $r_n + 1$ **subcolumns** of equal width. Denote a **sublevel** of C_n by $I_{n,j}^{[i]}$ where $0 \leq a < h_n$ is the height of the level within that column, and i represents the position of the subcolumn, where $i = 0$ represents the leftmost subcolumn and $i = r_n$ is the rightmost subcolumn. After cutting C_n into subcolumns, add extra intervals called **spacers** on top of each subcolumn to function as levels of the next column. The **spacer sequence**, $\{s_{n,i}\}$ such that $0 \leq i \leq r_n$ and $s_{n,i} \geq 0$, specifies how many sublevels to add above each subcolumn. Spacers are the same width as the sublevels, act as new levels in the column C_{n+1}, and are taken to be the leftmost intervals in $[1, \infty)$ not in C_n. After the spacers are added, stack the subcolumns with their spacers right on top of left, i.e. so that $I_{n,0}^{[i+1]}$ is directly above $I_{n,h_n-1}^{[i]}$. This gives the next column, C_{n+1}.

Each column C_n defines T on $\bigcup_{j=0}^{h_n-2} I_{n,j}$ and the partially defined map T on C_{n+1} agrees with that of C_n, extending the definition of T to a portion of the top level of C_n where it was previously undefined. Continuing this process gives the sequence of columns $\{C_1, \dots, C_n, C_{n+1}, \dots\}$ and T is then the limit of the partially defined maps.

Though this construction could result in Y being an infinite interval with infinite Lebesgue measure, Y has finite measure if and only if $\sum_n \frac{1}{(r_n+1)h_n} \sum_{i=0}^{r_n} s_{n,i} < \infty$, see [CS10]. All rank-one transformations we define satisfy this condition, and for convenience we renormalize so that $Y = [0, 1)$. Every rank-one transformation is ergodic and invertible.

The reader should be aware that we are making r_n cuts and obtaining $r_n + 1$ subcolumns (following Ferenczi [Fer96]), while other papers (e.g. [Cre21]) use r_n as the number of subcolumns.

For a rank-one transformation defined as above, we define a subshift $X(T)$ on the alphabet $\{0, 1\}$ which is measure-theoretically isomorphic to T provided T is on a finite measure space:

Definition 1.10. The **symbolic model** $X(T)$ of a rank-one transformation T is given by the sequence of words: $B_0 = 0$ and

$$B_{n+1} = B_n 1^{s_{n,0}} B_n 1^{s_{n,1}} \cdots B_n 1^{s_{n,r_n}} = \prod_{i=0}^{r_n} B_n 1^{s_{n,i}}$$

and $X(T)$ is the set of all bi-infinite sequences such that every subword is a subword of some B_n.

The words B_n are a symbolic coding of the column C_n: 0 represents C_1 and 1 represents the spacers. There is a natural measure associated to $X(T)$:

Definition 1.11. The **empirical measure** for a symbolic model $X(T)$ of a rank-one transformation T is the measure ν defined by, for each word w,

$$\nu([w]) = \lim_{n\to\infty} \frac{|\{1 \le j \le |B_n| - |w| + 1 \ : \ B_{n[j,j+|w|)} = w\}|}{|B_n| - |w| + 1}$$

Danilenko [Dan16] (combined with [dJ77] and [Kal84]) proved that the symbolic model $X(T)$ of a finite measure rank-one subshift, equipped with its empirical measure, is measure-theoretically isomorphic to the cut-and-stack construction (see [AFP17]; see [FGH$^+$23] for the full generality including odometers).

Due to this isomorphism, we move back and forth between rank-one and symbolic model terminology as needed and write $\mathcal{L}(T)$ for the language of $X(T)$.

2 Staircase-Chacon-like hybrid transformations

Definition 2.1. Given a word B_n representing a column in a rank-one transformation, say that B_{n+1} is of **staircase form** when

$$B_{n+1} = \left(\prod_{j=0}^{r_n-1} B_n 1^{c_n+j} \right) B_n$$

for some $r_n \ge 2$ and $c_n \ge 0$.

Definition 2.2. Given a word B_n representing a column in a rank-one transformation, say that B_{n+1} is of **Chacon-like form** when

$$B_{n+1} = \left(\prod_{j=0}^{r_n-1} B_n 1^{c_n+\iota_{n,j}M_n} \right) B_n$$

where $\iota_{n,j} \in \mathbb{N} \cup \{0\}$ for all n and $r_n \ge 2$ and $c_n \ge 0$ and $M_n \ge 1$.

Definition 2.3. Let $\{\text{Form}_n\}$ be a sequence of forms all either staircase or Chacon-like and let $\{M_n\}$ be a sequence of positive integers for those n for which B_{n+1} is of Chacon-like form and $\{r_n\}$ a sequence of positive integers.

Set $B_0 = 0$ and $c_0 = 1$ and $r_0 = 3$. Given, B_n, construct B_{n+1} to be of the specified form.

If B_{n+1} and B_{n+2} are of staircase form, set $c_{n+1} = c_n + r_n$.

If B_{n+1} is of staircase form and B_{n+2} is of Chacon-like form, set $c_{n+1} \ge c_n + r_n$ minimal such that M_{n+1} divides $|B_{n+1}| + c_{n+1}$.

If B_{n+1} is of Chacon-like form and B_{n+2} is of staircase form, let $c_{n+1} \ge |B_\ell| + 1 + M_{n+1}$ be any integer where $\ell < n$ is the largest stage that was of staircase form.

If B_{n+1} and B_{n+2} are of Chacon-like form, set $c_{n+1} = c_n$.

The resulting transformation is an **elevated staircase-Chacon-like hybrid transformation**.

Notation 2.4. For an elevated staircase-Chacon-like hybrid transformation,

$$\tilde{h}_n = |B_n| + c_n$$

Lemma 2.5. *Let T be an elevated staircase-Chacon-like hybrid transformation. If B_{n+1} is constructed using the staircase operation then*

$$\tilde{h}_{n+1} = (r_n + 1)\tilde{h}_n + \frac{1}{2} r_n(r_n - 1) + c_{n+1} - c_n$$

and if B_{n+1} is constructed using a Chacon-like operation then

$$\tilde{h}_{n+1} = (r_n + 1)\tilde{h}_n + s_n M_n + c_{n+1} - c_n$$

where s_n is the number of subcolumns over which non-elevation spacers are placed.

Proof. In general, if $s_{n,j}$ is the number of non-elevation spacers added above each subcolumn then $\tilde{h}_{n+1} = (r_n+1)h_n + r_n c_n + \sum_{j=0}^{r_n-1} s_{n,j} + c_{n+1}$ since no spacers (elevating or otherwise) are added above the rightmost subcolumn.

For n such that B_{n+1} is constructed using the staircase operation, $s_{n,j} = j$ so $\tilde{h}_{n+1} = (r_n+1)\tilde{h}_n + \frac{1}{2}r_n(r_n-1) + c_{n+1} - c_n$.

For n such that B_{n+1} is constructed using a Chacon-like operation, $\sum s_{n,j} = s_n M_n$ so the claim follows similarly. □

Definition 2.6. Say that B_{n+1} is constructed using the γ-Chacon-like form for $\gamma \geq 2$ if

$$B_{n+1} = \left(B_n 1^{c_n} 1^{M_n}\right)^\gamma B_n 1^{c_n} \left(B_n 1^{c_n} 1^{M_n}\right)^\gamma B_n$$

for some $M_n \geq 1$.

3 Word complexity estimates

Proposition 3.1. *Let T be an elevated staircase-Chacon-like hybrid transformation and n be such that B_{n+1} is of staircase form. Then $p(|B_{n+1}| + c_{n+1})$ is completely determined by the cutting and stacking operations up to B_{n+1}.*

Proof. Let w be a right-special word with $|w| \leq |B_{n+1}| + c_{n+1}$. Since $1^{|w|}$ will appear regardless, we may assume w has 01^z as a suffix for some $z \geq 1$ (as 00 is not in the language because $c_n \geq 1$).

If $z < c_{n+1}$ then w must be a subword of $1^{c_n} B_{n+1}$ because $1^{c_{n+1}} v$ is not right-special for any proper prefix v of B_{n+1} (as $1^{c_{n+1}}$ is always followed by B_{n+1}).

If $z > c_{n+1}$ then, since $1^{c_{n+1}+M_{n+1}}$ is not right-special and $c_N \geq |B_{n+1}|$ for the first $N > n+1$ for which B_{N+1} is constructed using the staircase operation, then $w = 1^{|w|}$.

If $z = c_{n+1}$ then w has $01^{c_{n+1}}$ as a suffix hence shares a suffix with $1^{c_{n+1}} B_{n+1} 1^{c_{n+1}}$ and is therefore a suffix of that word.

Therefore all right-special words of length at most $|B_{n+1}| + c_{n+1}$ are determined by B_{n+1} and c_{n+1}. □

Proposition 3.2. *Let w be a right-special word in the language of an elevated staircase-Chacon-like hybrid transformation such that $|B_n| < |w|$ for n such that B_{n+1} is of Chacon-like form and such that $|w| \leq C_N$ for the smallest $N > n$ such that B_N is of staircase form. Then one of the following holds:*

- *$w = 1^{|w|}$*
- *w shares a suffix with $1^{c_n} B_n 1^{c_n}$*

Proof. If $w = 1^{|w|}$ then we are in the first case so assume that w has 01^z as a suffix for some $z \geq 0$. By construction, $00 \notin \mathcal{L}$ so $z \geq 1$.

Let m such that $c_m \leq z < c_{m+1}$. Here m is unique by the construction of $\{c_n\}$. This follows as when B_{m+1} is of staircase form, c_m is strictly larger than any spacer amounts added at any prior stage and c_{m+1} is strictly greater than any spacer amounts added in B_{m+1} (and any values outside the ranges between c_n and c_{n+1} can never be the suffix of a right-special word).

Suppose $m \geq N$. Then $|w| \leq c_N \leq c_m$ so $w = 1^{|w|}$. Therefore $m < N$.

Suppose $m < n$ and B_{m+1} is constructed using the staircase operation. Then w ends properly inside a copy of B_{m+1}. Then w has a suffix of the form $u1^{c_{m+1}}v$ for some word u and some prefix v of B_m and v has 1^{c_m+j} as a suffix for some $0 \leq j < r_m$. This follows as $|w| > |B_n|$. Then w has $1^{c_{m+1}} B_m 1^{c_m} B_m 1^{c_m+1} \dots B_m 1^{c_m+j-1} B_m 1^{c_m+j}$ as a suffix. But that word is never followed by a 1 so is not right-special. Therefore $m < n$ is impossible in this case.

Likewise, if $m < n$ and m is prior to a staircase operation t which is prior to n then w must contain $1^{c_{t+1}}B_t1^{c_t}B_t1^{c_t+1}\dots B_t1^{c_t+j-1}B_t1^{c_t+j}$ followed by a prefix of B_t. But that word is never right-special.

Therefore m and n are not separated by a staircase operation. So we may assume that $c_n \le z < c_N$. Therefore z is either c_n or $c_n + M_n$ by construction. Since $01^{c_n+M_n+1}$ is not in the language (as $C_N > c_n + M_n + 1$), then $z = c_n$. So w has 01^{c_n} as a suffix. Then by construction w shares a suffix with $B_n1^{c_n}$ hence with $1^{c_n}B_n1^{c_n}$. □

Lemma 3.3. *Let w be a right-special word in the language of an elevated staircase-Chacon-like hybrid transformation such that $|B_n| < |w|$ for n such that all of B_n and B_{n+1} and B_{n+2} are of γ-Chacon-like form and such that w shares a suffix with $11^{c_n}B_n1^{c_n}$. Then one of the following holds:*

- *w shares a suffix with $B_{n+1}1^{c_n}$*
- *w is a suffix of $\left(B_n1^{c_n}1^{M_n}\right)^{2\gamma})B_n1^{c_n}$*

Proof. Here w shares a suffix with $1^{c_n}B_n1^{c_n+M_n}B_n1^{c_n}$ since $11^{c_n}B_n1^{c_n}$ only appears as a suffix of that word. By construction, $c_n = c_{n+1}$ and $M_n = M_{n+1}$.

Suppose w shares a suffix with $01^{c_n}B_n1^{c_n+M_n}B_n1^{c_n}$. Then w and $1^{c_n}B_n1^{c_n}B_n1^{c_n+M_n}B_n1^{c_n}$ have a common suffix but $1^{c_n}B_n1^{c_n}B_n1^{c_n+M_n}B_n1^{c_n}B_n$ is not in the language (since $\gamma > 1$) so this would mean w is not right-special.

So w shares a suffix with $11^{c_n}B_n1^{c_n+M_n}B_n1^{c_n}$ hence with $1^{c_n+M_n}B_n1^{c_n+M_n}B_n1^{c_n}$.

If w shares a suffix with $01^{c_n}(B_n1^{c_n+M_n})^tB_n1^{c_n+M_n}B_n1^{c_n}$ for any $t \ge 1$ then $w0$ shares a suffix with $01^{c_n}(B_n1^{c_n+M_n})^tB_n1^{c_n+M_n}B_n1^{c_n}0$ which only appears as a subword of $(B_n1^{c_n})^2(B_n1^{c_n+M_n})^{t+1}(B_n1^{c_n})^2$ meaning that $t \ge \gamma$. Therefore w shares a suffix with $(B_n1^{c_n+M_n})^\gamma B_n1^{c_n}$.

Consider when w shares a suffix with $11^{c_n}(B_n1^{c_n+M_n})^\gamma B_n1^{c_n+M_n}B_n1^{c_n}$. That word only appears as a subword of $B_{n+1}1^{c_n+M_n}B_{n+1}1^{c_n}$ and is always preceded by $(B_n1^{c_n+M_n})^\gamma B_n1^{M_n-1}$. Therefore w shares a suffix with $(B_n1^{c_n+M_n})^{2\gamma+1}B_11^{c_n}$.

Since $(B_n1^{c_n+M_n})^{2\gamma+1}B_11^{c_n}$ is not right-special (because $(B_n1^{c_n+M_n})^{2\gamma+2}$ is not in the language), then w is a suffix of $(B_n1^{c_n+M_n})^{2\gamma}B_11^{c_n}$ and we are in the second claimed case.

Now consider the case when w shares a suffix with $01^{c_n}(B_n1^{c_n+M_n})^\gamma B_n1^{c_n}$. Then w shares a suffix with $B_n1^{c_n}(B_n1^{c_n+M_n})^\gamma B_n1^{c_n}$ which is always preceded by $(B_n1^{c_n+M_n})^\gamma B_n1^{c_n}$, in which case we are in the second case claimed, or by preceded $1^{c_n+M_n}$ in which case w shares a suffix with $((B_n1^{c_n+M_n})^\gamma B_n1^{c_n})^2$ so with $B_{n+1}1^{c_n}$ and we are in the first claimed case. □

Lemma 3.4. *Let w be a right-special word in the language of an elevated staircase-Chacon-like hybrid transformation such that $|B_n| < |w|$ for n such that all of B_n and B_{n+1} and B_{n+2} are of γ-Chacon-like form and such that w shares a suffix with $01^{c_n}B_n1^{c_n}$.*

Then w is a suffix of $(B_{n-1}1^{c_{n-1}}1^{M_{n-1}})^{\gamma-1}B_{n-1}1^{c_{n-1}}B_n1^{c_n}$.

Proof. Here w shares a suffix with $1^{c_n}B_n1^{c_n}B_n1^{c_n}$. By construction, $c_{n-1} = c_n$ and $M_{n-1} = M_n$.

Since $B_n1^{c_n}B_n$ only appears as a subword of $1^{c_n+M_n}B_n1^{c_n}B_n1^{c_n+M_n}$, then $1^{c_n}B_n1^{c_n}B_n1^{c_n}B_n$ is not in the language so $1^{c_n}B_n1^{c_n}B_n1^{c_n}0$ is not so w must be a proper suffix of $1^{c_n}B_n1^{c_n}B_n1^{c_n}$.

Consider when w shares a suffix with $01^{c_n}(B_{n-1}1^{c_n+M_n})^\gamma B_{n-1}1^{c_n}B_n1^{c_n}$. That word only appears when the leading zero is the tail zero in the first $(B_{n-1}1^{c_n+M_n})^\gamma B_{n-1}$ which is in the first B_n of $B_n1^{c_n}B_n1^{c_n}$. Then $01^{c_n}(B_{n-1}1^{c_n+M_n})^\gamma B_{n-1}1^{c_n}B_n1^{c_n}0$ is not in the language so w is a suffix of $1^{c_n}(B_{n-1}1^{c_n+M_n})^{\gamma-1}B_{n-1}1^{c_n}B_n1^{c_n}$ as claimed.

Now consider when w shares a suffix with $11^{c_n}(B_{n-1}1^{c_n+M_n})^\gamma B_{n-1}1^{c_n}B_n1^{c_n}$. Then it shares a suffix with $1^{c_n+M_n}(B_{n-1}1^{c_n+M_n})^\gamma B_{n-1}1^{c_n}B_n1^{c_n}$ so, as $1^{c_n+M_n}$ is always preceded by B_{n-1} (as it is a suffix of B_n), then w shares a suffix with $1^{c_n}B_{n-1}1^{c_n+M_n}(B_{n-1}1^{c_n+M_n})^\gamma B_{n-1}1^{c_n}B_n1^{c_n}$ but $(B_{n-1}1^{c_n+M_n})^{\gamma+1}$ is not in the language so this is impossible. □

Proposition 3.5. *Let w be a right-special word in the language of an elevated staircase-Chacon-like hybrid transformation such that $|B_n| < |w|$ for n such that all of B_n and B_{n+1} and B_{n+2} are of γ-Chacon-like form and such that $|w| \leq C_N$ for the smallest $N > n$ such that B_N is of staircase form. Then one of the following holds:*

- $w = 1^{|w|}$
- *w shares a suffix with $B_{n+1}1^{c_n}$*
- *w is a suffix of $\left(B_n 1^{c_n} 1^{M_n}\right)^{2\gamma} B_n 1^{c_n}$*
- *w is a suffix of $(B_{n-1}1^{c_{n-1}}1^{M_{n-1}})^{\gamma-1}B_{n-1}1^{c_{n-1}}B_n 1^{c_n}$*

Proof. By Proposition 3.2, if $w \neq 1^{|w|}$ then w shares a suffix with $1^{c_n}B_n 1^{c_n}$. By Lemmas 3.3 and 3.4 then the claim follows. □

Proposition 3.6. *Let T be an elevated staircase-Chacon-like hybrid transformation such that all of B_n and B_{n+1} and B_{n+2} are of γ-Chacon-like form. Then*

$$p(q+1) - p(q) \leq \begin{cases} 3 & \text{if } \tilde{h}_n < q \leq \frac{3}{2}\tilde{h}_n \\ 2 & \text{if } \frac{3}{2}\tilde{h}_n < q \leq \frac{1}{2}\tilde{h}_{n+1} \\ 3 & \text{if } \frac{1}{2}\tilde{h}_{n+1} < q \leq \tilde{h}_{n+1} - \tilde{h}_n \\ 2 & \text{if } \tilde{h}_{n+1} - \tilde{h}_n < q \leq \tilde{h}_{n+1} \end{cases}$$

Proof. Let $\tilde{h}_n < q \leq \tilde{h}_{n+1}$. The words 1^q and the suffix of $B_{n+1}1^{c_n}$ of length q are both right-special. By Proposition 3.5, the only remaining right-special words are suffixes of $(B_n 1^{c_n+M_n})^{2\gamma}B_n 1^{c_n}$ and $(B_{n-1}1^{c_n+M_n})^{\gamma-1}B_{n-1}1^{c_n}B_n 1^{c_n}$.

Since $(B_n 1^{c_n+M_n})^{2\gamma}B_n 1^{c_n}$ and $B_{n+1}1^{c_n}$ share a common suffix of length $(\gamma+1)\tilde{h}_n + \gamma M_n = \frac{1}{2}\tilde{h}_{n+1}$, the only new right-special words from that word have length $\frac{1}{2}\tilde{h}_{n+1} < q \leq (2\gamma+1)\tilde{h}_n + 2\gamma M_n = \tilde{h}_{n+1} - \tilde{h}_n$.

Since $(B_{n-1}1^{c_n+M_n})^{\gamma-1}B_{n-1}1^{c_n}B_n 1^{c_n}$ shares a suffix of length $\tilde{h}_n + c_n$ with $B_{n+1}1^{c_n}$, the only new right-special words from that word have length $\tilde{h}_n < q \leq \gamma\tilde{h}_{n-1} + \gamma M_n + \tilde{h}_n = \frac{1}{2}\tilde{h}_n$. □

Proposition 3.7. *Let T be an elevated staircase-Chacon-like hybrid such that the γ-Chacon-like operation is used to construct B_m through B_{n+2} for $n > m$. Then*

$$\frac{p(\frac{1}{2}\tilde{h}_{n+1})}{\frac{1}{2}\tilde{h}_{n+1}} < 2 + \frac{1}{\gamma+1} + 3^{m-n}\frac{p(\tilde{h}_m)}{\tilde{h}_m}$$

Proof. Since B_n and B_{n+1} and B_{n+2} are all constructed using the γ-Chacon-like operation, by Proposition 3.6,

$$p(\tilde{h}_{n+1}) - p(\tilde{h}_n) \leq 2(\tilde{h}_{n+1} - \tilde{h}_n) + \frac{1}{2}\tilde{h}_n + \frac{1}{2}\tilde{h}_n + \frac{1}{2}\tilde{h}_{n+1} - \tilde{h}_n = \frac{5}{2}\tilde{h}_{n+1} - \frac{5}{2}\tilde{h}_n$$

and therefore, by the telescoping nature of the series,

$$p(\tilde{h}_n) \leq \frac{5}{2}\tilde{h}_n - \frac{5}{2}\tilde{h}_m + p(\tilde{h}_m)$$

Again by Proposition 3.6,

$$p\left(\frac{1}{2}\tilde{h}_{n+1}\right) - p(\tilde{h}_n) \leq 2\left(\frac{1}{2}\tilde{h}_{n+1} - \tilde{h}_n\right) + \frac{1}{2}\tilde{h}_n$$

and therefore

$$p\left(\frac{1}{2}\tilde{h}_{n+1}\right) \leq \tilde{h}_{n+1} - 2\tilde{h}_n + \frac{1}{2}\tilde{h}_n + \frac{5}{2}\tilde{h}_n - \frac{5}{2}\tilde{h}_m + p(\tilde{h}_m) \leq \tilde{h}_{n+1} + \tilde{h}_n + p(\tilde{h}_m)$$

meaning that

$$\frac{p(\frac{1}{2}\tilde{h}_{n+1})}{\frac{1}{2}\tilde{h}_{n+1}} \leq 2 + \frac{\tilde{h}_n}{(\gamma+1)\tilde{h}_n + \gamma M_n} + \frac{p(\tilde{h}_m)}{\frac{1}{2}\tilde{h}_{n+1}} < 2 + \frac{1}{\gamma+1} + \frac{\tilde{h}_m}{\tilde{h}_n}\frac{p(\tilde{h}_m)}{\tilde{h}_m}$$

and the claim follows since $\tilde{h}_{t+1} > 3\tilde{h}_t$ (as $\gamma \geq 2$) for all t. □

4 Mixing spans and meshes

For a column, C_n, recall that $I_{n,0}$ is the base of the column and $I_{n,j} = T^j I_{n,0}$ for $0 \leq j < \tilde{h}_n$ are the levels (including the elevating levels).

For a level $I_{n,j}$ and $1 \leq i \leq r_n$, recall that $I_{n,j}^{[i]}$ is the i^{th} sublevel, i.e. the $1/r_n+1$ fraction of $I_{n,j}$ in the i^{th} subcolumn.

Notation 4.1. For a measurable set B, let $\chi_B = \mathbb{1}_B - \mu(B)$.

For $J_{n,0} \subseteq I_{n,0}$, let $J_{n,0}^{[i]} := J_{n,0} \cap I_{n,0}^{[i]}$ for $0 \leq i \leq r_n$.

Definition 4.2. Let $I_{n,0}$ be the base level in a tower C_n of a rank-one transformation. If $J_{n,0} \subseteq I_{n,0}$ such that for all $0 \leq i \leq r_n$, it holds that $J_{n,0} \cap I_{n,0}^{[i]}$ has the property that $J_{n,0}^{[i]} = T^{i\tilde{h}_n + \sum_{z=0}^{i-1} s_{n,z}} J_{n,0}^{[0]}$ then $J_{n,0}$ is a **mesh prototype** for $I_{n,0}$.

Given a mesh prototype $J_{n,0}$ for $I_{n,0}$, for $0 \leq j < \tilde{h}_n$ and $0 \leq i \leq r_n$, write

$$J_{n,j} = T^j J_{n,0} \qquad \text{and} \qquad J_{n,j}^{[i]} = T^j \left(J_{n,0} \cap I_{n,0}^{[i} \right)$$

and say that $\bigsqcup_{j=0}^{\tilde{h}_n - 1} J_{n,j}$ is a **mesh** for C_n and that $J_{n,j}$ are the **mesh levels**.

Note that C_n is a mesh for itself by putting $J_{n,j} = I_{n,j}$ and also that the intersection of two meshes for C_n is again a mesh for C_n.

Definition 4.3. For a mesh $\bigsqcup_{j=0}^{\tilde{h}_n - 1} J_{n,j}$ of some column C_n of a rank-one transformation, the **size** of the mesh is

$$\frac{\mu\left(\bigsqcup_{j=0}^{\tilde{h}_n - 1} J_{n,j}\right)}{\mu(C_n)} = \frac{\mu(J_{n,0})}{\mu(I_{n,0})}$$

Lemma 4.4. *Let $J_{n,0}$ be a mesh prototype for some base $I_{n,0}$ of some column C_n of a rank-one transformation. Let A be a union of levels in C_n. Then for $0 \leq j < \tilde{h}_n$ and $0 \leq i \leq r_n$,*

$$\mu(J_{n,j}^{[i]} \cap A) = \frac{1}{r_n + 1}\mu(J_{n,j} \cap A)$$

Proof. Since A is a union of levels in C_n, either $I_{n,j} \subseteq A$ or $I_{n,j} \cap A = \varnothing$. Since $J_{n,j}^{[i]} \subseteq I_{n,j}^{[i]} \subseteq I_{n,j}$, if $I_{n,j} \cap A = \varnothing$ then $J_{n,j} \cap A = \varnothing$ so $\mu(J_{n,j}^{[i]} \cap A) = 0 = \frac{1}{r_n+1}0 = \frac{1}{r_n+1}\mu(J_{n,j} \cap A)$. Likewise, if $I_{n,j} \subseteq A$ then $J_{n,j} \subseteq A$ and since $\mu(J_{n,j}^{[i]}) = \mu(J_{n,0}^{[0]})$ (as T is measure-preserving and from the definition of a mesh) is constant over all $0 \leq i \leq r_n$ and j (for a fixed n), then $\mu(J_{n,j}^{[i]} \cap A) = \mu(J_{n,j}^{[i]}) = \frac{1}{r_n+1}\mu(J_{n,j}) = \frac{1}{r_n+1}\mu(J_{n,j} \cap A)$. □

Lemma 4.5. *Let $\bigsqcup J_{n+1,j}$ be a mesh of C_{n+1} for some column of a rank-one transformation. Then $C_n \cap \bigsqcup J_{n+1,j}$ is a mesh of C_n.*

Proof. Let $s_{n,i}$ for $1 \leq i \leq r_n$ be the number of non-elevation spacers added above each subcolumn of

C_n. Then for $0 \le j < \tilde{h}_n$,

$$I_{n,j} = \bigsqcup_{i=0}^{r_n} I_{n,j}^{[i]} = \bigsqcup_{i=0}^{r_n} I_{n+1,j+i\tilde{h}_n+\sum_{t=0}^{i-1} s_{n,t}}$$

and set

$$J_{n,j} := I_{n,j} \cap \bigsqcup_{m=0}^{\tilde{h}_{n+1}-1} J_{n+1,m} = \bigsqcup_{i=0}^{r_n} \left(I_{n+1,j+i\tilde{h}_n+\sum_{t=0}^{i-1} s_{n,t}} \cap J_{n+1,j+i\tilde{h}_n+\sum_{t=0}^{i-1} s_{n,t}} \right)$$

Then

$$J_{n,j}^{[i]} = I_{n+1,j+i\tilde{h}_n+\sum_{t=0}^{i-1} s_{n,t}} \cap J_{n+1,j+i\tilde{h}_n+\sum_{t=0}^{i-1} s_{n,t}} = T^{i\tilde{h}_n+\sum_{0=1}^{i-1} s_{n,t}} J_{n,j}^{[0]}$$

meaning $\bigsqcup J_{n,j}$ form a mesh of C_n. □

Lemma 4.6. *Let T be a rank-one transformation and A be a measurable set and let $\epsilon > 0$ and $K \in \mathbb{N}$. Let $\bigsqcup_{j=0}^{\tilde{h}_n-1} J_{n,j}$ be a mesh for C_n for some n. If for all $1 \le k \le K$,*

$$\sum_{j=0}^{\tilde{h}_n-1} \left| \mu(T^{k\tilde{h}_n} J_{n,j} \cap A) - \mu J_{n,j} \mu A \right| < \epsilon$$

then for all $\tilde{h}_n \le t \le K\tilde{h}_n$,

$$\sum_{j=0}^{\tilde{h}_n-1} \left| \mu(T^{t} J_{n,j} \cap A) - \mu J_{n,j} \mu A \right| < 2\epsilon$$

Proof. Let $\tilde{h}_n \le t < K\tilde{h}_n$. Write $t = k\tilde{h}_n + z$ for the unique $0 \le z < \tilde{h}_n$ and $1 \le k < K$. Then

$$\begin{aligned}
&\sum_{j=0}^{\tilde{h}_n-1} \left| \mu(T^{t} J_{n,j} \cap A) - \mu J_{n,j} \mu A \right| \\
&\quad = \sum_{j=\tilde{h}_n-z}^{\tilde{h}_n-1} \left| \mu(T^{t} J_{n,j} \cap A) - \mu J_{n,j} \mu A \right| + \sum_{j=0}^{\tilde{h}_n-z-1} \left| \mu(T^{t} J_{n,j} \cap A) - \mu J_{n,j} \mu A \right| \\
&\quad = \sum_{j=0}^{z-1} \left| \mu(T^{(k+1)\tilde{h}_n} J_{n,j} \cap A) - \mu J_{n,j} \mu A \right| + \sum_{j=z}^{\tilde{h}_n-1} \left| \mu(T^{k\tilde{h}_n} J_{n,j} \cap A) - \mu J_{n,j} \mu A \right| < \epsilon + \epsilon
\end{aligned}$$ □

Lemma 4.7. *Let T be a rank-one transformation and n be such that B_{n+1} is constructed using the staircase operation. Let A be a union of levels in C_n and $\bigsqcup_{j=0}^{\tilde{h}_n-1} J_{n,j}$ be a mesh for C_n. Then for $\tilde{h}_n > j \ge r_n$,*

$$\left| \mu(T^{\tilde{h}_n} J_{n,j} \cap A) - \mu J_{n,j} \mu A \right| \le \int_{J_{n,j}} \left| \frac{1}{r_n+1} \sum_{i=0}^{r_n-1} \chi_A \circ T^{-i} \right| d\mu + \frac{1}{r_n+1} \mu J_{n,j}$$

Proof. For $j \ge r_n$ and $0 \le i < r_n$,

$$T^{\tilde{h}_n} J_{n,j}^{[i]} = T^{-i} J_{n,j}^{[i+1]} = J_{n,j-i}^{[i+1]}$$

and since A is a union of levels, by Lemma 4.4,

$$\mu(T^{\tilde{h}_n} J_{n,j}^{[i]} \cap A) = \mu(J_{n,j-i}^{[i+1]} \cap A) = \frac{1}{r_n+1} \mu(J_{n,j-i} \cap A) = \frac{1}{r_n+1} \mu(T^{-i} J_{n,j} \cap A)$$

and therefore

$$\mu(T^{\tilde{h}_n} J_{n,j} \cap A) = \frac{1}{r_n+1} \sum_{i=0}^{r_n-1} \mu(T^{-i} J_{n,j} \cap A) + \mu(T^{\tilde{h}_n} J_{n,j}^{[r_n]} \cap A)$$

Then, since $\mu(T^{\tilde{h}_n} J_{n,j}^{[r_n]} \cap A) \le \frac{1}{r_n+1}\mu J_{n,j}$,

$$\begin{aligned}\left|\mu(T^{\tilde{h}_n} J_{n,j} \cap A) - \mu J_{n,j}\mu A\right| &\le \left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-1} \mu(T^{-i} J_{n,j} \cap A) - \mu J_{n,j}\mu A\right| + \frac{1}{r_n+1}\mu J_{n,j} \\ &= \left|\int_{J_{n,j}} \frac{1}{r_n+1}\sum_{i=0}^{r_n-1} \chi_A \circ T^{-i}\, d\mu\right| + \frac{1}{r_n+1}\mu J_{n,j} \\ &\le \int_{J_{n,j}} \left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-1} \chi_A \circ T^{-i}\right| d\mu + \frac{1}{r_n+1}\mu J_{n,j}\end{aligned}$$ □

Lemma 4.8. *Let T be a rank-one transformation and n be such that B_{n+1} is constructed using the staircase operation. Let A be a union of levels in C_n and $\tilde{C}_n = \bigsqcup_{j=0}^{\tilde{h}_n-1} J_{n,j}$ be a mesh for C_n. Then for $\tilde{h}_n > j \ge r_n$,*

$$\left|\mu(T^{\tilde{h}_n} I_{n,j} \cap A \cap \tilde{C}_n) - \mu J_{n,j}\mu A\right| \le \int_{J_{n,j}} \left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-1} \chi_A \circ T^{-i}\right| d\mu + \mu(T^{\tilde{h}_n} I_{n,j}^{[r_n]} \cap A \cap \tilde{C}_n)$$

Proof. For $j \ge r_n$ and $0 \le i < r_n$,

$$T^{\tilde{h}_n} I_{n,j}^{[i]} = T^{-i} I_{n,j}^{[i+1]} = I_{n,j-i}^{[i+1]}$$

since $I_{n,j-i}^{[i+1]} \cap A \cap \tilde{C}_n \subseteq J_{n,j-i}^{[i+1]}$ and since A is a union of levels, by Lemma 4.4,

$$\mu(T^{\tilde{h}_n} I_{n,j}^{[i]} \cap A \cap \tilde{C}_n) = \mu(I_{n,j-i}^{[i+1]} \cap A \cap \tilde{C}_n) = \mu(J_{n,j-i}^{[i+1]} \cap A) = \frac{1}{r_n+1}\mu(J_{n,j-i} \cap A) = \frac{1}{r_n+1}\mu(T^{-i} J_{n,j} \cap A)$$

and the claim follows as in Lemma 4.7. □

Lemma 4.9. *Let T be a rank-one transformation and n be such that B_{n+1} is constructed using the staircase operation. Let A be a union of levels in C_n and $\tilde{C}_n = \bigsqcup_{j=0}^{\tilde{h}_n-1} J_{n,j}$ be a mesh for C_n. Then*

$$\sum_{j=0}^{\tilde{h}_n-1} \left|\mu(T^{\tilde{h}_n} J_{n,j} \cap A) - \mu J_{n,j}\mu A\right| \le \int_{\tilde{C}_n} \left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-1} \chi_A \circ T^{-i}\right| d\mu + \left(\frac{1}{r_n+1} + \frac{r_n}{\tilde{h}_n}\right)\mu\tilde{C}_n$$

and

$$\sum_{j=0}^{\tilde{h}_n-1} \left|\mu(T^{\tilde{h}_n} I_{n,j} \cap A \cap \tilde{C}_n) - \mu J_{n,j}\mu A\right| \le \int_{\tilde{C}_n} \left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-1} \chi_A \circ T^{-i}\right| d\mu + \sum_{j=0}^{\tilde{h}_n-1} \mu(T^{\tilde{h}_n} I_{n,j}^{[r_n]} \cap \tilde{C}_n) + \frac{r_n}{\tilde{h}_n}\mu\tilde{C}_n$$

Proof. Since the bottom r_n levels in the mesh have measure at most $\frac{r_n}{\tilde{h}_n}\mu\tilde{C}_n$, by Lemma 4.7,

$$\begin{aligned}\sum_{j=0}^{\tilde{h}_n-1} \left|\mu(T^{\tilde{h}_n} J_{n,j} \cap A) - \mu J_{n,j}\mu A\right| &\le \sum_{j=r_n}^{\tilde{h}_n-1} \left|\mu(T^{\tilde{h}_n} J_{n,j} \cap A) - \mu J_{n,j}\mu A\right| + \frac{r_n}{h_n}\mu\tilde{C}_n \\ &\le \sum_{j=r_n}^{\tilde{h}_n-1} \int_{J_{n,j}} \left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-1} \chi_A \circ T^{-i}\right| d\mu + \frac{1}{r_n+1}\mu\tilde{C}_n + \frac{r_n^2}{\tilde{h}_n}\mu\tilde{C}_n \\ &\le \int_{\tilde{C}_n} \left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-1} \chi_A \circ T^{-i}\right| d\mu + \frac{1}{r_n+1}\mu\tilde{C}_n + \frac{r_n}{\tilde{h}_n}\mu\tilde{C}_n\end{aligned}$$

where we used that the levels are disjoint to consolidate the integral. Similarly, using Lemma 4.8,

$$\begin{aligned}\sum_{j=0}^{\tilde{h}_n-1}\left|\mu(T^{\tilde{h}_n}I_{n,j}\cap A\cap\tilde{C}_n)-\mu J_{n,j}\mu A\right| &\le \sum_{j=r_n}^{\tilde{h}_n-1}\left|\mu(T^{\tilde{h}_n}I_{n,j}\cap A\cap\tilde{C}_n)-\mu J_{n,j}\mu A\right|+\frac{r_n}{\tilde{h}_n}\mu\tilde{C}_n\\ &\le \sum_{j=r_n}^{\tilde{h}_n-1}\int_{J_{n,j}}\left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-1}\chi_A\circ T^{-i}\right|d\mu+\sum_{j=0}^{\tilde{h}_n-1}\mu(T^{\tilde{h}_n}I_{n,j}^{[r_n]}\cap\tilde{C}_n)+\frac{r_n^2}{\tilde{h}_n}\mu\tilde{C}_n\\ &\le \int_{\tilde{C}_n}\left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-1}\chi_A\circ T^{-i}\right|d\mu+\sum_{j=0}^{\tilde{h}_n-1}\mu(T^{\tilde{h}_n}I_{n,j}^{[r_n]}\cap\tilde{C}_n)+\frac{r_n}{\tilde{h}_n}\mu\tilde{C}_n\end{aligned}$$ □

Lemma 4.10. *Let T be a rank-one transformation and n be such that B_{n+1} is constructed using the staircase operation. Let A be a union of levels in C_n and $\tilde{C}_n^{(q)}$ be a collection of disjoint meshes of C_n. Then*

$$\sum_q\sum_{j=0}^{\tilde{h}_n-1}\left|\mu(T^{\tilde{h}_n}J_{n,j}^{(q)}\cap A)-\mu J_{n,j}^{(q)}\mu A\right|\le\int\left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-1}\chi_A\circ T^{-i}\right|d\mu+\frac{1}{r_n+1}+\frac{r_n}{\tilde{h}_n}$$

and

$$\sum_q\sum_{j=0}^{\tilde{h}_n-1}\left|\mu(T^{\tilde{h}_n}I_{n,j}\cap A\cap\tilde{C}_n^{(q)})-\mu I_{n,j}\mu(A\cap\tilde{C}_n^{(q)})\right|\le\int\left|\frac{1}{r_n}\sum_{i=1}^{r_n-1}\chi_A\circ T^{-i}\right|d\mu+\frac{1}{r_n}+\frac{r_n}{h_n}$$

Proof. The first is immediate from the first statement in Lemma 4.9 as the meshes are disjoint. For the second, observe that $\mu J_{n,j}^{(q)}=\rho\mu I_{n,j}$ where ρ is the size of $\tilde{C}_n^{(q)}$ and that, since A is a union of levels, $\mu(A\cap\tilde{C}_n^{(q)})=\rho\mu(A)$. Then the claim follows from the second statement in Lemma 4.9 and that, as the meshes are disjoint,

$$\sum_q\sum_{j=0}^{\tilde{h}_n-1}\mu(T^{\tilde{h}_n}I_{n,j}^{[r_n]}\cap\tilde{C}_n^{(q)})=\sum_{j=0}^{\tilde{h}_n-1}\mu(T^{\tilde{h}_n}I_{n,j}^{[r_n]})=\frac{1}{r_n+1}\sum_{j=0}^{\tilde{h}_n-1}\mu(T^{\tilde{h}_n}I_{n,j})\le\frac{1}{r_n+1}$$ □

Lemma 4.11. *Let T be a rank-one transformation and n be such that B_{n+1} is constructed using the staircase operation. Let $\tilde{C}_n=\bigsqcup_{j=0}^{\tilde{h}_n-1}J_{n,j}$ be a mesh for C_n. Let $1\le k<r_n$ and A be a union of levels in C_n. Then for $j\ge kr_n+\frac{1}{2}k(k-1)$,*

$$\begin{aligned}\left|\mu(T^{k\tilde{h}_n}J_{n,j}\cap A)-\mu J_{n,j}\mu A\right|\le&\int_{T^{-\frac{1}{2}k(k-1)}J_{n,j}}\left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k}\chi_A\circ T^{-ik}\right|d\mu\\&+\left|\sum_{i=r_n-k+1}^{r_n}\mu(T^{k\tilde{h}_n}J_{n,j}^{[i]}\cap A)-\mu J_{n,j}^{[i]}\mu A\right|\end{aligned}$$

and therefore

$$\left|\mu(T^{k\tilde{h}_n}J_{n,j}\cap A)-\mu J_{n,j}\mu A\right|\le\int_{T^{-\frac{1}{2}k(k-1)}J_{n,j}}\left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k-1}\chi_A\circ T^{-ik}\right|d\mu+\frac{k}{r_n+1}\mu J_{n,j}$$

Proof. For $j\ge kr_n$ and $0\le i\le r_n-k$,

$$T^{k\tilde{h}_n}J_{n,j}^{[i]}=T^{-ik-\frac{1}{2}k(k-1)}J_{n,j}^{[i+k]}=J_{n,j-ik-\frac{1}{2}k(k-1)}^{[i+k]}$$

so by Lemma 4.4,

$$\mu(T^{k\tilde{h}_n}J_{n,j}^{[i]}\cap A)=\mu(J_{n,j-ik-\frac{1}{2}k(k-1)}^{[i+k]}\cap A)=\frac{1}{r_n+1}\mu(J_{n,j-ik-\frac{1}{2}k(k-1)}\cap A)=\frac{1}{r_n+1}\mu(T^{-ik-\frac{1}{2}k(k-1)}J_{n,j}\cap A)$$

and therefore

$$\mu(T^{k\tilde{h}_n}J_{n,j}\cap A)=\frac{1}{r_n+1}\sum_{i=0}^{r_n-k}\mu(T^{-ik-\frac{1}{2}k(k-1)}J_{n,j}\cap A)+\sum_{i=r_n-k+1}^{r_n}\mu(T^{k\tilde{h}_n}J_{n,j}^{[i]}\cap A)$$

Then

$$\begin{aligned}&\left|\mu(T^{k\tilde{h}_n}J_{n,j}\cap A)-\mu J_{n,j}\mu A\right|\\&\quad\le\left|\int_{T^{-\frac{1}{2}k(k-1)}J_{n,j}}\frac{1}{r_n+1}\sum_{i=0}^{r_n-k}\chi_A\circ T^{-ik}\,d\mu\right|+\left|\sum_{i=r_n-k+1}^{r_n}\mu(T^{k\tilde{h}_n}J_{n,j}^{[i]}\cap A)-\mu J_{n,j}^{[i]}\mu A\right|\\&\quad\le\int_{T^{-\frac{1}{2}k(k-1)}J_{n,j}}\left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k}\chi_A\circ T^{-ik}\right|d\mu+\left|\sum_{i=r_n-k+1}^{r_n}\mu(T^{k\tilde{h}_n}J_{n,j}^{[i]}\cap A)-\mu J_{n,j}^{[i]}\mu A\right|\end{aligned}$$ □

Lemma 4.12. *Let T be a rank-one transformation and n be such that B_{n+1} is constructed using the staircase operation. Let $\tilde{C}_n=\bigsqcup_{j=0}^{\tilde{h}_n-1}J_{n,j}$ be a mesh for C_n. Let $1\le k<r_n$ and A be a union of levels in C_n. Then for $j\ge kr_n+\frac{1}{2}k(k-1)$,*

$$\begin{aligned}\left|\mu(T^{k\tilde{h}_n}I_{n,j}\cap A\cap\tilde{C}_n)-\mu I_{n,j}\mu(A\cap\tilde{C}_n)\right|&\le\int_{T^{-\frac{1}{2}k(k-1)}J_{n,j}}\left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k}\chi_A\circ T^{-ik}\right|d\mu\\&\quad+\left|\sum_{i=r_n-k+1}^{r_n}\mu(T^{k\tilde{h}_n}I_{n,j}^{[i]}\cap A\cap\tilde{C}_n)-\mu I_{n,j}^{[i]}\mu(A\cap\tilde{C}_n)\right|\end{aligned}$$

and therefore

$$\left|\mu(T^{k\tilde{h}_n}I_{n,j}\cap A\cap\tilde{C}_n)-\mu I_{n,j}\mu(A\cap\tilde{C}_n)\right|\le\int_{T^{-\frac{1}{2}k(k-1)}J_{n,j}}\left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k}\chi_A\circ T^{-ik}\right|d\mu+\frac{k}{r_n+1}\mu I_{n,j}\mu\tilde{C}_n$$

Proof. For $j\ge kr_n$ and $1\le i\le r_n-k$,

$$T^{k\tilde{h}_n}I_{n,j}^{i}=T^{-ik-\frac{1}{2}k(k-1)}I_{n,j}^{[i+k]}=I_{n,j-ik-\frac{1}{2}k(k-1)}^{[i+k]}$$

so by Lemma 4.4, as $I_{n,j-ik-\frac{1}{2}k(k-1)}^{[i+k]}\cap\tilde{C}_n=J_{n,j-ik}^{[i+k]}$,

$$\begin{aligned}\mu(T^{k\tilde{h}_n}I_{n,j}\cap A\cap\tilde{C}_n)&=\mu(I_{n,j-ik-\frac{1}{2}k(k-1)}^{[i+k]}\cap A\cap\tilde{C}_n)\\&=\mu(J_{n,j-ik-\frac{1}{2}k(k-1)}^{[i+k]}\cap A)=\frac{1}{r_n+1}\mu(J_{n,j-ik}\cap A)=\mu(T^{-ik-\frac{1}{2}k(k-1)}J_{n,j}\cap A)\end{aligned}$$

so the claim follows as in the proof of Lemma 4.11 making use of the facts that $\mu J_{n,j}\mu A=\mu I_{n,j}\mu(A\cap\tilde{C}_n)$ and $\mu J_{n,j}=\mu I_{n,j}\frac{\mu\tilde{C}_n}{\mu C_n}\le\mu I_{n,j}\mu\tilde{C}_n$. □

Lemma 4.13. *Let T be a rank-one transformation and n be such that B_{n+1} is constructed using the staircase operation. Let $\tilde{C}_{n+1}=\bigsqcup_{j=0}^{\tilde{h}_{n+1}-1}J_{n+1,j}$ be a mesh for C_{n+1} (so by Lemma 4.5, $\tilde{C}_n=C_n\cap\tilde{C}_{n+1}=\bigsqcup_{j=0}^{\tilde{h}_n-1}J_{n,j}$ is a mesh for C_n). Let $1\le k<r_n$. Then for $j\ge r_n(r_n-1)+c_{n+1}-c_n$ and $r_n-k<i\le r_n$,*

$$T^{k\tilde{h}_n}J_{n,j}^{[i]}=T^{\tilde{h}_{n+1}}J_{n+1,j+\frac{1}{2}i(i-1)-\frac{1}{2}r_n(r_n-1)-c_{n+1}+c_n}$$

Proof. Since $J_{n,j}^{[i]}=J_{n+1,j+i\tilde{h}_n+\frac{1}{2}i(i-1)}$ and $\tilde{h}_{n+1}=(r_n+1)\tilde{h}_n+\frac{1}{2}r_n(r_n-1)+c_{n+1}-c_n$, this follows from the fact that

$$i\tilde{h}_n+\frac{1}{2}i(i-1)+k\tilde{h}_n=\tilde{h}_{n+1}+(i+k-r_n-1)\tilde{h}_n+\frac{1}{2}i(i-1)-\frac{1}{2}r_n(r_n-1)-c_{n+1}+c_n$$ □

Lemma 4.14. *Let T be a rank-one transformation and n be such that B_{n+1} and B_{n+2} are constructed using the staircase operation with $c_{n+1} = c_n + r_n$. Let $\tilde{C}_{n+1} = \bigsqcup_{j=0}^{\tilde{h}_{n+1}-1} J_{n+1,j}$ be a mesh for C_{n+1} (so by Lemma 4.5, $\tilde{C}_n = C_n \cap \tilde{C}_{n+1} = \bigsqcup_{j=0}^{\tilde{h}_n-1} J_{n,j}$ is a mesh for C_n). Then for $1 \le k < r_n$ and any union of levels A in C_n,*

$$\sum_{j=0}^{\tilde{h}_n-1} \left|\mu(T^{k\tilde{h}_n} J_{n,j} \cap A) - \mu J_{n,j}\mu A\right| \le \int_{T^{-\frac{1}{2}k(k-1)}\tilde{C}_n} \left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k} \chi_A \circ T^{-ik}\right| d\mu + \int_{\tilde{C}_{n+1}} \left|\frac{1}{r_{n+1}+1}\sum_{i=0}^{r_{n+1}-1} \chi_A \circ T^{-i}\right| d\mu + \frac{r_n^2}{\tilde{h}_n}\mu\tilde{C}_n + \frac{1}{r_{n+1}+1}\mu\tilde{C}_{n+1}$$

Proof. For $r_n - k < i \le r_n$ and $r_n(r_n-1) + c_{n+1} - c_n \le j < h_n$, by Lemma 4.13 then Lemma 4.7,

$$\begin{aligned}
&\left|\mu(T^{k\tilde{h}_n} J^{[i]}_{n,j} \cap A) - \mu J^{[i]}_{n,j}\mu A\right| \\
&\quad= \left|\mu(T^{\tilde{h}_{n+1}} J_{n+1,j+\frac{1}{2}i(i-1)-\frac{1}{2}r_n(r_n-1)-c_{n+1}+c_n} \cap A) - \mu J_{n+1,j+\frac{1}{2}i(i-1)-\frac{1}{2}r_n(r_n-1)}\mu A\right| \\
&\quad\le \int_{J_{n+1,j+\frac{1}{2}i(i-1)-\frac{1}{2}r_n(r_n-1)-c_{n+1}+c_n}} \left|\frac{1}{r_{n+1}+1}\sum_{i=0}^{r_{n+1}-2} \chi_A \circ T^{-i}\right| d\mu \\
&\qquad+ \frac{1}{r_{n+1}+1}\mu(J_{n+1,j+\frac{1}{2}i(i-1)-\frac{1}{2}r_n(r_n-1)-c_{n+1}+c_n})
\end{aligned}$$

Then, since the levels in C_{n+1} are disjoint over j and i,

$$\begin{aligned}
&\sum_{j=r_n(r_n-1)+c_{n+1}}^{h_n} \left|\sum_{i=r_n-k+1}^{r_n} \mu(T^{k\tilde{h}_n} J^{[i]}_{n,j} \cap A) - \mu J^{[i]}_{n,j}\mu A\right| \\
&\qquad\le \int_{\tilde{C}_{n+1}} \left|\frac{1}{r_{n+1}+1}\sum_{i=0}^{r_{n+1}-1} \chi_A \circ T^{-i}\right| d\mu + \frac{1}{r_{n+1}+1}\mu\tilde{C}_{n+1}
\end{aligned}$$

Then by Lemma 4.11,

$$\begin{aligned}
&\sum_{j=r_n(r_n-1)+c_{n+1}}^{h_n} \left|\mu(T^{k\tilde{h}_n} J_{n,j} \cap A) - \mu J_{n,j}\mu A\right| \\
&\le \int_{T^{-\frac{1}{2}k(k-1)}\tilde{C}_n} \left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k} \chi_A \circ T^{-ik}\right| d\mu + \int_{\tilde{C}_{n+1}} \left|\frac{1}{r_{n+1}+1}\sum_{i=0}^{r_{n+1}-1} \chi_A \circ T^{-i}\right| d\mu + \frac{1}{r_{n+1}+1}\mu\tilde{C}_{n+1}
\end{aligned}$$

As the bottom $r_n(r_n-1) + c_{n+1} - c_n$ levels in $\tilde{C}_n$ have measure at most $\frac{r_n(r_n-1)+c_{n+1}-c_n}{h_n}\mu\tilde{C}_n$, since $i \le r_n - k$, then $ik + \frac{1}{2}k(k-1) \le r_nk - \frac{1}{2}k^2 - \frac{1}{2}k < r_n(r_n-1)$ and $c_{n+1} = c_n + r_n$, the claim follows. □

Lemma 4.15. *Let T be a rank-one transformation and n be such that B_{n+1} and B_{n+2} are constructed using the staircase operation with $c_{n+1} = c_n + r_n$. Let $\tilde{C}_{n+1} = \bigsqcup_{j=0}^{\tilde{h}_{n+1}-1} J_{n+1,j}$ be a mesh for C_{n+1} (so by Lemma 4.5, $\tilde{C}_n = C_n \cap \tilde{C}_{n+1} = \bigsqcup_{j=0}^{\tilde{h}_n-1} J_{n,j}$ is a mesh for C_n). Then for $1 \le k < r_n$ and any union of levels A in C_n,*

$$\begin{aligned}
&\sum_{j=0}^{\tilde{h}_n-1} \left|\mu(T^{k\tilde{h}_n} I_{n,j} \cap A \cap \tilde{C}_{n+1}) - \mu I_{n,j}\mu(A \cap \tilde{C}_{n+1})\right| \\
&\qquad\le \int_{T^{-\frac{1}{2}k(k-1)}\tilde{C}_n} \left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k} \chi_A \circ T^{-ik}\right| d\mu
\end{aligned}$$

$$+\int_{\tilde{C}_{n+1}}\left|\frac{1}{r_{n+1}+1}\sum_{i=0}^{r_{n+1}-1}\chi_A\circ T^{-i}\right|d\mu+\frac{r_n^2}{h_n}\mu\tilde{C}_n+\frac{1}{r_{n+1}+1}\mu\tilde{C}_{n+1}$$

Proof. For $r_n-k\le i<r_n$ and $r_n(r_n-1)+c_{n+1}-c_n\le j<h_n$, by Lemma 4.13 then Lemma 4.8, and using that $\mu I_{n+1,j}\mu(A\cap\tilde{C}_{n+1})=\mu J_{n+1,j}\mu A$,

$$\begin{aligned}
&\left|\mu(T^{k\tilde{h}_n}I_{n,j}^{[i]}\cap A\cap\tilde{C}_{n+1})-\mu I_{n,j}^{[i]}\mu(A\cap\tilde{C}_{n+1})\right|\\
&\quad=\left|\mu(T^{\tilde{h}_{n+1}}I_{n+1,j+\frac{1}{2}i(i-1)-\frac{1}{2}r_n(r_n-1)-c_{n+1}+c_n}\cap A\cap\tilde{C}_{n+1})-\mu J_{n+1,j+\frac{1}{2}i(i-1)-\frac{1}{2}r_n(r_n-1)-c_{n+1}+c_n}\mu A\right|\\
&\quad\le\int_{J_{n+1,j+\frac{1}{2}i(i-1)-\frac{1}{2}r_n(r_n-1)-c_{n+1}+c_n}}\left|\frac{1}{r_{n+1}+1}\sum_{i=0}^{r_{n+1}-2}\chi_A\circ T^{-i}\right|d\mu\\
&\qquad+\frac{1}{r_{n+1}+1}\mu(J_{n+1,j+\frac{1}{2}i(i-1)-\frac{1}{2}r_n(r_n-1)-c_{n+1}+c_n})
\end{aligned}$$

and then the claim follows as in the proof of Lemma 4.14. □

Lemma 4.16. *Let T be a rank-one transformation and n be such that B_{n+1} and B_{n+2} are constructed using the staircase operation with $c_{n+1}=c_n+r_n$. Let $\tilde{C}_{n+1}^{(q)}$ be a collection of disjoint meshes of C_{n+1} and A be a union of levels in C_n. Then for $1\le k<r_n$,*

$$\begin{aligned}
&\sum_q\sum_{j=0}^{\tilde{h}_n-1}\left|\mu(T^{k\tilde{h}_n}J_{n,j}^{(q)}\cap A)-\mu J_{n,j}^{(q)}\mu A\right|\\
&\qquad\le\int\left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k}\chi_A\circ T^{-ik}\right|d\mu+\int\left|\frac{1}{r_{n+1}+1}\sum_{i=0}^{r_{n+1}}\chi_A\circ T^{-i}\right|d\mu+\frac{r_n^2}{\tilde{h}_n}+\frac{1}{r_{n+1}+1}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_q\sum_{j=0}^{\tilde{h}_n-1}\left|\mu(T^{k\tilde{h}_n}I_{n,j}\cap A\cap\tilde{C}_{n+1})-\mu I_{n,j}\mu(A\cap\tilde{C}_{n+1})\right|\\
&\qquad\le\int\left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k}\chi_A\circ T^{-ik}\right|d\mu+\int\left|\frac{1}{r_{n+1}+1}\sum_{i=0}^{r_{n+1}-1}\chi_A\circ T^{-i}\right|d\mu+\frac{r_n^2}{\tilde{h}_n}+\frac{1}{r_{n+1}+1}
\end{aligned}$$

Proof. This is immediate from Lemmas 4.14 and 4.15 using the disjointness. □

Lemma 4.17. *Let T be a rank-one transformation and n be such that B_{n+1} is constructed using the staircase operation and B_{n+2} through B_{n+z} using Chacon-like operations with the same M_{n+1} and $c_{n+1}<c_n+r_n+M_{n+1}$ and B_{n+z+1} be constructed using the staircase operation. Let $\tilde{C}_n^{(q)}=\bigsqcup_{j=0}^{\tilde{h}_n-1}J_{n,j}^{(q)}$ be disjoint meshes for C_n. Then for $1\le k<r_n$ and any union of levels A in C_n,*

$$\begin{aligned}
\sum_q\sum_{j=0}^{\tilde{h}_n-1}\left|\mu(T^{k\tilde{h}_n}J_{n,j}^{(q)}\cap A)-\mu J_{n,j}^{(q)}\mu A\right|&\le\int\left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k}\chi_A\circ T^{-ik}\right|d\mu+\int\left|\frac{1}{r_n+1}\sum_{i=0}^{k}\chi_A\circ T^{i(r_n-k)}\right|d\mu\\
&\quad+\int\left|\frac{1}{r_{n+z}+1}\sum_{i=0}^{r_{n+z}-1}\chi_A\circ T^{-i}\right|d\mu+\frac{1}{r_{n+z}}+\frac{r_n^2+c_n+2M_{n+1}}{h_n}
\end{aligned}$$

Proof. Let $\tilde{C}_n=\bigsqcup_{j=0}^{\tilde{h}_n-1}J_{n,j}$ be one of the meshes. Let $S^\star$ be the spacer levels added above C_{n+z}. Let $Q=\{x\in I_{n+1,0}:\exists\ 0<y\le\tilde{h}_{n+1}\text{ s.t. }T^yx\in S^\star\}$. Let $L_{n+1}=\{x\in I_{n+1,0}\setminus Q:T^{\tilde{h}_{n+1}}x\in I_{n+1,0}\}$ and $L_{n+1}^\star=\{x\in I_{n+1,0}\setminus Q:T^{\tilde{h}_{n+1}+M_{n+1}}x\in I_{n+1,0}\}$. Since B_{n+2} through B_{n+z} are constructed using Chacon-like operations all with the same M_{n+1} and c_{n+1}, $I_{n+1,0}=L_{n+1}\sqcup L_{n+1}^\star\sqcup Q$.

Let $L_n = \bigsqcup_{i=1}^{r_n} T^{i\tilde{h}_n+\frac{1}{2}i(i-1)}L_{n+1}$ and $L_n^\star = \bigsqcup_{i=1}^{r_n} T^{(i\tilde{h}_n+\frac{1}{2}i(i-1)}L_{n+1}^\star$. For $x \in L_{n+1}$, since $x \notin Q$ and $T^{\tilde{h}_{n+1}}x \in I_{n+1,0}$, it follows that if $x \in T^{i\tilde{h}_n+\frac{1}{2}i(i-1)}L_{n+1}$ then $T^{\tilde{h}_{n+1}}x \in T^{i\tilde{h}_n+\frac{1}{2}i(i-1)}L_{n+1}$. Let $K_{n,0} = T^{\tilde{h}_{n+1}}(J_{n,0} \cap L)$. Then $K_{n,0} \subseteq I_{n,0}$ and $K_{n,0}^{[i]} = T^{i\tilde{h}_n+\frac{1}{2}i(i-1)}K_{n,0}^{[0]}$ so $K_{n,0}$ is a mesh prototype for C_n. Likewise, $K_{n,0}^\star = T^{\tilde{h}_{n+1}}(J_{n,0} \cap L^\star)$ is a mesh prototype for C_n.

For $r_n - k < i \le r_n$ and $0 \le j < \tilde{h}_n$, since $\tilde{h}_{n+1} = (r_n+1)\tilde{h}_n + \frac{1}{2}r_n(r_n-1) + c_{n+1} - c_n$,

$$\begin{aligned} T^{k\tilde{h}_n}J_{n,j}^{[i]} &= T^{k\tilde{h}_n}T^{(i\tilde{h}_n+\frac{1}{2}i(i-1)}J_{n,j}^{[0]} = T^{\tilde{h}_{n+1}}T^{(i+k-r_n-1)\tilde{h}_n+\frac{1}{2}i(i-1)-\frac{1}{2}r_n(r_n-1)-c_{n+1}+c_n}J_{n,j}^{[0]} \\ &= T^{\tilde{h}_{n+1}}T^{\frac{1}{2}i(i-1)-\frac{1}{2}r_n(r_n-1)-c_{n+1}+c_n-\frac{1}{2}(i+k-r_n)(i+k-r_n-1)}J_{n,j}^{[i+k-r_n]} \\ &= T^{\tilde{h}_{n+1}}T^{i(r_n-k)-x_{n,k}}J_{n,j}^{[i+k-r_n]} \end{aligned}$$

where $x_{n,k} = r_n(r_n-1) - \frac{1}{2}k(r_n-k+1) + c_{n+1} - c_n$. Since $c_{n+1} < c_n + r_n + M_{n+1}$, $x_{n,k} < r_n^2 + M_{n+1}$ Then for $r_n^2 + M_{n+1} \le j < \tilde{h}_n$,

$$T^{k\tilde{h}_n}J_{n,j}^{[i]} = T^{\tilde{h}_{n+1}}J_{n,j+i(r_n-k)-x_{n,k}}^{[i+k-r_n]}$$

Now observe that

$$\begin{aligned} T^{\tilde{h}_{n+1}}J_{n,j} = T^{\tilde{h}_{n+1}}T^jJ_{n,0} &= T^j\left(T^{\tilde{h}_{n+1}}\left((J_{n,0}\cap L_n)\sqcup(J_{n,0}\cap L_n^\star)\sqcup(J_{n,0}\cap Q)\right)\right) \\ &= T^j\left(K_{n,0}\sqcup T^{-M_{n+1}}K_{n,0}^\star \sqcup T^{\tilde{h}_{n+1}}(J_{n,0}\cap Q)\right) \end{aligned}$$

Since $J_{n,0}$, $K_{n,0}$ and $K_{n,0}^\star$ are all meshes, then for $j \ge r_n^2 + 2M_{n+1}$,

$$T^{\tilde{h}_{n+1}}J_{n,j+i(r_n-k)-x_{n,k}}^{[i+k-r_n]} = K_{n,j+i(r_n-k)-x_{n,k}}^{[i+k-r_n]} \sqcup (K_{n,j+i(r_n-k)-x_{n,k}-M_{n+1}}^\star)^{[i+k-r_n]} \sqcup T^{\tilde{h}_{n+1}}Z_{i,j}$$

where $Z_{i,j}$ is a collection of levels in C_{n+z} such that $T^{\tilde{h}_{n+1}}$ pushes each level through the top of C_{n+z}. Since $K_{n,0}$ and $K_{n,0}^\star$ are mesh prototypes for C_n and A is a union of levels in C_n, then by Lemma 4.4,

$$\mu(K_{n,j+i(r_n-k)-x_{n,k}}^{[i+k-r_n]} \cap A) = \frac{1}{r_n+1}\mu(K_{n,j+i(r_n-k)-x_{n,k}} \cap A) = \frac{1}{r_n+1}\mu(T^{i(r_n-k)-x_{n,k}}K_{n,j} \cap A)$$

and

$$\begin{aligned} \mu((K_{n,j+i(r_n-k)-x_{n,k}-M_{n+1}}^\star)^{[i+k-r_n]} \cap A) &= \frac{1}{r_n+1}\mu(K_{n,j+i(r_n-k)-x_{n,k}-M_{n+1}} \cap A) \\ &= \frac{1}{r_n+1}\mu(T^{i(r_n-k)-x_{n,k}-M_{n+1}}K_{n,j} \cap A) \end{aligned}$$

Therefore

$$\begin{aligned} \left|\sum_{i=r_n-k+1}^{r_n}\mu(T^{k\tilde{h}_n}J_{n,j}^{[i]} \cap A) - \mu J_{n,j}^{[i]}\mu A\right| &\le \left|\frac{1}{r_n+1}\sum_{i=r_n-k+1}^{r_n}\mu(T^{i(r_n-k)-x_{n,k}}K_{n,j} \cap A) - \mu K_{n,j}\mu A\right| \\ &+ \left|\frac{1}{r_n+1}\sum_{i=r_n-k+1}^{r_n}\mu(T^{i(r_n-k)-x_{n,k}-M_{n+1}}K_{n,j} \cap A) - \mu K_{n,j}\mu A\right| \\ &+ \sum_{\text{levels in } C_{n+z}}\mu(T^{\tilde{h}_{n+z}}I_{n+z,y} \cap A) \end{aligned}$$

Using that T is measure-preserving,

$$\left|\frac{1}{r_n+1}\sum_{i=r_n-k+1}^{r_n}\mu(T^{i(r_n-k)-x_{n,k}}K_{n,j} \cap A) - \mu K_{n,j}\mu A\right|$$

$$= \left| \frac{1}{r_n+1} \sum_{i=1}^{k} \mu(T^{i(r_n-k)+(r_n-k)^2-x_{n,k}} K_{n,j} \cap A) - \mu K_{n,j} \mu A \right|$$
$$\leq \int_{T^{(r_n-k)^2-x_{n,k}} K_{n,j}} \left| \frac{1}{r_n+1} \sum_{i=0}^{k} \chi_A \circ T^{i(r_n-k)} \right| d\mu$$

and similarly

$$\left| \frac{1}{r_n+1} \sum_{i=r_n-k+1}^{r_n} \mu(T^{i(r_n-k)-x_{n,k}-M_{n+1}} K_{n,j} \cap A) - \mu K^\star_{n,j} \mu A \right|$$
$$\leq \int_{T^{(r_n-k)^2-x_{n,k}-M_{n+1}} K^\star_{n,j}} \left| \frac{1}{r_n+1} \sum_{i=1}^{k} \chi_A \circ T^{i(r_n-k)} \right| d\mu$$

Since $K_{n,0}$ and $T^{-M_n} K^\star_{n,0}$ are disjoint and contained in $T^{\tilde{h}_{n+1}} J_{n,0}$, then

$$\int_{T^{(r_n-k)^2-x_{n,k}} K_{n,j}} \left| \frac{1}{r_n+1} \sum_{i=0}^{k} \chi_A \circ T^{i(r_n-k)} \right| d\mu + \int_{T^{(r_n-k)^2-x_{n,k}-M_{n+1}} K^\star_{n,j}} \left| \frac{1}{r_n+1} \sum_{i=0}^{k} \chi_A \circ T^{i(r_n-k)} \right| d\mu$$
$$\leq \int_{T^{(r_n-k)^2-x_{n,k}+\tilde{h}_{n+1}} J_{n,j}} \left| \frac{1}{r_n+1} \sum_{i=0}^{k} \chi_A \circ T^{i(r_n-k)} \right| d\mu$$

Therefore if $\tilde{C}^{(q)}_n$ are disjoint meshes of C_n then by Lemma 4.10 applied to the levels of C_{n+z} pushed through the top of C_{n+z},

$$\sum_q \left| \sum_{i=r_n-k+1}^{r_n} \mu(T^{k\tilde{h}_n} (J^{(q)}_{n,j})^{[i]} \cap A) - \mu(J^{(q)}_{n,j})^{[i]} \mu A \right| \leq \int \left| \frac{1}{r_n+1} \sum_{i=0}^{k} \chi_A \circ T^{i(r_n-k)} \right| d\mu + \frac{r_n^2+2M_{n+1}}{h_n}$$
$$+ \int \left| \frac{1}{r_{n+z}+1} \sum_{i=0}^{r_{n+z}-1} \chi_A \circ T^{-i} \right| d\mu + \frac{1}{r_{n+z}}$$

so the claim follows from Lemma 4.12. □

Lemma 4.18. *Let T be a rank-one transformation and n be such that B_{n+1} is constructed using the staircase operation and B_{n+2} through B_{n+z} using Chacon-like operations with the same M_{n+1} and $c_{n+1} < c_n + r_n + M_{n+1}$ and B_{n+z+1} be constructed using the staircase operation. Let $\tilde{C}^{(q)}_n = \bigsqcup_{j=0}^{\tilde{h}_n-1} J^{(q)}_{n,j}$ be disjoint meshes for C_n. Then for $1 \leq k < r_n$ and any union of levels A in C_n,*

$$\sum_q \sum_{j=0}^{\tilde{h}_n-1} \left| \mu(T^{k\tilde{h}_n} I_{n,j} \cap A \cap \tilde{C}^{(q)}_n) - \mu I_{n,j} \mu(A \cap \tilde{C}^{(q)}_n) \right|$$
$$\leq \int \left| \frac{1}{r_n+1} \sum_{i=0}^{r_n-k} \chi_A \circ T^{-ik} \right| d\mu + \int \left| \frac{1}{r_n+1} \sum_{i=0}^{k} \chi_A \circ T^{i(r_n-k)} \right| d\mu$$
$$+ \int \left| \frac{1}{r_{n+z}} \sum_{i=1}^{r_{n+z}-1} \chi_A \circ T^{-i} \right| d\mu + \frac{1}{r_{n+z}} + \frac{r_n^2+2M_{n+1}}{\tilde{h}_n}$$

Proof. Let $\tilde{C}_n = \bigsqcup_{j=0}^{\tilde{h}_n-1} J_{n,j}$ be one of the meshes. Let $L_{n,0}$ and $L^\star_{n,0}$ and Q be as in the proof of Lemma 4.17. Then $L_{n,0}$ and $L^\star_{n,0}$ are mesh prototypes. As in the proof of Lemma 4.17, for $r_n - k < i \leq r_n$ and $r_n^2 + 2M_{n+1} \leq j < \tilde{h}_{n+1}$,

$$T^{k\tilde{h}_n} I^{[i]}_{n,j} = T^{\tilde{h}_{n+1}} I^{[i+k-r_n]}_{n,j+i(r_n-k)-x_{n,k}} = L^{[i+k-r_n]}_{n,j+i(r_n-k)-x_{n,k}} \sqcup (L^\star_{n,j+i(r_n-k)-x_{n,k}-M_{n+1}})^{[i+k-r_n]} \sqcup T^{\tilde{h}_{n+1}} Z_{i,j}$$

where $Z_{i,j}$ is a collection of levels in C_{n+z} such that $T^{\tilde{h}_{n+1}}$ pushes each level through the top of C_{n+z}.

Since $L_{n,0}$ is a mesh prototype and $\tilde{C}_n$ is a mesh, $\tilde{C}_n \cap \bigsqcup L_{n,j}$ is a mesh (and likewise with $L^\star_{n,0}$) so by Lemma 4.4,

$$\begin{aligned}\mu(L^{[i+k-r_n]}_{n,j+i(r_n-k)-x_{n,k}} \cap A \cap \tilde{C}_n) &= \frac{1}{r_n+1}\mu(L_{n,j+i(r_n-k)-x_{n,k}} \cap A \cap \tilde{C}_n)\\ &= \frac{1}{r_n+1}\mu(T^{i(r_n-k)-x_{n,k}}(L_{n,j} \cap J_{n,j}) \cap A)\end{aligned}$$

and similarly

$$\mu((L^\star_{n,j+i(r_n-k)-x_{n,k}-M_{n+1}})^{[i+k-r_n]} \cap A \cap \tilde{C}_n) = \frac{1}{r_n+1}\mu(T^{i(r_n-k)-x_{n,k}-M_{n+1}}(L^\star_{n,j} \cap J_{n,j}) \cap A)$$

The claim now follows similar to the proof of Lemma 4.17. □

Lemma 4.19. *Let T be a rank-one transformation such that the staircase operation is used infinitely often and $r_n \to \infty$ along the sequence of n for which the staircase operation is used. Then $\{\tilde{h}_n\}$ is a mixing sequence along those n for which the staircase operation is used.*

Proof. Let A and D be unions of levels in some column C_{n_0}. Then for $n \geq n_0$ such that the staircase operation is used, as A and D can be written as unions of levels in C_n, by Lemma 4.10 applied to the single mesh $\bigsqcup I_{n,j}$,

$$\begin{aligned}\left|\mu(T^{\tilde{h}_n}D \cap A) - \mu D\mu A\right| &\leq \sum_{j=0}^{\tilde{h}_n-1}\left|\mu(T^{\tilde{h}_n}I_{n,j} \cap A) - \mu I_{n,j}\mu A\right|\\ &\leq \int\left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-1}\chi_A \circ T^{-i} - \mu A\right|d\mu + \frac{1}{r_n+1} + \frac{r_n}{\tilde{h}_n}\end{aligned}$$

which tends to zero as T is ergodic and $r_n \to \infty$ (and $\frac{r_n}{\tilde{h}_n} \to 0$ since T is on a finite measure space). □

Lemma 4.20. *Let T be a totally ergodic rank-one transformation such that the staircase operation is used infinitely often and $r_n \to \infty$ along the sequence of n for which the staircase operation is used. Then for each fixed k, the sequence $\{k\tilde{h}_n\}$ is a mixing sequence along those n for which the staircase operation is used.*

Proof. Let $k \geq 1$ be fixed. Let A and D be unions of levels in some column C_{n_0}. Then for $n \geq n_0$ such that the staircase operation is used, as A and D can be written as unions of levels in C_n, by Lemma 4.11 (the second statement) applied to the mesh $\bigsqcup I_{n,j}$,

$$\begin{aligned}&\left|\mu(T^{k\tilde{h}_n}D \cap A) - \mu D\mu A\right|\\ &\quad\leq \sum_{j=0}^{\tilde{h}_n-1}\left|\mu(T^{k\tilde{h}_n}I_{n,j} \cap A) - \mu I_{n,j}\mu A\right|\\ &\quad\leq \sum_{j=kr_n+\frac{1}{2}k(k-1)}^{\tilde{h}_n-1}\left(\int_{T^{-\frac{1}{2}k(k-1)}I_{n,j}}\left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k}\chi_A \circ T^{-ik}\right|d\mu + \frac{k}{r_n}\mu I_{n,j}\right) + \frac{kr_n+\frac{1}{2}k(k-1)}{\tilde{h}_n}\\ &\quad\leq \int\left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k}\chi_A \circ T^{-ik}\right|d\mu + \frac{k}{r_n+1} + \frac{kr_n+\frac{1}{2}k(k-1)}{\tilde{h}_n}\end{aligned}$$

which tends to zero as k is fixed, $r_n \to \infty$, $\frac{r_n}{\tilde{h}_n} \to 0$ and T^k is ergodic. □

Lemma 4.21 ([Ada98]; see Lemma A.16 [CPR23])**.** *Let T be a transformation and n, p, k, L be nonzero integers with $n, L > 0$ and $\chi \in L^1$. Then*

$$\int \left| \frac{1}{n} \sum_{i=1}^{n} \chi \circ T^{-ik} \right| d\mu \leq \int \left| \frac{1}{L} \sum_{i=0}^{L} \chi \circ T^{-ipk} \right| d\mu + \frac{|p|L}{n}$$

Lemma 4.22 ([Ada98]; see Lemma A.15 [CPR23])**.** *Let T be a transformation and J a measurable set. Let $\epsilon > 0$ and L such that $\frac{1}{L} < \epsilon$. Let p such that for all $1 \leq i < L$,*

$$\left| \mu(T^{ip} J \cap J) - \mu J \mu J \right| < \epsilon$$

Then

$$\int \left| \frac{1}{L} \sum_{i=0}^{L-1} \chi_J \circ T^{-ip} \right| d\mu < \sqrt{2\epsilon}$$

Definition 4.23. Let T be a rank-one transformation and A be a union of levels in some column C_{n_0}. Let $\epsilon > 0$ and $n \in \mathbb{N}$. Then A is **uniformly mixed within ϵ at stage n** if for any collection of disjoint meshes $\tilde{C}_n^{(q)}$ of C_n and all $(1 - \frac{1}{2}\epsilon)\tilde{h}_n \leq t < (1 - \frac{1}{2}\epsilon)\tilde{h}_{n+1}$,

$$\sum_q \sum_{j=0}^{\tilde{h}_n - 1} \left| \mu(T^t J_{n,j}^{(q)} \cap A) - \mu J_{n,j}^{(q)} \mu A \right| < \epsilon$$

and

$$\sum_q \sum_{j=0}^{\tilde{h}_n - 1} \left| \mu(T^t I_{n,j} \cap A \cap \tilde{C}_n^{(q)}) - \mu I_{n,j} \mu(A \cap \tilde{C}_n^{(q)}) \right| < \epsilon$$

Lemma 4.24. *Let T be a rank-one transformation and A be a union of levels in some column C_{n_0}. Let $\epsilon > 0$ and $n \in \mathbb{N}$ and let A be uniformly mixed within ϵ at stage n. Then for any collection of disjoint meshes $\tilde{C}_n^{(q)}$ of C_n and any union of levels D in C_n and all $(1 - \frac{1}{2}\epsilon)\tilde{h}_n \leq t < (1 - \frac{1}{2}\epsilon)\tilde{h}_{n+1}$,*

$$\sum_q \left| \mu(T^t(D \cap \tilde{C}_n^{(q)}) \cap A) - \mu(D \cap \tilde{C}_n^{(q)}) \mu A \right| < \epsilon$$

Proof. Since D is a union of levels in C_n, there exists $\mathcal{D} \subseteq \{0, 1, \ldots, \tilde{h}_n - 1\}$ such that $D = \bigsqcup_{j \in \mathcal{D}} I_{n,j}$. Then

$$\begin{aligned} \sum_q \left| \mu(T^t(D \cap \tilde{C}_n^{(q)}) \cap A) - \mu(D \cap \tilde{C}_n^{(q)}) \mu A \right| &\leq \sum_q \sum_{j \in \mathcal{D}} \left| \mu(T^t(I_{n,j} \cap \tilde{C}_n^{(q)}) \cap A) - \mu(I_{n,j} \cap \tilde{C}_n^{(q)}) \mu A \right| \\ &\leq \sum_q \sum_{j=0}^{\tilde{h}_n - 1} \left| \mu(T^t J_{n,j}^{(q)} \cap A) - \mu J_{n,j}^{(q)} \mu A \right| < \epsilon \end{aligned}$$

□

Proposition 4.25. *Let T be an elevated staircase-Chacon-like hybrid transformation on a finite measure space such that the staircase operation is used infinitely often and that along the sequence of n for which B_{n+1} is constructed using the staircase operation,*

- $r_n \to \infty$
- $\frac{r_n^2}{\tilde{h}_n} \to 0$
- $r_n = \tilde{h}_\ell$ *for $\ell < n$ such that $B_{\ell+1}$ was constructed using a staircase operation*
- $c_{n+1} = c_n + r_n$

and along the sequence of n for which B_{n+1} is constructed using a Chacon-like operation,

- $c_n < c_{n-1} + r_{n-1} + M_n$

- *for all consecutive Chacon-like operations prior to the next staircase operation, the same c_n and M_n are used.*

and such that $\frac{M_n}{h_{n-1}} \to 0$ along the sequence of n for which B_{n+1} is constructed using a Chacon-like operation and B_n was constructed using a staircase operation.

Let $\epsilon_0 > 0$ and A be a union of levels in some column C_{n_0}. Then there exists $n_1 > n_0$ such that for all $n \geq n_1$ for which B_{n+1} is constructed using the staircase operation, A is uniformly mixed within ϵ_0 at stage n.

Proof. By Lemma 4.19, the sequence $\{\tilde{h}_n\}$ restricted to those n for which the staircase operation is used forms a mixing sequence so T is weakly mixing. Then T is totally ergodic so by Lemma 4.20, for each fixed k, again along the staircase heights, $\{k\tilde{h}_n\}$ is mixing.

Choose ϵ such that $4\sqrt{\epsilon} + 3\sqrt{2\epsilon} + 5\epsilon < \frac{1}{4}\epsilon_0$. Choose $L \geq 1$ such that $\frac{1}{L} < \epsilon$. Take n such that the staircase operation used to construct B_{n+1} to be large enough that $\int \left| \frac{1}{r_n} \sum_{i=0}^{r_n-k} \chi \circ T^{-ik} \right| d\mu < \epsilon$ for all $1 \leq k \leq L^2$ and that $\frac{1}{r_n} < \epsilon$ and $\frac{r_n^2+2M_{n+1}}{h_n} < \epsilon$ (or just $\frac{r_n^2}{h_n}$ if the following stage is also a staircase operation)

Since $\{k\tilde{h}_n\}$ is a mixing sequence along n for which B_{n+1} is constructed using the staircase operation, by Lemma 4.6, we may also take n large enough that for all $\tilde{h}_\ell \leq t \leq 4L\tilde{h}_\ell$,

$$\left|\mu(T^t A \cap A) - \mu A \mu A\right| < \epsilon$$

since $\ell \to \infty$ as $n \to \infty$ because $r_n \to \infty$.

By Lemma 4.16, then for $1 \leq k \leq L^2$,

$$\sum_q \sum_{j=0}^{\tilde{h}_n - 1} \left| \mu(T^{k\tilde{h}_n} J_{n,j}^{(q)} \cap A) - \mu J_{n,j}^{(q)} \mu A \right| < 4\epsilon < \frac{1}{4}\epsilon_0$$

and

$$\sum_q \sum_{j=0}^{\tilde{h}_n - 1} \left| \mu(T^{k\tilde{h}_n} I_{n,j} \cap A \cap \tilde{C}_n^{(q)}) - \mu I_{n,j} \mu(A \cap \tilde{C}_n^{(q)}) \right| < 4\epsilon < \frac{1}{4}\epsilon_0$$

Let k such that $L^2 \leq k < r_n = h_\ell$. Choose p to be the minimal integer such that $pk \geq \tilde{h}_\ell$. Then $pk < 2\tilde{h}_\ell$. Write $\chi = \chi_A$. By Lemma 4.21,

$$\int \left| \frac{1}{r_n + 1} \sum_{i=0}^{r_n - 1} \chi \circ T^{-ik} \right| d\mu \leq \int \left| \frac{1}{L} \sum_{i=0}^{L-1} \chi \circ T^{-ipk} \right| d\mu + \frac{pL}{r_n + 1}$$

Since $pk < 2\tilde{h}_\ell$ and $r_n = \tilde{h}_\ell$ and $k \geq L^2$,

$$\frac{pL}{r_n + 1} < \frac{pL}{r_n} = \frac{pkL}{\tilde{h}_\ell k} < \frac{2\tilde{h}_\ell L}{\tilde{h}_\ell k} = \frac{2L}{k} \leq \frac{2L}{L^2} = \frac{2}{L} < 2\epsilon$$

Now $\tilde{h}_\ell \leq ipk < 2L\tilde{h}_\ell$ for all $1 \leq i \leq L$ so by Lemma 4.22,

$$\int \left| \frac{1}{L} \sum_{i=0}^{L-1} \chi \circ T^{-ipk} \right| d\mu < \sqrt{2\epsilon} + 2\epsilon$$

Therefore if B_{n+2} is constructed using the staircase operation, then by Lemma 4.16,

$$\sum_q \sum_{j=0}^{\tilde{h}_n - 1} \left| \mu(T^{k\tilde{h}_n} J_{n,j}^{(q)} \cap A) - \mu J_{n,j}^{(q)} \mu A \right| < \sqrt{2\epsilon} + 2\epsilon + \epsilon + \epsilon + \epsilon < \frac{1}{4}\epsilon_0$$

and

$$\sum_q \sum_{j=0}^{\tilde{h}_n-1} \left|\mu(T^{k\tilde{h}_n} I_{n,j} \cap A \cap \tilde{C}_n^{(q)}) - \mu I_{n,j}\mu(A \cap \tilde{C}_n^{(q)})\right| < \frac{1}{4}\epsilon_0$$

Now consider when B_{n+2} through B_{n+z} are constructed using the same Chacon-like operation and B_{n+z+1} using the staircase operation.

If $k < \sqrt{\epsilon} r_n$ then $\int \left|\frac{1}{r_n}\sum_{i=1}^{k} \chi \circ T^{i(r_n-k)}\right| d\mu < \sqrt{\epsilon}$ so we may assume $k \geq \sqrt{\epsilon} r_n$. Likewise, if $r_n - k < \frac{1}{2}L^2$ then since we have taken n large enough already, $\int \left|\frac{1}{r_n}\sum_{i=1}^{k} \chi \circ T^{i(r_n-k)}\right| d\mu < \epsilon$.

Take p minimal such that $p(r_n - k) \geq \tilde{h}_\ell$. Then $p(r_n - k) < 2\tilde{h}_\ell$ and

$$\frac{pL}{k} < \frac{pL}{\sqrt{\epsilon} r_n} = \frac{p(r_n-k)L}{\sqrt{\epsilon}(r_n-k)kr_n} < \frac{2\tilde{h}_\ell L}{\sqrt{\epsilon}L^2\tilde{h}_\ell} = \frac{2}{\sqrt{\epsilon}L} < 2\sqrt{\epsilon}$$

So by Lemma 4.21,

$$\int \left|\frac{1}{r_n+1}\sum_{i=0}^{k} \chi \circ T^{i(r_n-k)}\right| d\mu \leq \int \left|\frac{1}{L}\sum_{i=0}^{L-1} \chi \circ T^{i(r_n-k)p}\right| d\mu + 2\sqrt{\epsilon}$$

and since $\tilde{h}_\ell \leq 2i(r_n - k)p < 4L\tilde{h}_\ell$ for all $1 \leq i < L$, by Lemma 4.22,

$$\int \left|\frac{1}{r_n+1}\sum_{i=0}^{k} \chi \circ T^{i(r_n-k)}\right| d\mu < \sqrt{2\epsilon} + 2\sqrt{\epsilon}$$

Then by Lemma 4.17,

$$\sum_q \sum_{j=0}^{\tilde{h}_n-1} \left|\mu(T^{k\tilde{h}_n} J_{n,j}^{(q)} \cap A) - \mu J_{n,j}^{(q)}\mu A\right| < \sqrt{2\epsilon} + 2\epsilon + \epsilon + 2\left(\sqrt{2\epsilon} + 2\sqrt{\epsilon}\right) + \epsilon + \epsilon < \frac{1}{4}\epsilon_0$$

and by Lemma 4.18,

$$\sum_q \sum_{j=0}^{\tilde{h}_n-1} \left|\mu(T^{k\tilde{h}_n} I_{n,j} \cap A \cap \tilde{C}_n^{(q)}) - \mu I_{n,j}\mu(A \cap \tilde{C}_n^{(q)})\right| < \frac{1}{4}\epsilon_0$$

Therefore, by Lemma 4.6, for every $\tilde{h}_n \leq t < (r_n - 1)\tilde{h}_n$,

$$\sum_q \sum_{j=0}^{\tilde{h}_n-1} \left|\mu(T^{t} J_{n,j}^{(q)} \cap A) - \mu J_{n,j}^{(q)}\mu A\right| < 2\left(\frac{1}{4}\epsilon_0 + \frac{1}{4}\epsilon_0\right) = \epsilon_0$$

and likewise

$$\sum_q \sum_{j=0}^{\tilde{h}_n-1} \left|\mu(T^{t} I_{n,j} \cap A \cap \tilde{C}_n^{(q)}) - \mu I_{n,j}\mu(A \cap \tilde{C}_n^{(q)})\right| < \epsilon_0$$

Finally, consider $(1 - \frac{1}{2}\epsilon_0)\tilde{h}_n \leq t < \tilde{h}_n$. Then by Lemma 4.10,

$$\begin{aligned}\sum_q \sum_{j=\tilde{h}_n-t}^{h_n} \left|\mu(T^t J_{n,j}^{(q)} \cap A) - \mu J_{n,j}^{(q)}\mu A\right| &\leq \sum_q \sum_{j=r_n}^{t} \left|\mu(T^{\tilde{h}_n} J_{n,j}^{(q)} \cap A) - \mu J_{n,j}^{(q)}\mu A\right| + \sum_q \sum_{j=0}^{r_n-1} \mu J_{n,j}^{(q)} \\ &\leq \int \left|\frac{1}{r_n+1}\sum_{i=0}^{r_n-1} \chi \circ T^{-i}\right| d\mu + \frac{1}{r_n+1} + \frac{r_n}{\tilde{h}_n}\end{aligned}$$

and therefore

$$\sum_q \sum_{j=0}^{\tilde{h}_n-1} \left|\mu(T^t J_{n,j}^{(q)} \cap A) - \mu J_{n,j}^{(q)} \mu A\right| < \epsilon + \frac{h_n - t}{h_n} + \epsilon + \epsilon < \epsilon + \frac{1}{2}\epsilon_0 + 2\epsilon < \epsilon_0$$

Similar reasoning using the second statement in Lemma 4.10 gives

$$\sum_q \sum_{j=0}^{\tilde{h}_n-1} \left|\mu(T^t I_{n,j} \cap A \cap \tilde{C}_n^{(q)}) - \mu I_{n,j} \mu(A \cap \tilde{C}_n^{(q)})\right| < \epsilon_0 \qquad \square$$

5 Partial mixing

Lemma 5.1. *Let T be a rank-one transformation such that B_{n+1} through B_{n+m} are constructed using Chacon-like operations with $c_{n+z} = c_n$ and $M_{n+z} = M_n$ for all $0 \le z < m$ where M_n and c_n are such that M_n divides $\tilde{h}_n$.*

For any $1 \le \ell < \frac{\tilde{h}_{n+m}}{M_n}$, there exist disjoint mesh prototypes $J_{n-1,0}^{(q)}$ of $I_{n-1,0}$ for each $-\frac{\tilde{h}_n}{2M_n} < q \le \frac{\tilde{h}_n}{2M_n}$ and a set $Q_{n-1,0}$ which is a union of levels in C_{n+m} such that $T^j Q_{n-1,0} \cap Q_{n-1,0} = \varnothing$ for $1 \le j < \tilde{h}_n$ and

$$T^{\ell M_n} I_{n-1,0} = T^{\tilde{h}_{n+m}} Q_{n-1,0} \sqcup \bigsqcup_{q=-\tilde{h}_n/2M_n+1}^{\tilde{h}_n/2M_n} T^{qM_n} J_{n-1,0}^{(q)}$$

If there exists $0 < \beta < 1$ such that $\mu(T^{\ell M_n} I_{n,0} \cap I_{n,0}) \le \beta\mu(I_{n,0})$ and $\mu(T^{(\ell-1)M_n} I_{n,0} \cap I_{n,0}) \le \beta\mu(I_{n,0})$ then $\mu(J_{n-1,0}^{(0)}) \le \beta\mu(I_{n-1,0})$ and $\mu(J_{n-1,0}^{(-1)}) \le \beta\mu(I_{n-1,0})$.

Proof. Since $\tilde{h}_{n+1} = (r_n + 1)\tilde{h}_n + s_n M_n + c_{n+1} - c_n$ where s_n is the number of subcolumns with M_n non-elevation spacers placed over them, and since $c_{n+1} = c_n$, it follows that $\tilde{h}_{n+1}$ is divisible by M_n as by hypothesis $\tilde{h}_n$ is. The same reasoning gives that $\tilde{h}_{n+z}$ is divisible by M_n for all $0 \le z < m$.

Note that $\ell M_n < \tilde{h}_{n+m}$. Let S be the union of all non-elevation spacers levels placed above C_n through C_{n+m-1} and let $S^\star$ be the non-elevation spacer levels placed above C_{n+m}. Let

$$Q_{n,0} = \{x \in I_{n,0} : \exists\ 0 < y \le \ell M_n \text{ s.t. } T^y x \in S^\star\}$$

be the set of points in $I_{n,0}$ which $T^{\ell M_n}$ pushes through the top of C_{n+m}. For $-\frac{\tilde{h}_n}{2M_n} < q \le \frac{\tilde{h}_n}{2M_n}$, let

$$Z_{n,0}^{(q)} = \{x \in I_{n,0} \setminus Q_{n,0} : T^{-qM_n + \ell M_n} x \in I_{n,0}\}$$

be the set of points in $I_{n,0}$ such that $T^{\ell M_n}$ does not push them through the top of C_{n+m} and leaves them exactly qM_n away from $I_{n,0}$. Since M_n divides $\tilde{h}_{n+z}$ for all $0 \le z < m$,

$$I_{n,0} \setminus Q_{n,0} \subseteq \bigsqcup_{q=1-\tilde{h}_n/2M_n}^{\tilde{h}_n/2M_n} Z_{n,0}^{(q)}$$

For $x \in Q_{n,0}$, if $x \in I_{n+m,j}$ then since $T^y x \in S^\star$ for some $0 < y \le \ell M_n < \tilde{h}_{n+m}$, it follows that the same holds for all $y \in I_{n+m,j}$ since if any point in a level is pushed through the top of the column then all points in the level are (and no point is pushed through twice as $\ell M_n < \tilde{h}_{n+m}$). Therefore $Q_{n,0}$ can be written as union of levels in C_{n+m}. For any $I_{n+m,j} \subseteq Q_{n,0}$, since $T^{\ell M_n}$ pushes $I_{n+m,j}$ through the top of C_{n+m} exactly once, it must be that $j \ge \tilde{h}_{n+m} - \ell M_n$.

Set $Q'_{n,0} = T^{\ell M_n - \tilde{h}_{n+m}} Q$. Then $Q'_{n,0}$ is a union of levels in C_{n+m} and $T^{\ell M_n} Q_{n,0} = T^{\tilde{h}_{n+m}} Q'_{n,0}$. If $x \in T^j Q_{n,0} \cap Q_{n,0}$ for $0 \le j < \tilde{h}_n$ then $x \in I_{n,j} \cap I_{n,0}$ so $j = 0$.

By construction, $T^{\ell M_n} Z_{n,0}^{(q)} \subseteq T^{qM_n} I_{n,0}$. Set

$$J_{n,0}^{(q)} = T^{-qM_n}\left(T^{\ell M_n} Z_{n,0}^{(q)}\right) \subseteq I_{n,0}$$

and set, writing $s_{n-1,y}$ for the non-elevation spacer amounts added above C_{n-1},

$$J_{n-1,0}^{(q)} = \bigsqcup_{i=0}^{r_n-1} T^{i\tilde{h}_{n-1}+\sum_{y=0}^{i-1} s_{n-1,y}} J_{n,0}^{(q)}$$

Then $J_{n-1,0}^{(q)}$ is a mesh prototype for $I_{n,0}$ (since $I_{n-1,0}^{[i]} = T^{i\tilde{h}_{n-1}+\sum_{y=0}^{i-1} s_{n-1,y}} I_{n,0}$) and

$$\begin{aligned} T^{\ell M_n} I_{n-1,0} &= T^{\ell M_n}\left(\bigsqcup_{i=0}^{r_{n-1}} T^{i\tilde{h}_{n-1}+\sum_{y=0}^{i-1} s_{n-1,y}} I_{n,0}\right) \\ &= \bigsqcup_{i=0}^{r_{n-1}} T^{i\tilde{h}_{n-1}+\sum_{y=0}^{i-1} s_{n-1,y}}\left(T^{\tilde{h}_{n+m}} Q'_{n,0} \sqcup \bigsqcup_q T^{qM_n} J_{n,0}^{(q)}\right) \\ &= T^{h_{n+m}}\left(\bigsqcup_{i=0}^{r_{n-1}} T^{i\tilde{h}_{n-1}+\sum_{y=0}^{i-1} s_{n-1,y}} Q'_{n,0}\right) \sqcup \bigsqcup_q T^{qM_n} J_{n-1,0}^{(q)} \end{aligned}$$

Set $Q_{n-1,0} = \bigsqcup_{i=0}^{r_{n-1}} T^{i\tilde{h}_{n-1}+\sum_{y=0}^{i-1} s_{n-1,y}} Q'_{n,0}$. Then $Q_{n-1,0}$ is a union of levels in C_{n+m} since $i\tilde{h}_{n-1} + \sum_{y=0}^{i-1} s_{n-1,y} < \tilde{h}_n$ for all $0 \le i < r_{n-1}$.

Since for $q < q'$, $J_{n,0}^{(q)} \cap J_{n,0}^{(q')} \subseteq T^{\ell M_n + qM_n}(Z_{n,0}^{(q)} \cap T^{(q'-q)M_n} Z_{n,0}^{(q')})$, if the intersection were nonempty then $(q'-q)M_n$ would have to be a multiple of $\tilde{h}_n$ as $Z_{n,0}^{(q)} \subseteq I_{n,0}$. But that is impossible so it follows that the $J_{n,0}^{(q)}$ and hence the $J_{n-1,0}^{(q)}$ are mutually disjoint.

Assume now that there exists $0 < \beta < 1$ such that $\mu(T^{\ell M_n} I_{n,0} \cap I_{n,0}) \le \beta\mu(I_{n,0})$ and $\mu(T^{(\ell-1)M_n} I_{n,0} \cap I_{n,0}) \le \beta\mu(I_{n,0})$. Since $J_{n,0}^{(0)} \subseteq I_{n,0} \cap T^{\ell M_n} I_{n,0}$, it follows that $\mu(J_{n,0}^{(0)}) = \mu(J_{n,0}^{(0)} \cap I_{n,0} \cap T^{\ell M_n} I_{n,0}) \le \beta\mu(I_{n,0})$. Likewise, since $J_{n,0}^{(-1)} \subseteq I_{n,0} \cap T^{(\ell-1)M_n} I_{n,0}$, the same holds for $J_{n,0}^{(-1)}$. □

Proposition 5.2. *Let T be an elevated staircase-Chacon-like hybrid transformation and A and D be unions of levels in some column C_n and let $\epsilon > 0$. If for some $m > 2$,*

- *A and D are uniformly mixed within ϵ at stages $n-2$ and $n-1$*
- *A is uniformly mixed within ϵ at stage $n+m$*
- *B_{n+1} through B_{n+m} are constructed using Chacon-like operations*
- *$M_{n+z} = 2\tilde{h}_{n-2}$ for all $0 \le z < m$*
- *for all $0 \le z < m$, $c_{n+z} = c_n$ is the unique minimal positive integer such that $c_n \ge c_{n-1} + r_{n-1}$ and $2\tilde{h}_{n-2}$ divides $h_n + c_n = \tilde{h}_n$*
- *there exists $0 < \beta < 1$ such that $\mu(T^{\ell M_n} I_{n,0} \cap I_{n,0}) \le \beta\mu(I_{n,0})$ for all $\frac{\tilde{h}_n}{M_n} - 1 \le \ell < \frac{\tilde{h}_{n+m}}{M_n}$*

then for all $(1-\frac{1}{2}\epsilon)\tilde{h}_{n-2} \le t < (1-\frac{1}{2}\epsilon)\tilde{h}_{n+m}$,

$$\mu(T^t D \cap A) \ge (1-\beta)\mu D\mu A - 5\epsilon$$

Proof. The claim is immediate for $(1-\frac{1}{2}\epsilon)\tilde{h}_{n-2} \le t < (1-\frac{1}{2}\epsilon)\tilde{h}_n$ since

$$\left|\mu(T^t D \cap A) - \mu D\mu A\right| \le \sum_{j=0}^{h_n-1} \left|\mu(T^t I_{n,j} \cap A) - \mu I_{n,j}\mu A\right| < \epsilon$$

so $\mu(T^t D \cap A) \ge \mu D\mu A - \epsilon$.

Let $\tilde{h}_n \le t < (1-\frac{1}{2}\epsilon)\tilde{h}_{n+m}$. Let ℓ be the unique integer such that $\ell M_n \le t < (\ell+1)M_n$. Then

$\frac{\tilde{h}_n}{M_n} \le \ell < \frac{\tilde{h}_{n+m}}{M_n}$ and let $y = t - \ell M_n$ be such that $0 \le y < M_n$. Let $J^{(q)}_{n-1,0}$ be the mesh prototypes and $Q_{n-1,0}$ be the set from Lemma 5.1. Then

$$T^t I_{n-1,0} = T^y T^{\ell M_n} I_{n-1,0} = T^{\tilde{h}_{n+m}+y} Q_{n-1,0} \sqcup \bigsqcup_{q=1-\tilde{h}_n/2M_n}^{\tilde{h}_n/2M_n} T^{qM_n+y} J^{(q)}_{n-1,0}$$

Let $\tilde{C}^{(q)}_{n-1} = \bigsqcup_{j=0}^{\tilde{h}_{n-1}-1} J^{(q)}_{n-1,j}$ be the disjoint meshes from the prototypes. By Lemma 4.5, $C_{n-2} \cap \tilde{C}^{(q)}_{n-1}$ is a mesh of C_{n-2} which we will write as $\tilde{C}^{(q)}_{n-2} = \bigsqcup_{j=0}^{\tilde{h}_{n-2}-1} J^{(q)}_{n-2,j}$.

Since $Q_{n-1,0}$ is a union of levels in C_{n+m} and $Q_{n-1,0} \cap T^j Q_{n-1,0} = \varnothing$ for $1 \le j < \tilde{h}_n$, set $Q_{n-1,j} = T^j Q_{n-1,0}$ for each $0 \le j < \tilde{h}_{n-1}$ and $Q_{n-1} = \bigsqcup_{j=0}^{\tilde{h}_{n-1}} Q_{n-1,j}$ is a union of levels in C_{n+m}.

Since D can be written as a union of levels in C_{n-1}, there exists $\mathcal{D} \subseteq \{0, 1, \dots, \tilde{h}_{n-1} - 1\}$ such that $D = \bigsqcup_{j\in\mathcal{D}} I_{n-1,j}$. Then

$$\begin{aligned} T^t D = \bigsqcup_{j\in\mathcal{D}} T^t I_{n,j} &= \bigsqcup_{j\in\mathcal{D}} \left(T^{\tilde{h}_{n+m}+y} Q_{n-1,j} \sqcup \bigsqcup_q T^{qM_n+y} J^{(q)}_{n-1,j} \right) \\ &= T^{\tilde{h}_{n+m}+y} (D \cap Q_{n-1}) \sqcup \bigsqcup_q T^{qM_n+y} \left(D \cap \tilde{C}^{(q)}_{n-1} \right) \end{aligned}$$

and therefore, since $D \subseteq C_{n-2}$,

$$\begin{aligned} \mu(T^t D \cap A) \ge \mu(T^{\tilde{h}_{n+m}+y}(D \cap Q_{n-1}) \cap A) &+ \sum_{q:\tilde{h}_{n-2} \le |qM_n+y| < \tilde{h}_{n-1}} \mu(T^{qM_n+y}(D \cap \tilde{C}^{(q)}_{n-2}) \cap A) \\ &+ \sum_{q:|qM_n+y| \ge \tilde{h}_{n-1}} \mu(T^{qM_n+y}(D \cap \tilde{C}^{(q)}_{n-1}) \cap A) \end{aligned}$$

Since $Q_{n-1,j}$ are disjoint sets which are unions of levels in C_{n+m}, then

$$\begin{aligned} \left| \mu(T^{\tilde{h}_{n+m}+y}(D \cap Q_{n-1}) \cap A) - \mu(D \cap Q_{n-1}) \mu A \right| &\le \sum_{j=0}^{\tilde{h}_{n-1}-1} \left| \mu(T^{\tilde{h}_{n+m}+y} Q_{n-1,j} \cap A) - \mu Q_{n-1,j} \mu A \right| \\ &= \sum_{j=0}^{\tilde{h}_{n-1}-1} \left| \sum_{j': I_{n+m,j'} \subseteq Q_{n-1,j}} \mu(T^{\tilde{h}_{n+m}+y} I_{n+m,j'} \cap A) - \mu I_{n+m,j'} \mu A \right| \\ &\le \sum_{j=0}^{\tilde{h}_{n+m}-1} \left| \mu(T^{\tilde{h}_{n+m}+y} I_{n+m,j} \cap A) - \mu I_{n+m,j} \mu A \right| < \epsilon \end{aligned}$$

since A is uniformly mixed at stage $n+m$ and $\tilde{h}_{n+m} \le \tilde{h}_{n+m} + y < \tilde{h}_{n+m} + M_n < (1-\epsilon)\tilde{h}_{n+m+1}$.

By Lemma 4.24, since A is uniformly mixed within ϵ at stage $n-2$ (and at stage $n-1$ for when $qM_n + y \ge (1 - \frac{\epsilon}{2})\tilde{h}_{n-2}$),

$$\sum_{q:\tilde{h}_{n-2} \le qM_n+y < \tilde{h}_{n-1}} \left| \mu(T^{qM_n+y}(D \cap \tilde{C}^{(q)}_{n-2}) \cap A) - \mu(D \cap \tilde{C}^{(q)}_{n-2}) \mu A \right| < \epsilon$$

and since A is uniformly mixed at stage $n-1$ and $|qM_n + y| \le \frac{1}{2}\tilde{h_n} + M_n = \frac{1}{2}\tilde{h}_n + 2\tilde{h}_{n-2} < (1 - \frac{1}{2}\epsilon)\tilde{h}_n$,

$$\sum_{q:\tilde{h}_{n-1} \le qM_n+y} \left| \mu(T^{qM_n+y}(D \cap \tilde{C}^{(q)}_{n-1}) \cap A) - \mu(D \cap \tilde{C}^{(q)}_{n-1}) \mu A \right| < \epsilon$$

Since D is uniformly mixed at stages $n-2$ and $n-1$, using the second condition of uniformly mixed

within ϵ,

$$\sum_{q:\tilde{h}_{n-2}\le -(qM_n+y)<\tilde{h}_{n-1}} \left|\mu(T^{qM_n+y}(D\cap\tilde{C}^{(q)}_{n-2})\cap A)-\mu(D\cap\tilde{C}^{(q)}_{n-2})\mu A\right|$$
$$= \sum_{q:\tilde{h}_{n-2}\le -(qM_n+y)<\tilde{h}_{n-1}} \left|\mu(D\cap\tilde{C}^{(q)}_{n-2}\cap T^{-(qM_n+y)}A)-\mu(D\cap\tilde{C}^{(q)}_{n-2})\mu A\right| < \epsilon$$

and likewise since D is uniformly mixed at stage $n-1$,

$$\sum_{q:\tilde{h}_{n-1}\le -(qM_n+y)} \left|\mu(T^{qM_n+y}(D\cap\tilde{C}^{(q)}_{n-1})\cap A)-\mu(D\cap\tilde{C}^{(q)}_{n-1})\mu A\right| < \epsilon$$

Since $y<M_n$, for every $q\neq 0,-1$ it holds that $|qM_n+y|\ge M_n$ and also that one of $|-M_n+y|\ge\frac{1}{2}M_n$ or $|y|\ge\frac{1}{2}M_n$. Therefore $\sum_{q:|qM_n+y|<\tilde{h}_{n-2}}\mu(D\cap\tilde{C}^{(q)}_{n-1})\le\mu(D\cap\tilde{C}^{(q')}_{n-1})$ where $q'\in\{0,-1\}$. From Lemma 5.1, $\mu J^{(q')}_{n-1,0}\le\beta\mu I_{n-1,0}$ and so $\mu J^{(q')}_{n-1,j}\le\beta\mu I_{n-1,j}$ for all $0\le j<\tilde{h}_{n-1}-1$. Therefore, as D is a union of levels,

$$\mu\left(D\cap\tilde{C}^{(q')}_{n-1}\right)\le\beta\mu D$$

Since $T^{qM_n}I_{n,0}=T^{\tilde{h}_{n+m}}Q_{n-1,0}\sqcup\bigsqcup_q T^{qM_n}J^{(q)}_{n-1,0}$ and T is measure-preserving, $\mu I_{n,0}=\mu Q_{n-1,0}+\sum_q\mu J^{(q)}_{n-1,0}$. and therefore

$$\mu(D)=\mu\left(D\cap\left(Q_{n-1}\sqcup\bigsqcup_q\tilde{C}^{(q)}_{n-1}\right)\right)=\mu\left(D\cap\left(Q_{n-1}\sqcup\bigsqcup_{q\neq q'}\tilde{C}^{(q)}_{n-1}\right)\right)+\mu\left(D\cap\tilde{C}^{(q')}_{n-1}\right)$$

meaning that

$$\mu\left(D\cap\left(Q_{n-1}\sqcup\bigsqcup_{q\neq q'}\tilde{C}^{(q)}_{n-1}\right)\right)=\mu(D)-\mu\left(D\cap\tilde{C}^{(q')}_{n-1}\right)\ge\mu D-\beta\mu D$$

Putting all of this together (and using that $D\cap\tilde{C}^{(q)}_{n-2}=D\cap\tilde{C}^{(q)}_{n-1}$), since the meshes are disjoint and disjoint from Q_{n-1},

$$\mu(T^tD\cap A)\ge\mu(D\cap Q_{n-1})\mu A+\sum_{q\neq q'}\mu(D\cap\tilde{C}^{(q)}_{n-1})\mu A-5\epsilon$$
$$=\mu\left(D\cap\left(Q_{n-1}\sqcup\bigsqcup_{q\neq q'}\tilde{C}^{(q)}_{n-1}\right)\right)\mu A-5\epsilon\ge(1-\beta)\mu D\mu A-5\epsilon \qquad \square$$

6 Low complexity partial mixing

Fix an integer $\gamma\ge 2$. We will construct an elevated staircase-Chacon-like hybrid transformation T_γ.

Let $B_0=0$ be the starting word. Let $S_0=-1$ and $Q_0=0$. Set $r_{-1}=3$ and $c_0=1$ and $h_{-1}=3$.

Assume that the transformation has been constructed up to B_{Q_n} and that c_{Q_n} has been specified. Construct B_{Q_n+1} and B_{Q_n+2} using the staircase operation with $r_{Q_n}=r_{Q_n+1}=\tilde{h}_{S_n}$ and $c_{Q_n+1}=c_{Q_n}+r_{Q_n}$. Set $S_{n+1}=Q_n+2$.

Let $M_{S_{n+1}}=2\tilde{h}_{Q_n}$. Let $c_{S_{n+1}}$ be the minimal integer such that $c_{S_{n+1}}>c_{S_{n+1}-1}+r_{S_{n+1}-1}$ and $M_{S_{n+1}}$ divides $h_{S_{n+1}}+c_{S_{n+1}}$. By Proposition 3.1, $p(\tilde{h}_{S_{n+1}})$ is now completely determined. Choose $y_{n+1}>S_{n+1}+2$ such that

$$3^{S_{n+1}-y_{n+1}}\frac{p(\tilde{h}_{S_{n+1}})}{\tilde{h}_{S_{n+1}}}<\frac{1}{n+1}$$

Construct B_t for $S_n < t \le y_{n+1}$ using the γ-Chacon-like operation with $M_t = M_{S_{n+1}}$ and $c_t = c_{S_{n+1}}$. Choose $Q_{n+1} > y_{n+1} + n + 1$ such that $\tilde{h}^2_{S_{n+1}} < \frac{1}{n+1} 3^{Q_{n+1}}$. Construct B_t for $y_{n+t} < t \le Q_{n+1}$ using the γ-Chacon-like operation with $M_t = M_{S_{n+1}}$ and $c_t = c_{S_{n+1}}$. Set $c_{Q_{n+1}} = |B_{y_{n+1}}| + 1$.

Proceed inductively for all n.

6.1 Basic properties

Lemma 6.1. *T_γ satisfies*

$$\frac{|B_{y_{n+1}}|+1}{|B_{Q_{n+1}}|} < 3^{-n-1} \qquad \textit{and} \qquad \frac{r^2_{Q_{n+1}}}{|B_{Q_{n+1}}|} < \frac{1}{n+1} \qquad \textit{and} \qquad \frac{r^2_{Q_{n+1}+1}}{|B_{Q_{n+1}+1}|} < \frac{1}{n+1}$$

Proof. By construction, since $|B_{t+1}| \ge 3|B_t|$ for all t and $Q_{n+1} \ge y_{n+1} + n + 1$,

$$\frac{|B_{y_{n+1}}|+1}{|B_{Q_{n+1}}|} < 3^{-Q_{n+1}+y_{n+1}+1} < 3^{-n} \qquad \text{and} \qquad \frac{\tilde{h}^2_{S_n}}{|B_{Q_{n+1}}|} < \frac{\frac{1}{n+1}3^{Q_{n+1}}}{3^{Q_{n+1}}} = \frac{1}{n+1}$$

and we have chosen $r_{Q_{n+1}} = r_{Q_{n+1}+1} = \tilde{h}_{S_n}$. □

Lemma 6.2. *T_γ satisfies $\frac{M_{S_n}}{h_{S_n-1}} \le 3^{-n}$.*

Proof. Since $|B_{t+1}| > 3|B_t|$ (and $c_{n+1} > c_n$),

$$h_{S_n-1} > r_{S_n-2}\tilde{h}_{S_n-2} = r_{Q_{n-1}}\ \frac{1}{2}M_{S_n} = h_{S_{n-1}}\ \frac{1}{2}M_{S_n} \ge 3^{S_{n-1}}\ \frac{1}{2}M_{S_n}$$

and $S_{n-1} \ge n+1$ so the claim follows. □

Lemma 6.3. *T_γ satisfies $\sum \frac{c_n}{|B_n|} < \infty$.*

Proof. By Lemma 6.1, $\frac{c_{Q_n}}{|B_{Q_n}|} = \frac{|B_{y_n}|+1}{|B_{Q_n}|} < 3^{-n}$. By construction, using that $|B_t| > 3^t$ and $Q_n \ge n$,

$$\frac{c_{Q_n+1}}{|B_{Q_n+1}|} = \frac{c_{Q_n}+r_{Q_n}}{|B_{Q_n+1}|} < \frac{c_{Q_n}+r_{Q_n}}{r_{Q_n}|B_{Q_n}|} = \frac{1}{r_{Q_n}}\frac{c_{Q_n}}{|B_{Q_n}|} + \frac{1}{|B_{Q_n}|} < 3^{-n-1} + 3^{-n-1} < 3^{-n}$$

and similarly, by Lemma 6.2,

$$\frac{c_{S_{n+1}}}{|B_{S_{n+1}}|} < \frac{c_{Q_n+1}+r_{Q_n+1}+M_{S_{n+1}}}{|B_{S_{n+1}}|} < \frac{c_{Q_n+1}+r_{Q_n+1}+M_{S_{n+1}}}{r_{Q_n+1}|B_{Q_n+1}|} < 3^{-n-1} + 3^{-Q_n-1} + 3^{-n-1} < 3^{-n}$$

For $0 < t < Q_{n+1} - S_{n+1}$, $\frac{c_{S_{n+1}+t}}{|B_{S_{n+1}+t}|} < \frac{c_{S_{n+1}}}{3^t|B_{S_{n+1}}|} < 3^{-n-t}$. Therefore

$$\sum_{t=0}^{\infty}\frac{c_t}{|B_t|} = \sum_{n=0}^{\infty}\sum_{t=Q_n}^{Q_{n+1}-1}\frac{c_t}{|B_t|} < \sum_{n=0}^{\infty}\left(3\cdot 3^{-n} + \sum_{t=1}^{Q_{n+1}-S_{n+1}-1} 3^{-n-t}\right) < \sum_{n=0}^{\infty}\left(3^{-n+1} + 3^{-n}\frac{1}{2}\right) < \infty$$ □

Proposition 6.4. *T_γ is on a finite measure space.*

Proof. The total number of spacers added at each stage is c_t levels of the same size as those in B_t and either approximately $\frac{1}{2}r_t^2$ levels the size of those in B_{t+1} or M_t levels the size of those in B_{t+1}.

The measure of the elevation levels is then $\sum_t \frac{c_t}{h_t}$ which is finite by Lemma 6.3. By Lemma 6.1, $\frac{r_n^2}{h_n} \to 0$ for those n for which B_{n+1} is constructed using the staircase operation so the spacer levels in the staircase operations have finite total measure.

At each Chacon-like form operation, the non-elevation levels have measure at most $\frac{M_t}{|B_t|}$ since at most M_t spacers are added above each subcolumn. By Lemma 6.2, $\sum \frac{M_{S_n}}{|B_{S_n}|} < \infty$. Since $|B_{t+1}| > 3|B_t|$, the total measure added by the non-elevation spacers in Chacon-like form operations is, as $M_t = M_{S_{n+1}}$ for $S_{n+1} \le t < Q_{n+1}$,

$$\sum_{n=0}^{\infty} \sum_{t=S_{n+1}}^{Q_{n+1}-1} \frac{M_t}{|B_t|} \le \sum_{n=0}^{\infty} \sum_{t=0}^{Q_{n+1}-S_{n+1}-1} \frac{M_{S_{n+1}}}{|B_{S_{n+1}}|} \frac{|B_{S_{n+1}}|}{|B_{S_{n+1}+t}|} \le \sum_{n=0}^{\infty} \frac{M_{S_{n+1}}}{|B_{S_{n+1}}|} \sum_{t=0}^{\infty} 3^{-t} < \infty \qquad \square$$

6.2 Word complexity

Proposition 6.5. *T_γ satisfies* $\liminf {}^{p(q)}/_{q} \le 2 + {}^{1}/_{(\gamma+1)}$

Proof. By Proposition 3.7, since $|B_{y_n}| < c_{Q_n}$,

$$\frac{p\left(\frac{1}{2}\tilde{h}_{n+1}\right)}{\frac{1}{2}\tilde{h}_{n+1}} < 2 + \frac{1}{\gamma+1} + 3^{S_n - y_n} \frac{p(|B_{S_n}|)}{|B_{S_n}|}$$

so by our choice of y_n, then

$$\frac{p\left(\frac{1}{2}\tilde{h}_{n+1}\right)}{\frac{1}{2}\tilde{h}_{n+1}} \le 2 + \frac{1}{\gamma+1} + \frac{1}{n} \to 2 + \frac{1}{\gamma+1} \qquad \square$$

Remark 6.6. The T_γ also satisfy $\limsup {}^{p(q)}/_{q} = \infty$ which is the primary motivation for Conjecture 7.

The staircase construction, which necessarily occurs infinitely often in our constructions with cut parameter tending to infinity in order to ensure partial mixing, forces there to be a sequence of times along which $p(q)/q \to \infty$. Specifically, at each staircase stage, $B_n 1^{c_n+i}$ is a right-special word for each $0 \le i < r_n$ and therefore for $c_n + r_n \le q < |B_n|$ we effectively have $p(q+1) - p(q) \ge r_n - 1$ (we omit the details). As $|B_n| \to \infty$ rapidly, this forces $p(|B_n|)/|B_n| \to \infty$ for n such that a staircase construction is about to be implemented.

In essence, the tension here is that the Chacon-like stages are required to reduce the word complexity to as close to 2 as possible while the staircase constructions are required to ensure partial mixing. Conjecture 5 in essence states that this sort of tension should be unavoidable regardless of what subshift is exhibited since the partial mixing condition should effectively mean that "mixing occurs infinitely often" and, as shown in [Cre26], outright mixing implies $\lim p(q)/q = \infty$.

6.3 Partial mixing

Notation 6.7. For two words u and v of the same length, $d_H(u,v) = |\{i : u_{[i,i+1)} \ne v_{[i,i+1)}\}|$ is the Hamming distance.

Proposition 6.8. *Let $P \ge 1$ and let T be the subshift on the alphabet $\{0,1,\dots,P\}$ constructed using $B_0 = 023\dots P$ and $B_{n+1} = (B_n 1)^\gamma B_n (B_n 1)^\gamma B_n$. Let $W_n = (B_n 1)^\gamma B_n$ and for $1 \le i < |W_n|$, let $W_{n,i} = (W_n)_{[i,|W|_n]}(W_n)_{[1,i)}$ be the word obtained by shifting W_n to the left i places (with wraparound).*

Let W be the right-infinite word which has W_n at the origin for every n. Then there exists a constant $\kappa_\gamma > 0$ depending only on γ (and not on P) such that if $i \ge 1$ and n is such that $|W_n| > i$ then $d_H(W_{[1,|W_n|]}, (\sigma^i W)_{[1,|W_n|]}) \ge \kappa_\gamma |W_n|$.

Proof. Let $|u|_0$ be the number of 0s in a word u and likewise $|u|_1$ be the number of 1s. Since $|B_0|_0 = 1$ and $|B_{n+1}|_0 = (2\gamma+2)|B_n|_0$, then $|B_n|_0 = (2\gamma+2)^n$. Since $|B_0|_1 = 0$ and $|B_{n+1}|_1 = 2\gamma + (2\gamma+2)|B_n|_1$,

then $|B_n|_1 = 2\gamma(1 + (2\gamma+2) + \cdots + (2\gamma+2)^{n-1}) = 2\gamma\frac{(2\gamma+2)^n-1}{2\gamma+1}$. Then for $n \geq 1$,

$$\frac{|B_n|_1}{|B_n|_0} = \frac{2\gamma - \frac{2\gamma}{(2\gamma+2)^{n-1}}}{2\gamma+1} > \frac{2\gamma-1}{2\gamma+1} \qquad \text{and} \qquad \frac{P|B_n|_0}{|B_n|} = \frac{1}{1+\frac{|B_n|_1}{P|B_n|_0}} > \frac{1}{1+\frac{2\gamma}{P(2\gamma+1)}}$$

For a word u which is a concatenation of 1s and B_n blocks, since 0 is always followed by $23\ldots P$ in such a block, the number of letters in u which are not 1 is $P|u|_0$. For words u and v of the same length, let $d_0(u,v) = \frac{1}{P|u|_0}|\{i : u_{[i,i+1)} \neq 1, v_{[i,i+1)} \neq u_{[i,i+1)}\}|$.

Since B_n begins with 0 and 11 never appears, every occurrence of a 1 in $1B_n$ coincides with an occurrence of 0 in B_n1. As there are then at least P discrepancies between the words starting at that location,

$$d_0(B_n1, 1B_n) \geq \frac{P|B_n|_1}{P|B_n|_0} > \frac{2\gamma-1}{2\gamma+1}$$

and likewise $d_0(1B_n, B_n1) > \frac{2\gamma-1}{2\gamma+1}$.

We claim now that for every n and every word w in the language such that $|w| = |B_n|$, if $w \neq B_n$ then $d_0(B_n, w) \geq \frac{1}{2\gamma+2}\ \frac{2\gamma-1}{2\gamma+1}$. When $n = 0$, the claim is immediate since $d_0(a, 1) = 1$ for any letter $a \neq 1$.

Assume the claim holds for n. Let w be a word in the language such that $|w| = |B_{n+1}|$ and $w \neq B_{n+1}$. Then there exist words w_j for $1 \leq j \leq 2\gamma+2$ and z_j for $1 \leq j \leq \gamma$ and $\gamma+2 \leq j \leq 2\gamma+1$ with $|w_j| = |B_n|$ and $|z_j| = 1$ such that

$$w = \left(\prod_{j=1}^{\gamma} w_j z_j\right) w_{\gamma+1} \left(\prod_{j=\gamma+2}^{2\gamma+1} w_j z_j\right) w_{2\gamma+2}$$

Since $B_{n+1} = (B_n1)^\gamma B_n (B_n1)^\gamma B_n$ and d_0 ignores the locations where B_{n+1} reads 1,

$$d_0(B_{n+1}, w) = \frac{|\{i : (B_{n+1})_{[i,i+1)} \neq 1, w_{[i,i+1)} \neq, i \text{ is in a } B_n \text{ in } B_{n+1}\}|}{(2\gamma+2)P|B_n|_0} = \frac{1}{2\gamma+2}\sum_{j=1}^{2\gamma+2} d_0(B_n, w_j)$$

Suppose $w_j \neq B_n$ for all j. Then by the inductive assumption, $d_0(B_{n+1}, w) \geq \frac{1}{2\gamma+2}\sum_{j=1}^{2\gamma+2}\frac{1}{2\gamma+2}\ \frac{2\gamma-1}{2\gamma+1} = \frac{1}{2\gamma+2}\ \frac{2\gamma-1}{2\gamma+1}$. So we may assume that $w_j = B_n$ for some j. Take j to be the minimal location at which $w_j = B_n$.

Consider first the case when $j \neq 1, j \neq \gamma+1, j \neq 2\gamma+2$. Since B_n is always preceded by either B_n or B_n1, and since $w_{j-1} \neq B_n$, it must be that $w_{j-1}z_jw_j = aB_nB_n$ for some letter a. Since $B_nB_nB_n$ never appears, then $w_{j-1}z_jw_j = 1B_nB_n$ so w and B_{n+1} have at least one location where w reads $1B_n$ and B_{n+1} reads B_n1.

Now consider when $j = 1$. Let k minimal such that $w_k \neq B_n$. Then w has prefix $(B_n1)^{k-2}B_n$ but does not have prefix $(B_n1)^{k-1}$ so, as B_n is always followed by B_n or $1B_n$, then w has prefix $(B_n1)^{k-2}B_nB_n$ hence B_nB_n1 as B_nB_n is always followed by 1. If $k = \gamma+2$ then w would have prefix $(B_n1)^\gamma B_nB_n1$ but then $w_k = B_n$ so $k \neq \gamma+2$. Therefore z_{k-1} exists and $z_{k-1}w_k = B_n1$. So there is at least one location where w reads B_n1 and B_{n+1} reads $1B_n$.

Then in both cases,

$$d_0(B_{n+1}, w) \geq \frac{1}{2\gamma+2}d_0(B_n1, 1B_n) \geq \frac{1}{2\gamma+2}\ \frac{2\gamma-1}{2\gamma+1}$$

By induction this holds for all n and all words $w \neq B_n$ with $|w| = |B_n|$.

Since $B_{n+1} = W_nW_n$, for all words w with $|w| = |W_n|$, then $d_0(W_n, w) \geq \frac{1}{2\gamma+2}\ \frac{2\gamma}{2\gamma+1}$ and $\frac{P|W_n|_0}{|W_n|} = \frac{P|B_{n+1}|_0}{|B_{n+1}|}$ and therefore

$$\frac{d_H(W_n, W_{n,i})}{|W_n|} \geq d_0(W_n, W_{n,i})\frac{P|W_n|_0}{|W_n|} > \frac{1}{2\gamma+2}\ \frac{2\gamma-1}{2\gamma+1}\ \frac{1}{1+\frac{2\gamma}{P(2\gamma+1)}}$$

$$\geq \frac{1}{2\gamma+2}\ \frac{2\gamma-1}{2\gamma+1}\ \frac{1}{1+\frac{2\gamma}{2\gamma+1}} = \frac{2\gamma-1}{(2\gamma+2)(4\gamma+1)}$$

Let W be the right-infinite word that is the limit of W_n. Then for any $i \geq 1$, taking n sufficiently large that $|W_n| > i$, $\sigma^i W$ has $W_{n,i}$ at the origin. Then for such n,

$$d_H(W_{[1,|W_n|]}, (\sigma^i W)_{[1,|W_n|]} = d_H(W_n, W_{n,i}) > \frac{2\gamma-1}{(2\gamma+2)(4\gamma+1)}|W_n|$$

so the claim follows setting $\kappa_\gamma = \frac{2\gamma-1}{(2\gamma+2)(4\gamma+1)}$. □

Remark 6.9. The above transformations, in the case $P = 1$ so on a two letter alphabet, are precisely the examples exhibited in [Cre24] demonstrating rank-one weak mixing transformations with word complexity $\liminf {}^{p(q)}/_{q} = 1$ and $\limsup {}^{p(q)}/_{q} < 1.5 + \epsilon$.

Lemma 6.10. *T_γ has the property that for all n such that B_{n+1} through B_{n+t} are constructed using the γ-Chacon-like operation and all $\frac{\tilde{h}_n}{M_n} - 1 \leq \ell < \frac{\tilde{h}_{n+t}}{M_n}$,*

$$\mu(T^{\ell M_n} I_{n,0} \cap I_{n,0}) \leq (1-\kappa_\gamma)\mu(I_{n,0})$$

for κ_γ as in Proposition 6.8.

Proof. Since M_n divides $|B_n 1^{c_n}|$, treating $B_n 1^{c_n}$ as $023\ldots P$ where $P = \frac{\tilde{h}_n}{M_n}$ and 1^{M_n} as 1, the action of T^{M_n} on B_{n+1} through B_{n+t} generates the same right-infinite word W as the transformation in Proposition 6.8.

Suppose that $\mu(T^{\ell M_n} I_{n,0} \cap I_{n,0}) \geq (1-\kappa_\gamma+\delta)\mu I_{n,0}$ for some $\delta > 0$. Then the corresponding shift of W has the property that $d(W_{[1,|W_n|]}, (\sigma^\ell W)_{[1,|W_n|]}) \leq (\kappa_\gamma - \delta)|W_n|$ since it would force every block of p levels to line up at least $(1-\kappa_\gamma+\delta)$ of the time. But Proposition 6.8 states that $d(W_{[1,|W_n|]}, (\sigma^\ell W)_{[1,|W_n|]}) \geq \kappa_\gamma |W_n|$ so this is a contradiction. □

Proposition 6.11. *T_γ is at least κ_γ-partially mixing where $\kappa_\gamma = \frac{2\gamma-1}{(2\gamma+2)(4\gamma+1)}$.*

Proof. By construction, $r_{Q_n} = r_{Q_n+1} \to \infty$. Also, since $|B_{t+1}| \geq 3|B_t|$ for all t,

$$\frac{r_{Q_n}^2}{h_{Q_n}} = \frac{\tilde{h}_{S_{n-1}}^2}{h_{Q_n}} < \frac{1}{n}\frac{3^{Q_n}}{h_{Q_n}} \leq \frac{1}{n} \to 0$$

and therefore the conditions in Proposition 4.25 required on the n for which the staircase operation is used are satisfied.

By Lemma 6.2, $\frac{M_n}{h_{n-1}} \to 0$ along n for which the γ-Chacon-like operation is used. By construction, c_n and M_n satisfy the remaining conditions required by Proposition 4.25 on the n for which the Chacon-like operations are used. The conditions required by Proposition 5.2 are also satisfied using the above and Lemma 6.10.

Let D and A be unions of levels in some fixed column C_{m_0} and fix $\epsilon > 0$. By Proposition 4.25, there exists $m_1 > m_0$ such that for all $m \geq m_1$ for which B_{m+1} is constructed using the staircase operation, A and D are both uniformly mixed within ϵ at stage m. Therefore there is n_0 such that for all $n \geq n_0$, A and D are both uniformly mixed within ϵ at stages Q_n and $Q_n + 1$.

By Proposition 5.2, since the γ-Chacon-like operation is used to construct $B_{S_{n+1}+1}$ through $B_{Q_{n+1}}$ and $S_{n+1} - 2 = Q_n$, then for all $(1-\frac{1}{2}\epsilon)\tilde{h}_{Q_n} \leq t < (1-\frac{1}{2}\epsilon)\tilde{h}_{Q_{n+1}}$,

$$\mu(T^t D \cap A) \geq \kappa_\gamma \mu D \mu A - 5\epsilon$$

As this holds for all $n \geq n_0$, then the above holds for all $t \geq \tilde{h}_{Q_{n_0}}$. Since $\epsilon > 0$ was arbitrary, the claim is proved. □

Proof of Theorem 1. Let $\epsilon > 0$. Choose $\gamma \geq 2$ such that $\frac{1}{\gamma+1} < \epsilon$. By Proposition 6.5, T_γ satisfies $\liminf {}^{p(q)}/_{q} \leq 2 + \frac{1}{\gamma+1} < 2 + \epsilon$. By Proposition 6.11, T_γ is at least κ_γ-partially mixing. □

7 Lower bounds on word complexity of partially mixing rank-one subshifts

Definition 7.1. A rank-one transformation is in **canonical form** if B_n is a suffix of B_{n+1} for all n.

Lemma 7.2. *Let T be a rank-one transformation. Then T is measurably isomorphic to a rank-one transformation in canonical form which generates the same language as T.*

Proof. Since B_{n+1} is a concatenation of B_n and 1, there exists a unique c_n such that B_{n+1} has $B_n 1^{c_n}$ as a suffix. Set $\tilde{B}_{n+1}$ to be the prefix of B_{n+1} of length $|B_{n+1}| - c_n$. Then $B_{n+1} = \tilde{B}_{n+1} 1^{c_n}$. Clearly B_n and $\tilde{B}_n$ generate the same language. The measure isomorphism is likewise immediate (and can be viewed as saying that we place zero spacers over the rightmost subcolumn but instead place that many extra above every subcolumn at the next stage). □

For the remainder of this section, all rank-one transformations are assumed to be in canonical form, without loss of generality.

Definition 7.3. A rank-one transformation has **bounded spacers** if there exists some $t \geq 0$ such that for every n and j, $s_{n,j} \leq t$ and has **unbounded spacers** otherwise.

Lemma 7.4. *Let T be a nontrivial rank-one subshift in canonical form that is lightly mixing. Then T has unbounded spacers.*

Proof. Suppose that T has bounded spacers. Since T is in canonical form, T^{h_n} pushes every point in a level $I_{n,j}$ through exactly one spacer string (for points in the rightmost subcolumn, the spacers from a future stage). Then $T^{h_n} I_{n,j} \subseteq \bigcup_{i=0}^{t} T^{-i} I_{n,j}$ for every height h_n.

Let B be a union of levels in some fixed column such that every level in B is at least $2t+2$ away from every other level in B. Then $T^{h_n} B \cap T^{t+1} B = \varnothing$ since every point in $T^{h_n} B$ is within t levels of a level in B so none of those points can be exactly $t+1$ away as the levels in B are all at least $2t+2$ apart.

This would contradict light mixing so T must have unbounded spacers. □

Proof of Theorem 2. By Lemma 7.2, we may assume T is in canonical form. By Lemma 7.4, T has unbounded spacers so 1^∞ is a left-infinite right-special word. Therefore if for infinitely many q we had $p(q+1) - p(q) = 1$ then the unique right-special word of length q would be 1^q. The path from 1^q to itself starting with 1 is labeled 1, call the other path u_q. Then $1 + |u_q| = p(q) + 1$ so $|u_q| = p(q)$. Now u_q must have 1^q as a suffix so, since the total measure of the spacers added is finite, it must be that ${}^{q}/_{|u_q|} \to 0$. But then ${}^{p(q)}/_{q} \to \infty$. So $p(q+1) - p(q) \geq 2$ for all sufficiently large q.

Suppose that $\liminf {}^{p(q)}/_{q} \leq 2$. Fix $\epsilon > 0$. By Theorem 2.2 [Bos85], there exist infinitely many q such that $p(q+1) - p(q) \leq 2$ and $p(q) < (2+\epsilon)q$. For such q, by the above, $p(q+1) - p(q) = 2$ so there are two right-special words of length q. Since 1^q must be one of them, call the other w_q. Since $w_q \neq 1^q$, there is some $c_q, d_q \geq 1$ such that $w_q = 1^{d_q} z 1^{c_q}$ for some word z which starts and ends with 0.

Since 1^{d_q} and 1^{c_q} correspond to spacers being added at some n^{th} stage so $|w_q| \sim h_n$, and since T is on a finite measure space, then ${}^{c_q}/_{q} \to 0$ and ${}^{d_q}/_{q} \to 0$.

Consider the Rauzy graph for such a q. The path from w_q to 1^q must have label 1^{q-d_q} because w_q is followed by arbitrarily long strings of 1s in the language. The path from 1^q to w_q must have label $z1^{c_q}$. The path from 1^q to itself has label 1. Let v_q be the label of the path from w_q to itself.

Since the total length of the paths is $p(q) + 2$, then

$$|v_q| + q - c_q + q - d_q + 1 = p(q) + 2 < (2+\epsilon)q + 2$$

meaning that $|v_q| < \epsilon q + c_q + d_q + 1$.

As this holds for all $\epsilon > 0$, then ${}^{|v_q|}/_q \to 0$ over such q (as ${}^{c_q}/_q \to 0$ and ${}^{d_q}/_q \to 0$). By Lemma 5.6 [Cre26], $|v_q| < |w_q|$ implies v_q is a root of w_q, meaning that $w_q = v_q' v_q^{t_q}$ for some $t_q \geq 1$ and v_q' a suffix of v_q. Since ${}^{|v_q|}/_q = {}^{|v_q|}/_{|w_q|} \to 0$, then $t_q \to \infty$.

Now v_q is always followed by v_q or by 1^{q-d_q} as those are the only paths in the Rauzy graph. Note that if $\mu([1^{q-d_q}]) \geq \delta$ infinitely often then, as T is rank-one, this would imply the measure of the spacers added at each level does not go to zero so the transformation would not be on a finite measure space. So v_q is followed by 1^{q-d_q} at most ϵ_q of the time for some $\epsilon_q \to 0$.

For a fixed q_0 and $q > q_0$, it then follows by the shift-invariance of the measure that for $x \in [v_{q_0}]$, we have $T^{|v_q|}x \in [v_{q_0}]$ for at least $(1 - {}^1/_{t_q} - \epsilon_q)\mu([v_{q_0}])$ since every occurrence of v_{q_0} must be in some copy of v_q and then $T^{|v_q|}$ will take that occurrence to the same occurrence in the following v_q whenever v_q is followed by v_q (which must be at least $(1 - {}^1/_{t_q}) - \epsilon_q$ of the time by shift-invariance and that $\mu([1^{q-d_q}]) \leq \epsilon_q$). Therefore

$$\mu(T^{|v_q|}[v_{q_0}] \cap [v_{q_0}]) \geq \left(1 - \frac{1}{t_q} - \epsilon_q\right) \mu([v_{q_0}])$$

Since $\epsilon_q \to 0$ and $t_q \to \infty$, then

$$\mu(T^{|v_q|}[v_{q_0}] \cap [v_{q_0}]) \to \mu([v_{q_0}])$$

contradicting light mixing. □

Remark 7.5. In fact, the above proof shows that a rank-one transformation with $\liminf {}^{p(q)}/_q = 2$ and unbounded spacers is rigid. This follows since $\mu(\bigcup_{i=0}^{|v_{q_0}|} T^i[v_{q_0}]) \to 1$ since the spacer levels added have measure tending to zero. However, we do not elaborate as we do not need this and there exist nonrigid rank-one transformations with $\limsup \frac{p(q)}{q} < 2$, e.g. the examples in Section 4 [Cre24] (which, of course, have bounded spacers)

Proof of Theorem 3. By Lemma 7.2, we may assume T is in canonical form. We may and will assume $\liminf \frac{p(q)}{q} < 3 - 2\delta$ for some $\delta > 0$ since otherwise the claim is immediate. Then, by Theorem 2.2 [Bos85], there exists infinitely many q for which $p(q+1) - p(q) \leq 2$ and $p(q) \leq (3-\delta)q$ and we may assume that the $\liminf$ is attained along this sequence of q's. Since T must have unbounded spacers by Lemma 7.4, $p(q+1) - p(q) \geq 2$ for all q. So the $\liminf$ is attained along a sequence of q such that $p(q+1) - p(q) = 2$ and $p(q) \leq (3-\delta)q$.

For such q, let v_q be as in the proof of Theorem 2. The paths in the Rauzy graph have total length

$$|v_q| + 1 + q - c_q + q - d_q = p(q) + 2 \leq (3-\delta)q + 2$$

meaning that

$$|v_q| \leq (1-\delta)q + c_q + d_q + 1$$

so again v_q is a root of w_q (as ${}^{c_q}/_q \to 0$ and ${}^{d_q}/_q \to 0$) so $w_q = v_q' v_q^{t_q}$.

Let $C_q = \bigcup_{i=0}^{|v_q|-1} T^i[v_q]$ (note that we are not assuming the translates are disjoint). Since $\mu([1^{q-d_q}]) \to 0$, it follows that $\mu(C_q) \to 1$. As in the proof of Theorem 2, we have that for a fixed q_0 and any $q > q_0$ such that $p(q+1) - p(q) = 2$ and $p(q) \leq (3-\delta)q$,

$$\mu\left(T^{|v_q|}[v_{q_0}] \cap [v_{q_0}]\right) \geq \left(1 - \frac{1}{t_q} - \epsilon_q\right) \mu([v_{q_0}])$$

and therefore

$$\mu\left(T^{|v_q|}[v_{q_0}] \cap (C_{q_0} \setminus [v_{q_0}])\right) < \left(\frac{1}{t_q} + \epsilon_q\right) \mu([v_{q_0}])$$

Now $\mu(C_q) \to 1$ and $\mu([v_q]) \to 0$ (since $|v_q| \to \infty$ as otherwise the subshift is periodic), so $\mu(C_q \setminus [v_q]) \to 1$. Fix $\epsilon > 0$ and choose q_0 in the set of possible q such that $\mu(C_{q_0} \setminus [v_{q_0}]) > 1 - \epsilon$. Then

$$\mu\left(T^{|v_q|}[v_{q_0}] \cap (C_{q_0} \setminus [v_{q_0}])\right) < \left(\frac{1}{t_q} + \epsilon_q\right) \mu([v_{q_0}]) \frac{1}{1-\epsilon} \mu\left(C_{q_0} \setminus [v_{q_0}]\right)$$

Since T is α-partially mixing,

$$\mu\left(T^{|v_q|}[v_{q_0}] \cap (C_{q_0} \setminus [v_{q_0}])\right) \geq \alpha\mu([v_{q_0}])\mu\left(C_{q_0} \setminus [v_{q_0}]\right)$$

meaning that, for q large enough that $(1-\epsilon)\alpha > \epsilon_q$,

$$\alpha < \left(\frac{1}{t_q} + \epsilon_q\right)\frac{1}{1-\epsilon}$$

and therefore

$$t_q < \frac{1}{(1-\epsilon)\alpha - \epsilon_q}$$

Since $p(q) \geq |v_q| + q - c_q + q - d_q + 1 - 2$,

$$\begin{aligned}\frac{p(q)}{q} &\geq \frac{|v_q|}{q} + 2 - \frac{d_q + c_q + 1}{q} = \frac{|v_q|}{|w_q|} + 2 - \frac{d_q + c_q + 1}{q} \\ &> \frac{1}{t_q + 1} + 2 - \frac{d_q + c_q + 1}{q} > \frac{1}{\frac{1}{(1-\epsilon)\alpha - \epsilon_q} + 1} + 2 - \frac{d_q + c_q + 1}{q}\end{aligned}$$

Taking $q \to \infty$ (so $\epsilon_q \to 0$ and $c_q/q \to 0$ and $d_q/q \to 0$) and $\epsilon \to 0$, then, along the sequence of q such that $p(q+1) - p(q) = 2$,

$$\liminf \frac{p(q)}{q} \geq 2 + \frac{\alpha}{\alpha + 1}$$

so the claim follows as the lim inf is attained along that sequence. □

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