Commensurated Subgroups and the Dynamics of Group Actions on Quasi-Invariant Measure Spaces

DARREN CREUTZ

Doctoral Dissertation

University of California: Los Angeles

June 2011

Abstract

Margulis' Normal Subgroup and Arithmeticity Theorems classify completely the lattices in higher-rank algebraic groups as being (up to finite index) the simple groups consisting of integer points of algebraic groups. We generalize the Normal Subgroup Theorem to commensurators of lattices in arbitrary locally compact groups and in particular obtain that the commensurator of a uniform tree lattice in a regular tree is simple (up to finite index) if and only if a corresponding completion is. We also make progress on the Margulis-Zimmer Conjecture: a generalization of the Arithmeticity Theorem to commensurated–geometrically normal–subgroups by establishing the property (T) "half" of the conjecture.

In keeping with the methods of Margulis' work and of rigidity theory in general, we prove new results in the area of the dynamics of group actions with quasi-invariant measures and in the area of unitary representation theory that in turn lead to our structural results on commensurators of lattices in arbitrary groups and commensurated subgroups of lattices in algebraic groups.

The dynamics of group actions with quasi-invariant measures–the study of group actions on measure algebras preserving the ideal of null sets, but not necessarily the measure itself–is the natural setting for dynamics of nonamenable groups (such as those we are concerned with). We develop some foundational results on strongly approximately transitive actions–the dynamical opposite of measure-preserving–and prove a "SAT Factor Theorem" along the lines of Margulis' original boundary-based factor theorem that plays a key role in the proof of our Normal Subgroup Theorem.

In the realm of unitary representations we prove an existence and uniqueness result for harmonic cocycles–namely, in any reduced cohomology class there exists a unique harmonic representative. Consequences of this include our progress on the Margulis-Zimmer Conjecture.

Contents

Introduction

Originally motivating mathematical analysis, understanding the behavior of (physical) systems modeled by differential equations has grown into the entire field of dynamics: the behavior of abstract systems over abstract time (and symmetry). As the methods evolved, several distinct, related ideas emerged: by focusing on the differential equation as determining an action of the real numbers, i.e. "time", one is led to the study of symmetries of manifolds and Lie group actions; by focusing on the space acted on, one is led (via the ergodic hypothesis and similar notions) to the study of metric and measure spaces endowed with group actions (this is generally what is meant by dynamics); and by focusing on numerical approximations to the equation, one is led to the study of discrete approximations of the real numbers and of Lie groups in general–that is, to lattices.

A major result, due to Margulis in the 1970s, concerning the structure of lattices in Lie groups is the Normal Subgroup Theorem: any normal subgroup of a lattice in a (higher-rank) Lie group is either finite or has finite index in the lattice–geometrically (up to finite index) lattices in Lie groups are simple. Building on this structural result, Margulis proved the Arithmeticity Theorem: any lattice in a higher-rank Lie group is arithmetic–the integer points (in some appropriate sense and up to finite index) of the Lie group treated as an algebraic group over the rationals. Thus is in some sense approximating a Lie group (perhaps arising from a differential equation) is the same as approximating the real numbers and one will not see the discrete approximation vanish (i.e. have a nontrivial kernel) unless the group being approximated vanishes.

The remarkable strategy of proof is to show that, on the one hand, a lattice modulo an infinite normal subgroup has a very strong dynamical property: any action of such a group on a compact metric space admits an invariant (preserved) probability measure (the group is amenable); and on the other hand, that any continuous affine action on a real Hilbert space by a lattice modulo an infinite normal subgroup has a nontrivial fixed point (the group has property (T)). Together these imply that the group is finite.

Perhaps unsurprisingly, this deep result on the structure of lattices follows from the study of the actions of such lattices (on metric spaces endowed with measures and on Hilbert spaces). The salient feature of the dynamics involved is that one must work to show the existence of a probability measure preserved by the action of the lattice modulo the normal subgroup–the difficulty is in a large part due to the fact that Lie groups naturally act on many spaces (such as symmetric spaces) without admitting invariant measures. The dynamics of group actions with quasi-invariant measures, a major theme in our work, focuses on group actions on probability spaces preserving the measure algebra (though not necessarily the measure)–the natural setting for measurable dynamics of nonamenable groups such as Lie group and their lattices.

The Arithmeticity Theorem is proved in a similar fashion: given any lattice in a Lie group there exists an arithmetic lattice which shares a common subgroup with the lattice that has finite index in both (the finiteness of the index follows using the same ideas of dynamics and representation theory). Far-reaching generalizations of these results followed; the general program of study and the related advances in dynamics, particularly Furstenberg's boundary theory, have applications in many areas.

Formalizing the notion of "equivalence up to finite index" is the essence of commensurability. Two subgroups are commensurate when they share a common finite index subgroup (the intersection of the two groups has finite index in each group)–geometrically speaking, commensurate subgroups are the same. The commensurator is the geometric analogue of the normalizer: conjugation by elements of the commensurator of a group preserve the group up to finite index. Likewise, a commensurated subgroup is geometrically normalized: conjugation preserves the subgroup up to finite index.

The work presented here is a generalization of the results of Margulis to commensurators of lattices and commensurated subgroups of lattices, a notable deviation from other generalizations which have focused on relaxing restrictions on the ambient group but retaining that the object under consideration be a lattice. Our first major result is the Normal Subgroup Theorem for Commensurators of Lattices: for a lattice in any locally compact group (not necessarily Lie), the normal subgroups of the commensurator are (up to finite index) in one-one correspondence with the normal subgroups of a natural completion of the commensurator (which is in general much easier to study).

In another direction, Margulis' theorems (and his generalization of them to S-arithmetic lattices in arbitrary algebraic groups over local fields) give a more or less complete classification of the lattices in higher-rank algebraic groups and of their normal subgroups. However, the geometric approach taken in the proofs and in the resulting conclusions led to Margulis and Zimmer formulating a conjecture generalizing the result from normal subgroups to commensurated (geometrically normal) subgroups. The striking difficulty with this conjecture is that there are easy examples of commensurated subgroups of lattices (in contrast with the nonexistence of normal subgroups) and there is also the less apparent but much more problematic difficulty that one cannot take the quotient of a group by a commensurated subgroup so it is not immediately clear what object to even study.

Nonetheless, Margulis and Zimmer conjectured that irreducible lattices in higher-rank algebraic groups admit a standard description of commensurated subgroups–they are all (in some appropriate sense) S-arithmetic. We make progress on this conjecture by showing the property (T) "half" of the result. Shalom and Willis have recently identified the correct object of study and we show that this group indeed has property (T) as expected, leaving the remaining "half" of showing that it is also amenable as the only missing ingredient in the proof of the conjecture.

While not immediately apparent from the discussion so far, a significant portion of our work is in the realm of measurable dynamics: actions of locally compact groups on probability measure algebras. Correspondingly, we develop many new results about quasi-invariant actions of groups (those preserving the null sets but not the measure itself) and in particular focus heavily on the dynamical notion of strong approximate transitivity–the natural opposite of measure-preserving. The rigidity result we prove for such actions, the SAT Factor Theorem, is the key new fact that proves the Normal Subgroup Theorem for Commensurators of Lattices.

In the realm of representation theory, we prove a very general result about reduced cohomology and harmonic cocycles, namely that in any reduced cohomology class there exists a unique harmonic representative. Our progress on the Margulis-Zimmer Conjecture is a direct consequence of this fact.

Organization

The dissertation is organized in five sections and is not necessarily intended to be read linearly, though it certainly can be. The reader wishing to "get to the point" should skip to section III. Normal Subgroups of Commensurators and to section IV. Commensurated Subgroups of Lattices where the major new results are stated and proved, and return to the other parts as needed when reading the proofs.

I. Dynamics

The first section presents a somewhat comprehensive background on the dynamics of quasi-invariant group actions on measure spaces, in part since it is necessary for our work and in part to collect the basic ideas and results in one place. The reader familiar with and interested in dynamics may wish to skip immediately to Chapter 3: SAT Actions and refer back to the earlier chapters when needed.

The reader unfamiliar with aspects of group theory should refer to the appendices when needed. In particular, we make some use of placing measures on groups and heavy use of the amenability property of groups; the reader not accustomed to these ideas should perhaps begin with the appropriate parts of the appendix.

II. Representation Theory

The second section focuses on linear representations of groups as unitary operators on Hilbert spaces and on reduced cohomology in particular. The reader familiar with reduced cohomology theory should skip immediately to Chapter 6: Harmonic Cocycles, in particular Theorem 6.2.

The consequences of this Theorem then occupy the rest of the section–Chapter 7: Injectivity of Reduced Cohomology–and our work on Harmonic Cocycles is the main ingredient in our progress on the Margulis-Zimmer Conjecture (Chapter 13: The Property (T) "Half"). The reader unfamiliar with Property (T) should first consult the appendix on that subject before attempting to read our work on representation theory.

III. Normal Subgroups of Commensurators

The third section presents our Normal Subgroup Theorem for Commensurators of Lattices in locally compact groups (Chapter 9: Normal Subgroups of Commensurators of Lattices). In particular, we determine when the commensurator of a tree lattice is just infinite (Chapter 11: Commensurators of Tree Lattices) in terms of a relative profinite completion. The reader primarily interested in rigidity theory should start with this section and refer back to the previous sections as needed.

IV. Commensurated Subgroups of Lattices

Our final section presents our progress on the Margulis-Zimmer Conjecture (Chapter 13: The Property (T) "Half"), a direct consequence of our result on harmonic cocycles and the resulting injectivity of reduced cohomology for totally disconnected groups (Chapter 7: Injectivity of Reduced Cohomology). The reader primarily interested in algebraic groups and their lattices should begin with this section and refer back to the section on linear representations as needed.

Appendices on Group Theory

We conclude the dissertation with a collection of appendices on group theory: Appendix A: Group Theory presents basic background on groups, topological groups and the idea of placing measures on groups; Appendix B: Amenability and Property (T) presents in some detail the two most important properties of infinite groups–amenability and property (T); and Appendix C: Algebraic Groups appendix focuses on algebraic groups and arithmetic lattices.

Specific Results

The major new results in this dissertation are:

- The Normal Subgroup Theorem for Commensurators of Lattices (Corollary 9.11)
- The Property (T) Half of the Margulis-Zimmer Conjecture (Theorem 13.1)

Results of independent interest playing a key role in the proofs of the main theorems are:

- The SAT Factor Theorem (Theorem 3.9)
- The Harmonic Cocycle Theorem (Theorem 6.2)
- The Injectivity of Cohomology Theorems (Theorems 7.1 and 7.2)
- The Existence of Continuous Compact Models for G-Maps Theorem (Theorem 1.35)

Consequences of our Normal Subgroup Theorem of specific interest are:

- The Normal Subgroup Theorem for Irreducible Lattices in Products (Corollary 10.5)
- A Partial Characterization of Normal Subgroups of Commensurators of Tree Lattices (Theorem 11.4)

Our new contributions can be found in:

- I. Dynamics, Chapter 1: Quasi-Invariant Group Actions, Section 1.6
- I. Dynamics, Chapter 3: SAT Actions, Sections 3.4 – 3.6
- II. Representation Theory, Chapter 6: Harmonic Cocycles
- II. Representation Theory, Chapter 7: Injectivity of Reduced Cohomology
- III. Normal Subgroups of Commensurators (Chapter 9 – Chapter 11)
- IV. Commensurated Subgroups of Lattices, Chapter 13: The Property (T) "Half"

Attributions

The new results in this dissertation are joint work of the author and his advisor Yehuda Shalom. These results will be published in journal format shortly after the completion of this dissertation.

Material on dynamics and representations (I. Dynamics and II. Representation Theory) is both known results, usually attributed in the text, and new results to this work. The material in III. Normal Subgroups of Commensurators (excluding Chapter 8) and IV. Commensurated Subgroups of Lattices is new to this dissertation and comprises our main theorems. Background material in the Appendices is attributed in the text or is well-known and classical.

Acknowledgments

The author would like to thank his advisor, Yehuda Shalom, for providing direction; for always being there to advise when needed yet also willing to leave time for the author to work out ideas alone; for invaluable support and insight into the problems investigated; and for his encouragement throughout the doctoral program. One would not, and indeed could not, have asked for more in an advisor.

The author also expresses his thanks to the Mathematics Department of the University of California: Los Angeles, both for an environment conducive to learning mathematics and for the financial support, joint with the National Science Foundation, of the VIGRE Fellowship. The author also thanks his committee, without whom the doctoral degree would certainly not have been completed.

The complete list of individuals deserving thanks is perhaps too long to include but the author certainly wishes to thank the following people who contributed to his mathematical education and the dissertation (in roughly chronological order): Cesar Silva, Tom Liggett, Sorin Popa, Ed Effros, Yiannis Moschovakis, Itay Neeman, and Greg Hjorth. Also deserving thanks are the research group into functional analysis and dynamics and research group into logic, both professors and graduate students, at UCLA.

This dissertation is partly in memory of Greg Hjorth.

I

Dynamics

Chapter 1

Quasi-Invariant Group Actions

Dynamics refers to the study of groups acting on spaces with analytic structure. This ranges from the notion of a differential equation describing the behavior of a physical system over time to the more abstract setting of groups (for example symmetries) acting on metric spaces. Dynamics is characterized by relating the asymptotic behavior of the system to the structure of the space, the structure of the acting group and the nature of the action.

The reader unfamiliar with the aspects of group theory we make use of in this and the following chapters should consult Appendix A: Group Theory for definitions of group theoretic notions, in particular ideas about topological groups and measurable groups.

The material in this chapter is all well-known and either classical or due to Furstenberg, Glasner, Mackey, Weiss or Zimmer (among many many others) and is usually indicated in the text, with one exception: Theorem 1.35 is new to this work and plays a key role in our proofs as well as being of independent interest.

1.1 Metric Spaces

Modern analysis is characterized by the study of metric and measure spaces. We recall the basic definitions of metric spaces and group actions on them and the motivations and methods for placing a measure on a metric space. Most of the material presented here will be used implicitly in the sequel, particularly the relationship between the action of a group on a metric space and the corresponding actions on continuous functions and probability measures on that space.

Definition 1.1. A **metric space** is a set of **points** X and a **metric** $d : X \times X \to [0, \infty]$ that is symmetric ($d(x, y) = d(y, x)$), proper ($d(x, y) = 0$ if and only if $x = y$), complete ($d(x_n, x_m) \to 0$ implies the existence of x such that $d(x_n, x) \to 0$) and satisfies the triangle inequality ($d(x, y) \leq d(x, z) + d(z, y)$).

Metric spaces have the natural topology that $x_n \to x$ when $d(x_n, x) \to 0$. The Borel sets (see below) of this topology are denoted by $\mathcal{B}(X)$. Often we will omit d when the context makes clear which metric is being used.

Definition 1.2. A group G **acts on** a metric space X, written $G \curvearrowright X$, when there is a map: $G \times X \to X$ written gx such that $g(hx) = (gh)x$.

1.1.1 Continuous Actions

The natural setting for studying group actions on metric spaces is to have a group G acting on a metric space (X, d) continuously:

Definition 1.3. Let G be a group acting on a metric space X. The action is **continuous** when if $x_n \to x$ in X and $g_n \to g$ in G then $g_n x_n \to gx$, that is, the group action map $G \times X \to X$ is jointly continuous.

It is a classical fact that for a locally compact group G acting on a metric space X that if $G \times X \to X$ is separately continuous in each of G and X then it is in fact jointly continuous and hence the action is continuous.

In general there is no invariant metric for $G \curvearrowright X$ since the usual technique to obtain invariant metrics requires the map $G \times X \to X \times X$ by $(g, x) \mapsto (gx, x)$ be proper (meaning the preimage of compact sets is compact) which can only occur when G itself is compact.

1.1.2 Borel Sets

The most important aspect of metric topology for us will be the algebra of Borel sets.

Definition 1.4. Let X be a metric space. The **Borel sets** of X is the smallest σ-algebra of sets in X that contains the open sets. That is, $\mathcal{B}(X)$, the Borel sets, is the smallest collection of sets such that

- $B \in \mathcal{B}(X) \implies X \setminus B \in \mathcal{B}(X)$ (closed under complements);
- $B_1, B_2, \cdots \in \mathcal{B}(X) \implies \bigcup_n B_n \in \mathcal{B}(X)$ (closed under countable unions); and
- $U_{x_0,\epsilon} = \{x \in X : d(x, x_0) < \epsilon\} \in \mathcal{B}(X)$ (contains the open sets)

The most general natural setting for studies of Borel sets and group actions is that of Polish groups acting on Polish spaces and the reader is referred to Becker and Kechris [BK96] and to Kechris [Kec00] (among other sources) for a detailed exposition. We will not go into details here as our interest is primarily in placing a measure on the space but we will make use of facts about Borel sets in metric spaces at times.

1.1.3 Borel Actions

Often continuity of a group action is too much to require and we relax the condition to be the map being merely Borel. Of course, requiring any less than a Borel action in effect says that the group action does not respect the topology (hence the metric) at all and therefore the action is not "really" that of a group on a metric space.

Definition 1.5. A group action on a metric space $G \curvearrowright X$ is a **Borel action** when the map $G \times X \to X$ for the action is a Borel map (the preimage of Borel sets is Borel).

This definition, like that of continuity of an action, involves the topology of the group. When G is discrete and countable (has no topology in effect) the action is Borel precisely when each group element represents a Borel map $X \to X$. As with continuous actions, there is generally no invariant metric on X for the group action.

1.1.4 Continuous Functions

The space of continuous functions on a metric space plays a key role in dynamics.

Definition 1.6. Let X be a metric space. A function $f : X \to \mathbb{R}$ is **continuous** when for every $x \in X$ and $\epsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $|f(x) - f(y)| < \epsilon$.

Definition 1.7. Let X be a metric space. The **space of continuous functions** on X will be written $C(X)$.

Assume now that X is compact. The space of continuous functions is endowed the supremum metric topology, that is: for $f \in C(X)$ define

$$\|f\| = \sup_{x\in X} |f(x)|$$

Then define $D(f_1, f_2) = \|f_1 - f_2\|$. This is a metric on $C(X)$ called the **supremum metric**. Moreover $C(X)$ is separable (with this metric) when X is compact. Note that this means that $f_n \to f$ precisely when $\|f_n - f\| \to 0$.

Note that $C(X)$ separates points in the sense that if $x \neq y \in X$ then there is some $f \in C(X)$ such that $f(x) \neq f(y)$. This is an easy consequence of the fact that there are disjoint open sets containing x and y (take balls of diameter less than half the distance between the two points x and y).

The reader is referred to any general analysis and topology book for more information on continuous functions. We remark only that the ϵ-δ definition given is equivalent, in our case, to the usual open set definition: a function is continuous if and only if the preimage of any open set is open.

1.1.5 The Action on Functions

Given a group action on a metric space $G \curvearrowright X$ and $f : X \to Y$ a function on the metric space (to some other space), one can "compose" the function with the group action by setting

$$g \cdot f(x) = f(g^{-1}x)$$

for $x \in X$ and $g \in G$. Now

$$g \cdot (h \cdot f)(x) = h \cdot f(g^{-1}x) = f(h^{-1}g^{-1}x) = (gh) \cdot f(x)$$

so this in fact defines an action of the group on the space of functions.

Proposition 1.1.1. *If $G \curvearrowright X$ continuously and X is a compact metric space then $G \curvearrowright C(X)$ continuously (with the supremum metric).*

Proof. Observe that if $g \in G$ and $f \in C(X)$ then the action $g \cdot f(x) = f(g^{-1}x)$ is an action since $h \cdot g \cdot f(x) = g \cdot f(h^{-1}x) = f(g^{-1}h^{-1}x) = f((hg)^{-1}x) = hg \cdot f(x)$ and that clearly $g \cdot f \in C(X)$ when $f \in C(X)$ by the continuity of the G-action on X. Then if $g_n \to g$ in

G and $f \in C(X)$ is fixed then, by compactness, $\|g_n \cdot f - g \cdot f\| = \sup_x |f(g_n^{-1}x) - f(g^{-1}x)|$ is attained by some $x_n \in X$. Suppose that $\|g_n \cdot f - g \cdot f\| \geq \delta > 0$ infinitely often. Take a further subsequence of that subsequence, $\{n_j\}$, such that $x_{n_j} \to x_\infty$ for some $x_\infty \in X$ (again possible by compactness). Then

$$|f(g_{n_j}^{-1}x_{n_j}) - f(g^{-1}x_{n_j})| \geq \delta$$

but by the (joint) continuity of the G-action, $g_{n_j}^{-1}x_{n_j} \to g^{-1}x_\infty$ and $g^{-1}x_{n_j} \to g^{-1}x_\infty$ hence

$$|f(g_{n_j}^{-1}x_{n_j}) - f(g^{-1}x_{n_j})| \to |f(g^{-1}x_\infty) - f(g^{-1}x_\infty)| = 0$$

contradicting that $\delta > 0$. □

1.2 From Metric to Measure

A key idea in the early development of dynamics was the ergodic hypothesis: if one samples a system repeatedly and averages the results this should reflect the average behavior of the system as a whole. Concretely, one would like to say that if $T : X \to X$ describes the evolution of a system over time then for any measurement on the system $f : X \to \mathbb{R}$ and any given point $x_0 \in X$ (the initial configuration of the system) the average of the values $f(T^n(x_0))$ should converge to the "average value" of the measurement. Specifically we would like to say that

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n(x_0)) \to A(f)$$

where A represents the average value of f on the system. To make sense of this, the introduction of a measure on X is necessary.

1.2.1 Probability Measures

Definition 1.8. A (Borel) **probability measure** on a metric space X is a set function $\nu : \mathcal{B}(X) \to [0,1]$ satisfying:

- $\nu(X) = 1$;
- $\nu(X \setminus B) = 1 - \nu(B)$ for all $B \in \mathcal{B}$; and
- $\nu(\bigcup_j B_j) = \sum_j \nu(B_j)$ for all countable collections of disjoint $B_j \in \mathcal{B}$

The notion of integration is defined as usual: first for characteristic functions, then linear combinations of them and then for general Borel functions by approximation. Integration will be written $\int \cdot \, d\nu$. We will also use the shorthand

$$\nu(f) = \int_X f(x)\, d\nu(x)$$

when thinking of a probability measure as a functional on functions.

The **ergodic theorem** asserts the desired result: if there are no nontrivial invariant sets then for ν-almost every $x \in X$

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n(x)) \to \int f \, d\nu$$

where ν is any Borel probability measure such that $\nu(T^{-1}(B)) = \nu(B)$ for all Borel sets B (that is, ν is T-invariant).

1.2.2 The Space of Probability Measures

Since we will be concerned primarily with group actions that do not preserve a probability measure we will have the need to study the space of (Borel) probability measures as a topological space.

Definition 1.9. The space of **Borel probability measures**(Borel) probability measures on a metric space X will be written $P(X)$.

There are a few natural topologies on the Borel probability measures. It will be more convenient to define these topologies in terms of convergent sequences rather than open sets (though the reader should be able to easily translate to the open set versions):

Definition 1.10. Let X be a metric space. The **strong topology** on $P(X)$ is defined by $\nu_n \to \nu$ when $\nu_n(B) \to \nu(B)$ for every Borel set B.

Definition 1.11. Let X be a metric space. The **weak-* topology** on $P(X)$ is defined by $\nu_n \to \nu$ when $\nu_n(f) \to \nu(f)$ for every $f \in C(X)$.

We will exclusively use the weak-* topology on $P(X)$ in what follows. This is the more natural topology in our setting since it corresponds to treating $P(X)$ as the (continuous) dual of $C(X)$ which is itself the (continuous) dual of X.

Proposition 1.2.1. *Let X be a metric space. Then $P(X)$ is itself a metric space under the weak-* topology and will be compact when X is.*

The compactness of $P(X)$ is a consequence of the Banach-Alaoglu Theorem since $P(X)$ is the set of positive norm one elements of $C(X)^*$ and $C(X)$ is a separable Banach space when X is compact.

1.2.3 The Support of a Measure

The support of a measure is the smallest closed set on which the measure "lives" in the sense that any smaller closed set has measure strictly less than one.

Definition 1.12. Let X be a metric space and $\nu \in P(X)$ be a probability measure on X (or more generally any measure on X). The **support** of ν is the minimal closed set C such that $\nu(C) = 1$. The support is written as supp ν.

The support is well-defined since the collection $\{C \subseteq X \text{ closed} : \nu(C) = 1\}$ is nonempty ($\nu(X) = 1$ and X is closed) and since if $\nu(C_n) = 1$ for $n \in \mathbb{N}$ then $\nu(\cap C_n) = 1$ also and so Zorn's Lemma implies there is a unique minimal element in that collection which will necessarily be the support of ν by definition.

1.2.4 The Action on Functions and Measures

Let $G \curvearrowright X$ be a group acting continuously on a metric space. For $f \in C(X)$ or $f \in L^\infty(X, \nu)$ (for some measure ν) define gf by $(gf)(x) = f(g^{-1}x)$. This defined an action of G on functions which is continuous when the G action on X is.

Likewise, for $\nu \in P(X)$ define $g\nu$ by

$$\int f \; dg\nu = \int g^{-1} f \; d\nu = \int f(gx) \; d\nu(x)$$

and this defines an action $G \curvearrowright P(X)$ which will be continuous when $G \curvearrowright X$ is continuous.

1.3 Quasi-Invariant Actions

The focus of our study will be on quasi-invariant actions–groups acting on measure spaces preserving the space of null sets. Quasi-invariant actions are the most natural choice for dynamics when considering nonamenable groups in the sense that any group action on a metric space admits a quasi-invariant measure (as we shall see). The reader unfamiliar with the concept of amenability should consult Appendix B: Amenability and Property (T) for the definition and basic facts about amenable groups.

1.3.1 Invariant Measures

The usual context for dynamics on probability spaces is to assume the measure ν is invariant under the group action, that is $\nu(gB) = \nu(B)$ for all $g \in G$ and $B \in \mathcal{B}$. When the acting group is $\mathbb{Z}$ or $\mathbb{R}$, as is the case in classical dynamics, this presents no issues as such measures always exist. More generally, when the acting group is amenable then such measures exist:

Theorem 1.13. *Let X be a compact metric space and $T : X \to X$ be a Borel map. Then there exists a T-invariant Borel probability measure on X.*

Proof. Let $\nu_0 \in P(X)$ be any Borel probability measure on X. Consider the sequence of probability measures

$$\nu_N = \frac{1}{N} \sum_{n=0}^{N-1} \nu_0 \circ T^n$$

Observe that

$$\nu_N \circ T - \nu_N = \frac{1}{N}(\nu_0 \circ T^N - \nu_0)$$

Since X is compact, so is $P(X)$ so this sequence has a weak-* limit point, call it ν. Then

$$\nu \circ T - \nu = \lim_j \nu_{N_j} \circ T - \nu_{N_j} = \lim_j \frac{1}{N_j}(\nu_0 \circ T^{N_j} - \nu_0) = 0$$

so ν is invariant. □

Note that implied in how we have written the above is that for all $f \in C(X)$

$$\nu_N(f) = \frac{1}{N}\sum_{n=0}^{N-1} \nu_0 \circ T^n(f)$$

and likewise wherever we have written measures. We will often write in this fashion in what follows.

Theorem 1.14. *Let G be an amenable countable discrete group and let X be a compact metric space such that $G \curvearrowright X$ continuously. There exists a G-invariant Borel probability measure on X.*

Proof. Let $\nu_0 \in P(X)$ be any Borel probability measure on X. Let F_N be a sequence of Følner sets for G. Consider the sequence of probability measures

$$\nu_N = \frac{1}{|F_N|}\sum_{g \in F_N} g\nu_0$$

Observe that for any $g \in G$

$$g\nu_N - \nu_N = \frac{1}{|F_N|}\sum_{h \in gF_N} h\nu_0 - \frac{1}{|F_N|}\sum_{h \in F_N} h\nu_0$$

hence

$$\big|g\nu_N - \nu_N\big| \leq \frac{1}{|F_N|}\sum_{h \in gF_N \triangle F_N} |h\nu_0| \leq \frac{|gF_N \triangle F_N|}{|F_N|} \to 0$$

Since X is compact, so is $P(X)$ so this sequence has a weak-* limit point, call it ν. Then

$$g\nu - \nu = \lim_j g\nu_{N_j} - \nu_{N_j} = \lim_j g\nu_{N_j} - \nu_{N_j} = 0$$

so ν is invariant. □

Of course when G is locally compact second countable the same result holds.

1.3.2 Invariant Measures Need Not Exist

Let $\mathbb{F}_2$ be the free group on two generators and X be the space of (finite and infinite) words in a, b, a^{-1}, b^{-1} with cancellation. Clearly $\mathbb{F}_2 \curvearrowright X$ from the left by "concatenation with cancellation".

Suppose $\nu \in P(X)$ is an invariant probability measure on X. Let B_w be the Borel set of all words that begin with the finite word w. So

$$\nu(B_a) + \nu(B_b) + \nu(B_{a^{-1}}) + \nu(B_{b^{-1}}) + \nu(e) = 1$$

Now

$$B_a = a(B_e \setminus B_{a^{-1}})$$

so

$$\nu(B_a) = \nu(aB_e) - \nu(aB_{a^{-1}}) = 1 - \nu(B_{a^{-1}})$$

using the measure-preserving property. Then

$$\nu(B_a) + \nu(B_{a^{-1}}) = 1$$

and likewise

$$\nu(B_b) + \nu(B_{b^{-1}}) = 1$$

but

$$\nu(B_a) + \nu(B_b) + \nu(B_{a^{-1}}) + \nu(B_{b^{-1}}) + \nu(e) = 1$$

so we have a contradiction. There simply aren't any Borel probability measures on the space of words that are invariant.

1.3.3 Existence of Quasi-Invariant Actions

There is one way to fix the problem that no invariant measures exist: Let μ be a probability measure on the group with $\langle \text{supp } \mu \rangle = G$ (the notation $\langle \cdot \rangle$ means the subgroup generated by the set of elements under the group operations). For $\nu_0 \in P(X)$ define

$$\mu * \nu_0 = \int_G g\nu_0 \; d\mu(g)$$

the convolution. Then

$$\frac{1}{N} \sum_{n=0}^{N-1} \overbrace{\mu * \mu * \cdots * \mu}^{n \text{ times}} * \nu_0$$

has a weak-* limit point ν and necessarily

$$\mu * \nu = \nu$$

We say this measure is **stationary** under μ.

Note that for $g \in G$ and $B \in \mathcal{B}$

$$\nu(gB) = \int \mathbb{1}_{gB}(x) \; d\nu(x) = \int \mathbb{1}_B(g^{-1}x) \; d\nu(x) = g^{-1}\nu(B)$$

Now for B a measurable set with $\nu(B) > 0$ and $g \in \operatorname{supp} \mu$

$$g^{-1}\nu = g^{-1}\mu * \nu \geq g^{-1}\mu(g)g\nu = \mu(g)\nu$$

meaning that

$$\nu(gB) = g^{-1}\nu(B) \geq \mu(g)\nu(B) > 0$$

Of course for arbitrary g there is some convolution power of μ with $\mu^{(n)}(g) > 0$ so this is true in general.

Definition 1.15. Let $G \curvearrowright X$ a (compact) metric space. A probability measure $\nu \in P(X)$ is **quasi-invariant** when for all $g \in G$ and measurable B

$$\nu(gB) = 0 \qquad \text{if and only if} \qquad \nu(B) = 0$$

Our construction above of stationary measures using the weak-* compactness has shown that for any $\mu \in P(G)$ there exist μ-stationary measures. We can conclude that:

Theorem 1.16. *Let G be a locally compact group acting on a compact metric space X. Then there exists a quasi-invariant $\nu \in P(X)$.*

1.4 G-Spaces

The basic object of our study in dynamics will be:

Definition 1.17. A G**-space** is a Borel probability space (X, ν) such that $G \curvearrowright X$ in a Borel manner (the map $G \times X \to X$ is jointly Borel) and ν is quasi-invariant.

Having placed the measure ν on the standard Borel space X, we generally are only concerned with the measurable behavior of the group action. That is, we are interested in phenomena that can be seen in the algebra of measurable sets modulo null sets.

In particular, as we will explore in more detail below, if two compact metric spaces equipped with group actions and quasi-invariant measures are **measurably isomorphic** (meaning there is an equivariant map defined almost everywhere that preserves the measures of sets) then we will treat them as the same. In this case, the underlying metric spaces are both referred to as *compact models* for the measurable action.

We will see below that any measurable action of a locally compact second countable (in fact Polish) group admits a Borel compact model (that is, a compact metric space with a group action such that the action is Borel and the Borel sets form an isomorphic measure algebra to the given algebra).

We record here a classical result of Mackey indicating this is true:

Theorem 1.18 (Mackey). *Let (X, ν) be a standard Borel probability space. Write $\mathcal{B}(X)$ to denote the Borel sets. Assume that there is a continuous action of a (locally compact second countable) group G on $\mathcal{B}(X)$ preserving the boolean operations: union, complement and intersection.*

Then there exists a standard Borel probability space (Y, η) where G acts on Y in a Borel fashion and η is a quasi-invariant probability measure, that is (Y, η) is a G-space, and there exists a Borel measure-class-preserving surjective map $\phi : X_0 \to Y_0$ defined on conull Borel sets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ such that $\phi^* : \mathcal{B}(Y) \to \mathcal{B}(X)$, given by $\phi^*(B) = \phi^{-1}(B)$, is a homeomorphic G-map.

1.4.1 Ergodic G-Spaces

It is possible to decompose any G-space into "measurably minimal" components. A space is ergodic when there are no nontrivial invariant sets (in the measurable sense):

Definition 1.19. Let (X, ν) be a G-space. Then (X, ν) is **ergodic** or **G-ergodic** when for any measurable set B

$$gB = B \quad \forall g \in G \qquad \Longrightarrow \qquad \nu(B) \in \{0, 1\}$$

That is to say, if for some measurable set B we know that $\nu(B \triangle gB) = 0$ for all $g \in G$ then B is either null or conull (measurably trivial).

1.4.2 G-Maps

In the category of G-spaces, the morphisms are measurable maps between measure spaces that preserve (intertwine) the group action.

Definition 1.20. Let X and Y be Borel spaces and $\pi : X \to Y$ be a Borel map. The **push-forward** map $\pi_* : P(X) \to P(Y)$ is given by, for a Borel set $B \subseteq Y$ and $\nu \in P(X)$,

$$\pi_*\nu(B) = \nu(\pi^{-1}(B))$$

Definition 1.21. Let (X, ν) and (Y, η) be two Borel probability spaces. A measurable map $\pi : X \to Y$ such that $\pi_*\nu = \eta$ is called a **homomorphism** of probability spaces.

Definition 1.22. Let (X, ν) and (Y, η) be two G-spaces. Let $\pi : (X, \nu) \to (Y, \eta)$ be a homomorphism. If $\pi(gx) = g\pi(x)$ almost surely then π is **G-equivariant** and we say π is a **G-map**.

A basic fact, a proof of which can be found in the appendix of Zimmer [Zim84], is:

Proposition 1.4.1. *Let X and Y be Borel spaces and G a group acting in a Borel fashion on both. Let $\nu \in P(X)$ and $\eta \in P(Y)$ be Borel probability measures (the measurable sets are the Borel sets). Then for any measurable map $\pi : X \to Y$ such that $\pi_*\nu = \eta$ and $\pi(gx) = g\pi(x)$ for almost every x (for each g) there exists a Borel map $\pi_0 : X_0 \to Y_0$ defined on a ν-measure one set to an η-measure one set such that $(\pi_0)_*\nu = \eta$ and that $\pi_0(gx) = g\pi_0(x)$ for all $x \in X_0$ and $g \in G$ and such that $\pi = \pi_0$ almost everywhere.*

This fact allows us to talk unambiguously about G-maps in the context of measure algebras and also in the context of models for those algebras. Details on compact models are discussed later in the chapter.

When the need arises, a group acting on a measure algebra (that is, an algebra with boolean operations, usually written as union and complement) will be referred to as a **boolean system**. The reader is referred to Zimmer [Zim84], Appendix A, and to [FG10] section 7 for more information about the relationship between boolean systems and G-spaces.

1.4.3 (G, μ)-Spaces

We have already seen that given $G \curvearrowright X$ there always exist measures making X into a G-space. In fact we saw that for any $\mu \in P(G)$ there exist:

Definition 1.23. A (G, μ)**-space** is a G-space such that $\mu * \nu = \nu$.

While our focus will be on G-spaces, we remark that (G, μ)-spaces are in fact the more well-studied category. In fact, one of our objectives is to move away from the need to introduce a measure on the group in order to study dynamics. We remark that:

Theorem 1.24 (Bader-Shalom)**.** *Let G be a group and $\mu \in P(G)$ a probability measure on G. Let X be a compact metric space where G acts. The set of ergodic μ-stationary probability measures are extremal in the convex, compact set of all μ-stationary measures. In particular, if $\nu_1, \nu_2 \in P(X)$ are both μ-stationary and in the same ergodic measure class then $\nu_1 = \nu_2$.*

1.4.4 The Barycenter Map

Let (X, ν) be a G-space that is a convex subset of a (real or complex) complete metric vector space, meaning that if $x, y \in X$ then for any $t \in [0, 1]$ we have that $tx + (1 - t)y \in X$. The primary example of such a space will be $P(X)$ when $G \curvearrowright X$. Such a space is called a **convex G-space**. The barycenter is the center of a measure on such a space:

Definition 1.25. Let X be a convex space and $\nu \in P(X)$. Then the **barycenter** of ν is defined as

$$\mathrm{bar}(\nu) = \int_X x \; d\nu(x)$$

In particular, if $\nu \in P(P(X))$ for some compact space X then $\mathrm{bar}(\nu) \in P(X)$ and

$$\mathrm{bar}(\nu)(f) = \int_{P(X)} \eta(f) \; d\nu(\eta)$$

defines the barycenter measure. The convexity is required for the definition to make sense since the barycenter is defined to be a convex combination of elements in X. An obvious but useful observation is that the barycenter map is equivariant with respect to group actions:

Proposition 1.4.2. *Let X be a convex G-space. Then the barycenter map is G-equivariant.*

Proof. Let $\nu \in P(X)$ and $g \in G$. Since $G \curvearrowright X$ we can define an action of G on $P(X)$ by defining $g\nu \in P(X)$ to be

$$g\nu(f) = \int_X f(gx)\ d\nu(x)$$

for all $f \in C(X)$ (the usual action on probability measures). Then

$$g\ \mathrm{bar}(\nu) = g \int_X x\ d\nu(x) = \int_X gx\ d\nu(x) = \int_X x\ dg\nu(x) = \mathrm{bar}(g\nu)$$

□

1.5 Compact Models

In the preceding discussion we have been glossing over the details of the distinction between a measure algebra and the Borel sets of the underlying metric space. Here we spell out concretely how these interact and why this is justifiable. Most of the results presented here will be used implicitly in what follows. The reader wishing to know more about the interplay between topological models and measurable actions of groups should consult Glasner [Gla03] and the references provided there.

1.5.1 Compact Models of Group Actions

The formal definition of a compact model is as follows:

Definition 1.26. Let (Σ, ν) be a probability measure algebra and G a group acting on it quasi-invariantly (G acts on the algebra of measurable sets modulo null sets). A compact metric space X with a Borel probability measure $\nu \in P(X)$ is called a **compact model** of the G action on (Σ, ν) when the Borel sets of X are isomorphic to Σ in such a way that the ν agree over the isomorphism and the isomorphism is a G-map (is G-equivariant).

Recall that the **support** of a measure ν on a compact metric space X is defined to be the smallest closed set C such that $\nu(C) = 1$. Since C is closed it is also compact and is a metric space in its own right. Evidently the space (C, ν) is also a compact model for the same action that (X, ν) was. Therefore:

Proposition 1.5.1. *We may always assume that ν is fully supported on any compact model.*

1.5.2 Contractible Models

Contractible models, defined by Furstenberg and Glasner, form a useful characterization of SAT actions (which we will discuss in a later chapter):

Definition 1.27. Let G be a group and (X, ν) a G-space. Let Y be a compact model for the action. The model is called **contractible** when for every $y \in Y$ there exists a sequence $g_n \in G$ such that (in weak-*) $g_n\nu \to \delta_y$ (the point mass at y).

1.5.3 Proximal Models

Proximal models for actions, introduced by Furstenberg and studied extensively by Glasner [Gla76], Furman [Fur03] and Raja [Raj03], arise in an attempt to relate topological and measurable behavior of group actions.

Definition 1.28 (Furstenberg)**.** A compact metric space X with a group G acting continuously is **proximal** when for all $x, y \in X$ there exists a sequence $g_n \in G$ such that $\lim_n g_n x = \lim_n g_n y$.

Proximal spaces are the counterpart to the distal systems studied extensively by Furstenberg culminating in the topological structure theorem for distal systems. We are concerned with group actions that are far from distal so do not go into details here.

Definition 1.29 ([Gla76], [Fur03])**.** A compact metric space X where a group G acts is called **strongly proximal** when for every probability measure $\nu \in P(X)$ the closure of the G-orbit of ν (in the weak-* topology) contains a point mass.

Theorem 1.30 (Glasner)**.** *A compact metric space X with a G-action is strongly proximal if and only if the corresponding action on $P(X)$ is proximal.*

Theorem 1.31 (Glasner)**.** *Let X be a compact metric space with a G-action that is strongly proximal. Then $G \curvearrowright P(X)$ in a strongly proximal fashion.*

There are examples of proximal spaces which are not strongly proximal but the above Theorems show that there are no further levels of discrepancy in "how proximal" a space can be.

1.5.4 Minimal Models

Topologically speaking, minimal models are the smallest spaces one can study.

Definition 1.32. Let X be a compact metric space with a continuous group action by a group G. Then $G \curvearrowright X$ is **minimal** when the only closed G-invariant subsets of X are trivial (the empty set and X itself).

Unfortunately one cannot assume that minimal compact models always exist. In fact there exists a Polish group that admits no nontrivial minimal actions on a compact space but does admit quasi-invariant ergodic actions that are not measure-preserving (see [Gla98]).

1.5.5 Minimal Strongly Proximal Models

Minimal strongly proximal models are the natural topological extreme from distal spaces (which are the topological equivalent of measure-preserving spaces).

Proposition 1.5.2. *Let (X, ν) be a compact model for a G-space. If X is minimal and strongly proximal then X is contractible.*

Proof. Assume that X is minimal and strongly proximal. Since X is strongly proximal there is a point mass in the closure of the G-orbit of ν. Let $C \subseteq X$ be the points x such that δ_x is contained in the closure of the orbit. So $C \neq \emptyset$. Now C is closed since if $x_n \in C$ and $x_n \to x$ then $\delta_{x_n} \in \overline{G\nu}$ so $\delta_x = \lim \delta_{x_n} \in \overline{G\nu}$. Hence by minimality $C = X$. Thus (X, ν) is contractible. □

Note that the converse *does not* hold: Furstenberg showed that for any group G there exists a maximal compact space which is minimal and strongly proximal in the sense that any minimal strongly proximal compact G-space is a quotient of this maximal space. Furman [Fur03] showed that a group G is amenable if and only if this maximal space is trivial. However there exist amenable groups with nontrivial Poisson Boundary hence there exist groups that admit a nontrivial contractible action on a compact space but admit no nontrivial minimal strongly proximal actions on compact spaces. Therefore there exist contractible compact spaces that are not minimal and strongly proximal.

1.5.6 Existence of Compact Models

We make use of the following standard fact (see e.g. [Zim84]):

Lemma 1.5.3. *Let (Z, ζ) be a G-space, G a countable discrete or locally compact second countable group. Then there exists a compact model for (Z, ζ) where G acts in a Borel fashion.*

In fact, there exists a compact model where the action is continuous. We remark that the existence of a continuous model is equivalent to the existence of a countable collection of G-continuous functions which generate the σ-algebra (that is, separates points) but that for locally compact groups such a sequence is automatic. The reader is referred to [FG10] and [Zim84] for more details.

For completeness, and since we will make use of it in the next section on compact models of G-maps, we present a concrete construction of a compact model on which the action is continuous.

Theorem 1.33. *Let G be a locally compact second countable group and (X, ν) a G-space. There exists a compact metric space Y and $\eta \in P(Y)$ such that (X, ν) is measurably G-isomorphic to (Y, η) (the measurable isomorphism is G-equivariant almost everywhere, i.e. is a G-map) such that $G \curvearrowright Y$ continuously.*

Proof. Take (X_0, ν) to be a Borel model for (X, ν) so $G \curvearrowright X_0$ in a Borel fashion (and of course so that the measurable functions, and hence the continuous functions, separate points) which exists by Mackey's Theorem. Since X_0 is a Borel space with a Borel G-action, X_0 embeds as a Borel G-invariant subset into a Polish G-space, that is, a Polish space X' on which G acts continuously (see Becker and Kechris [BK96]). The continuous functions on X' are a separable Banach space and so we can take $\mathcal{F}_0$ to be a countable dense set in $C(X')$ restricted to having domain X_0 (which is a G-invariant subset of X'). Then the G-action on each element of $\mathcal{F}_0$ is continuous since the G-action on X' is continuous.

Let G_0 be a countable dense subset of G. Now let

$$\mathcal{F} = \bigcup_{g \in G_0} g\mathcal{F}_0 = \{g \cdot f : g \in G_0, f \in \mathcal{F}_0\}$$

be a countable collection of Borel functions on X_0. Let $\mathcal{C}$ be the C^*-algebra generated by $\mathcal{F}$ (that is, close $\mathcal{F}$ under the operations of addition, scalar multiplication and pointwise limits). Then $\mathcal{C}$ is a separable sub-C^*-algebra of the bounded Borel functions on X_0 (separable since $\mathcal{F}$ is countable). Note that since $\mathcal{C}$ is G_0-invariant, it is in fact a G-invariant sub-C^*-algebra since G_0 is dense in G (and the bounded Borel functions on X_0 is a G-invariant C^*-algebra).

Let Y be the Gelfand space for $\mathcal{C}$: the set of $*$-homomorphisms from $\mathcal{C}$ to the complex numbers $\mathbb{C}$. Denote the Gelfand Representation on $\mathcal{C}$ for $c \in \mathcal{C}$ by $\widehat{c} : Y \to \mathbb{C}$

$$\widehat{c}(y) = y(c)$$

By the Gelfand-Naimark Theorem the map $c \mapsto \widehat{c}$ is an isometric $*$-isomorphism $\mathcal{C} \to C(Y)$ (see e.g. Arveson [Arv81]). Since $1 \in \mathcal{C}$ we know Y is in fact compact and since $\mathcal{C}$ is separable Y is metrizable (using the weak-* topology).

We have a G-action on $\mathcal{C}$ by, for $g \in G$ and $c \in \mathcal{C}$,

$$g \cdot c(x) = c(g^{-1}x)$$

coming from the G-action on X_0. Clearly $g \cdot c \in \mathcal{C}$ since $\mathcal{C}$ is G-invariant. Now define a G-action on Y by letting

$$(g \cdot y)(c) = y(g \cdot c)$$

and then

$$\widehat{c}(g \cdot y) = (g \cdot y)(c) = y(g \cdot c) = \widehat{g \cdot c}(y)$$

so the map $c \mapsto \widehat{c}$ is G-equivariant. Let $\eta \in P(Y)$ be the push of ν over this map in the sense that ν defines a linear functional on $\mathcal{F}_0$ hence on $\mathcal{F}$ and therefore on $\mathcal{C}$ since ν is a continuous functional and so η is defined as the image of ν extended from $\mathcal{F}_0$ to $\mathcal{C}$. This is precisely the point in the proof where we use that ν is quasi-invariant to ensure that η is well-defined (if ν were not quasi-invariant under the action then ν would not necessarily extend in any reasonable sense to $\mathcal{C}$).

Then (Y, η) is a compact model for (X, ν) and the measurable isomorphism between them is a G-map as required (the Borel functions on Z must be isomorphic to those on X since the Borel algebra is the minimal algebra closed under the operations and separating points: every function in $C(Z)$ comes from a Borel function on X by construction). It remains to show that the G-action on Y is continuous. Let $y_n \to y$ in Y (recall the topology here is weak-*) and $g_n \to e$ in G. Then for $\widehat{c} \in C(Y)$ we have that

$$\left|\widehat{c}(g_n y_n) - \widehat{c}(y)\right| \leq \|\widehat{g_n \cdot c} - \widehat{c}\| + \left|\widehat{c}(y_n) - \widehat{c}(y)\right|$$

Now $\widehat{c}(y_n) \to \widehat{c}(y)$ since $\widehat{c}(y_n) = y_n(c) \to y(c) = \widehat{c}(y)$ for all $c \in \mathcal{F}$ by the definition of weak-*

convergence. Recall that we chose $\mathcal{F}_0$ to consist of G-continuous functions and therefore for all $c \in \mathcal{F}$ we know that $g_n \cdot c \to c$. Since $c \mapsto \widehat{c}$ is isometric this means that $\|\widehat{g_n \cdot c} - \widehat{c}\| \to 0$. Therefore we have that $\widehat{c}(g_n y_n) \to \widehat{c}(y)$ for all $c \in \mathcal{F}$ hence by construction for all $c \in \mathcal{C}$.

Therefore $G \curvearrowright Y$ continuously since $G \curvearrowright C(Y) \simeq \mathcal{C}$ continuously and $C(Y)$ separates points. So (Y, η) is a compact model for (X, ν) on which G acts continuously. □

The reader is referred to [GTW05] for more details (see Theorem 2.2 in that work) and to Becker and Kechris [BK96] for information on Borel actions and Arveson [Arv81] for information on C^*-algebras.

1.6 Continuous Models for G-Maps

In addition to compact models for G-spaces we can define:

Definition 1.34. Let $\pi : (X, \nu) \to (Y, \eta)$ be a G-map of G-spaces. A Borel map $\pi_0 : X_0 \to Y_0$ is a **compact model for** π when X_0 and Y_0 are compact models for (X, ν) and (Y, η) and π is measurably isomorphic to π_0.

We now show a much stronger fact than we did above that given a G-map between G-spaces, there exists a continuous compact model *for the map* in the sense that there are compact models for both spaces on which the G-action is continuous and further that the equivariant map between them is *continuous*.

The main ideas of our proof of this fact are due to Glasner, Tsirelson and Weiss [GTW05] where they show that a σ-algebra of functions admits a compact model with a continuous action if and only if the algebra has a dense sequence of G-continuous functions. Their work is stated in the context of measure-preserving actions but quasi-invariance is all they use.

Theorem 1.35. *Let G be a locally compact second countable group and $\varphi_n : (X, \nu) \to (Y, \eta)$ be G-maps between the same G-spaces for a finite (or countable) collection of n. Then there exists compact models for (X, ν) and (Y, η) where G acts continuously and such that the φ_n are continuous maps between compact metric spaces.*

Proof. First observe that we may assume already that $G \curvearrowright Y$ continuously since there are always compact models with a continuous action. Fix such a model for Y denoted Y_0. Likewise, choose X_0 to be a compact model for X where G acts continuously. Fix $G_0 \subseteq G$ a countable dense subset (possible since G is second countable).

Let $\mathcal{Y}$ be a countable collection of continuous functions on Y that generate the Borel sets (which exists by compactness and metrizability of Y_0). Define $\mathcal{Y}_0$ to be $\{g \cdot f : g \in G_0, f \in \mathcal{Y}\}$. Then $\mathcal{Y}_0$ is a G_0-invariant countable collection of continuous functions (the G action on Y is continuous) that generate the σ-algebra of Borel functions on Y.

Let $\mathcal{F}_n = \{f \circ \varphi_n : f \in \mathcal{Y}_0\}$. We are of course already assuming the φ_n are Borel maps (which always exist by Zimmer's result). Note that each $\mathcal{F}_n$ is a countable G_0-invariant collection of Borel functions on X since the φ_n are G-maps. Define

$$\mathcal{F}_0 = \bigcup_n \mathcal{F}_n$$

Then $\mathcal{F}_0$ is a G_0-invariant countable collection of Borel functions on X that generate the $\mathcal{B}(Y)$-measurable functions. Taking $\mathcal{X}_0$ as we did $\mathcal{Y}_0$ and setting

$$\mathcal{F} = \mathcal{F}_0 \cup \mathcal{X}_0$$

we obtain a G_0-invariant countable collection of Borel functions on X that generate the Borel functions on X.

Let A be the smallest G_0-invariant σ-algebra containing $\mathcal{F}$. Then A is a separable σ-algebra and is G_0-invariant. Since G_0 is dense in G this means that A is G-invariant.

Let Z be the compact metric Gelfand space for A: the space of norm one linear functionals on A where the map $A \to C(Z)$ by $f \mapsto \widehat{f}$ by $\widehat{f}(z) = z(f)$ is an isometric isomorphism of Banach algebras and the action $f \circ g$ on A defines a homomorphic action $\widehat{f} \circ g(z) = f(z \circ g)$ (see Theorem 1.33 for details).

Let $z_n \to z$ in Z and $g_n \to e$ in G. Then for $\widehat{f} \in C(Z)$ we have

$$|\widehat{f}(g_n z_n) - \widehat{f}(z)| \leq \|\widehat{f \circ g_n} - \widehat{f}\| + |\widehat{f}(z_n) - \widehat{f}(z)|$$

which tends to zero since the G action is continuous on $\mathcal{F}$ hence on $C(Z)$ since the Gelfand map is isometric.

Therefore $G \curvearrowright Z$ continuously. Now the Borel sets of Z are a G-invariant subset of $\mathcal{B}(X)$ and since $\mathcal{X}_0$ was chosen to be dense in the Borel functions, the Borel sets of Z must be all of the Borel sets of X under the isomorphism. Therefore Z with its Borel sets is a compact model for X. By construction G acts continuously on Z. Since all the functions in $\mathcal{F}$ are mapped to continuous functions on Z by the construction of the Gelfand space and the Gelfand Transform, the maps $\varphi_n : Z \to Y_0$ obtained over the isomorphism $X \to Z$ are necessarily continuous. Define η to be the extension of ν from $\mathcal{F}_0$ to $C(Z)$ as we did in the proof of Theorem 1.33 (using the quasi-invariance). The Theorem is then proved. □

That is to say, define:

Definition 1.36. Let $\pi : (X, \nu) \to (Y, \eta)$ be a G-map of G-spaces. Let $\pi_0 : X_0 \to Y_0$ be a compact model for π. When $G \curvearrowright X_0$ continuously, $G \curvearrowright Y_0$ continuously and π is continuous then π_0 is a **continuous model for** π.

and the above Theorem says that there always exist continuous models for G-maps.

1.7 Characterizing Amenability

Theorem 1.37. *Let G be a group. Then G is amenable if and only if for every compact metric space X such that $G \curvearrowright X$ continuously there exists a G-invariant Borel probability measure on X.*

The preceding sections have already shown that if G is amenable then any action on a metric space admits invariant probability measures. It also follows from the fact that $P(X)$ is a weakly closed convex invariant subset of the unit ball in the Banach space of measures on X so Theorem B.6 yields a fixed point.

The reverse holds since we can take X to be the positive cone in the unit ball of $L^\infty(G, \text{Haar})$ under the weak topology and obtain an invariant probability measure on it which is then an invariant mean (see Definition B.2).

Chapter 2

Stationary Dynamical Systems

Starting in the 1960s, Furstenberg developed the theory of boundaries as a means for studying harmonic functions on nonabelian Lie groups. Since then it has developed as a powerful tool in understanding the dynamics of nonamenable groups.

Furstenberg's original papers [Fur63], [Fur67], [Fur71] and [Fur73], updated by Furman [Fur02] and Bader and Shalom [BS05], should provide the interested reader with more information if desired. Applications of boundary theory appear in those works and also, for example, Nevo and Zimmer's structure theorem for actions of semisimple Lie groups [NZ02] and Raugi's work [Rau77].

Before presenting details of boundary theory, we elaborate on the category of (G, μ)-spaces introduced in the previous chapter. Boundaries are a very special type of (G, μ)-space, and the best understood, but it will be useful to collect some abstract facts about stationary systems before studying boundaries.

2.1 Stationary Systems

In the previous chapter we proved the existence of quasi-invariant probability measures on metric spaces equipped with group actions by showing that in fact there always exist stationary measures.

Locally compact second countable groups, the class of groups we will study, are topological groups with a topology that is separable (by second countability) and therefore whose Borel sets behave much like those of metric spaces. In particular, we can speak of measures on locally compact groups as we do measures on metric spaces.

Definition 2.1. Let G be a locally compact group and $\mu \in P(G)$ a probability measure on G. A G-space (X, ν) is a (G, μ)**-space** when $\mu * \nu = \nu$

Recall that the convolution $\mu * \nu$ is defined by, for $f \in C(X)$,

$$\mu * \nu(f) = \int_G g\nu(f)\ d\mu(g) = \int_G \int_X f(gx)\ d\nu(x)\ d\mu(g)$$

and that given $G \curvearrowright X$ a compact metric space there always exists $\nu \in P(X)$ such that $\mu * \nu = \nu$.

Definition 2.2. Let G be a locally compact group and $\mu \in P(G)$. A (G, μ)-space is also called a **stationary dynamical system**.

The category of stationary dynamical systems has (G, μ)-spaces as its objects and G-maps of G-spaces as its morphisms. This is justified by the easy observation that

Proposition 2.1.1. *Let (X, ν) be a (G, μ)-space and $\pi : (X, \nu) \to (Y, \eta)$ be a G-map of G-spaces. Then (Y, η) is a (G, μ)-space.*

Proof. We need only show that $\mu * \eta = \eta$. To see this, let $f \in C(Y)$ and observe that since $\pi_* \nu = \eta$ and π is G-equivariant

$$\begin{aligned}\mu * \eta(f) &= \int_G \int_Y f(gy)\ d\eta(y)\ d\mu(g) \\ &= \int_G \int_X f(g\pi(x))\ d\nu(x)\ d\mu(g) = \int_G \int_X f(\pi(gx))\ d\nu(x)\ d\mu(g) \\ &= \int_G \int_X f(\pi(x))\ dg\nu(x)\ d\mu(g) = \int_X f(\pi(x))\ d\mu * \nu(x) \\ &= \int_X f(\pi(x))\ d\nu(x) = \int_Y f(y)\ d\eta(y)\end{aligned}$$

That is, $\mu * \pi_* \nu = \pi_* \mu * \nu = \pi_* \nu$ (since π is equivariant so is π_*). Hence $\mu * \eta = \eta$. □

We also mention again a fact stated in the previous chapter:

Theorem 2.3 (Bader-Shalom)**.** *Let G be a group acting on a compact metric space X and $\mu \in P(G)$. The set of μ-stationary probability measures is denoted $P_\mu(X) = \{\nu \in P(X) : \mu * \nu = \nu\}$. The ergodic measures in $P_\mu(X)$ are the extreme points in the compact convex set $P_\mu(X)$.*

This indicates that in the context of stationary systems, the ergodic decomposition plays the same role as in the measure-preserving case.

2.2 Boundaries

Furstenberg developed boundary theory in the 1960s in an effort to generalize the classical Poisson Transform to general Lie groups. Since then boundary theory has been found to have applications both to Lie groups and to general locally compact groups and has been an active area for almost fifty years.

Our presentation is based heavily on the development of boundary theory put forth by Bader and Shalom [BS05] and we incorporate details from Furman [Fur02] and of course from Furstenberg's original papers. The reader is referred to those two excellent works for more information. Of course the vast majority of the ideas in what follows are due to Furstenberg and we should mention that Kaimanovich is responsible for a large amount of foundational work in the theory.

2.2.1 The Classical Poisson Transform

Before embarking on a discussion of general boundary theory we recall the basic idea of the classical Poisson Transform in complex analysis. Boundary theory generalizes this to Lie groups and the classical transform is the motivating example for what follows.

Let G be the group of fractional linear transformations of $\mathbb{C}$ which preserve the unit disc (equivalently that preserve the upper half plane if one then applies the standard conformal mapping technique, i.e. $G = \mathrm{PSL}_2(\mathbb{R})$) in the sense that for $g \in G$ we require that $g\mathbb{D} = \mathbb{D}$ where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $h : \mathbb{D} \to \mathbb{R}$ be a bounded harmonic function on the disc. Then the Poisson formula states that h can be written in terms of a bounded function f on the boundary of the disc:

$$h(z) = \int_{\partial\mathbb{D}} f(x)\mathrm{Re}\Big[\frac{1+z\overline{x}}{1-z\overline{x}}\Big]\ dm(x)$$

where m is the normalized Lebesgue measure on the boundary of the disc $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$.

Given such a bounded harmonic function $h : \mathbb{D} \to \mathbb{R}$ we can determine a bounded function on G using the formula

$$\tilde{h}(g) = h(g(0))$$

for $g \in G$. The Poisson formula tells us that $h(0) = \int f(x)\ dm(x)$: the value of a harmonic function at 0 is the average value along the boundary of the disc. Now $h_g(z) = h(g(z))$ defines another bounded harmonic function on the disc and due to the conformal nature of factional linear transformations it is clear the value of h_g at zero is the average along the boundary of the bounded function $f_g(x) = f(g(x))$ (that is, we can first shift the entire picture by g and take the average then shift back). Therefore the Poisson formula implies that

$$\tilde{h}(g) = \int_{\partial\mathbb{D}} f(g(x))\ dm(x) = \int_{\partial\mathbb{D}} f(x)\ dgm(x) = gm(f)$$

We also observe that since h is harmonic the Laplacian of h is zero on the disc and this in turn means that $\tilde{h}$ is annihilated by the corresponding differential operator on G. Therefore in some sense $\tilde{h}$ is a harmonic function on g. Since g is a fractional linear transformation that preserves the unit circle, in fact $g(0)$ determines the function g completely and so there is a one-one correspondence between harmonic functions on the group G and bounded functions on the unit circle.

The purpose of boundary theory is to generalize the above results on harmonic functions as functions on the Lie group G to arbitrary Lie groups (and more generally to locally compact groups) by constructing an appropriate "boundary" such that harmonic functions on the group are in one-one correspondence, via a "Poisson formula", to bounded functions on this boundary.

2.2.2 Harmonic Functions on Groups

In order to make sense of our above discussion we first define what we mean by a harmonic function on a group. We opt to present the most general approach first and then return to Lie groups later.

Definition 2.4. A function $\phi : G \to \mathbb{R}$ is **μ-harmonic** when for every $g' \in G$ we have that

$$\phi(g') = \int_G \phi(g'g)\ d\mu(g)$$

The space of all *bounded* μ-harmonic functions on G is denoted $\mathrm{Har}(G, \mu)$.

2.2.3 The Poisson Transform

Let (X, ν) be a (G, μ)-space. For $f \in L^\infty(X, \nu)$ define $\widehat{f} : G \to \mathbb{R}$ by

$$\widehat{f}(g) = \int_X f(gx)\ d\nu(x) = g\nu(f)$$

Then

$$\begin{aligned}\int_G \widehat{f}(g'g)\ d\mu(g) &= \int_G \int_X f(g'gx)\ d\nu(x)\ d\mu(g) = \int_G \int_X f(g'x)\ dg\nu(x)\ d\mu(g) \\ &= \int_X f(g'x)\ d(\mu * \nu)(x) = \int_X f(g'x)\ d\nu(x) = \widehat{f}(g')\end{aligned}$$

so $\widehat{f}$ is μ-harmonic. Moreover, $|\widehat{f}(g)| \leq \|f\|_{L^\infty}$ so $\widehat{f} \in \mathrm{Har}(G, \mu)$.

Definition 2.5. Let G be a group and $\mu \in P(G)$. Let (X, ν) be a (G, μ)-space. The mapping $L^\infty(X, \nu) \to L^\infty(G, \mathrm{Haar})$ by $f \mapsto \widehat{f}$ is the **Poisson Transform**.

2.2.4 The Universal (Poisson) Boundary

The Poisson boundary is the space on which the Poisson Transform just described can be inverted in a reasonable sense.

Consider the countable product $G^{\mathbb{N}}$ with measure $\mu^{\mathbb{N}}$ (the product measure). G acts on this space by $g(w_1, w_2, \ldots) = (gw_1, w_2, \ldots)$ (multiplication on the left in the first coordinate).

Let T be the map from $G^{\mathbb{N}}$ to itself given by $T(w_1, w_2, w_3, \ldots) = (w_1 w_2, w_3, \ldots)$ (the left shift combining the first two coordinates). We define the **Poisson Boundary** of G (relative to μ) to be the space of T-ergodic components of $G^{\mathbb{N}}$ with the push forward of the measure $T_* \mu^{\mathbb{N}}$. The Poisson boundary will (sometimes) be written $PB(G, \mu)$.

Since the G action commutes with T, the action descends to an action on the Poisson boundary. Moreover, since $\mu * (\mu \times \mu \times \cdots) = (\mu * \mu) \times \mu \times \cdots = T_* \mu^{\mathbb{N}}$ the measure ν on PB is stationary for μ. It is clear that since G acts continuously on itself (by left multiplication) the action of G on its Poisson boundary is continuous.

2.2.5 Harmonic Functions and Boundaries

Let $\varphi \in \mathrm{Har}(G,\mu)$. Define the maps $\varphi_n : G^{\mathbb{N}} \to \mathbb{R}$ by $\varphi_n(w_1, w_2, \ldots) := \varphi(w_1 w_2 \cdots w_n)$. Let $\mathcal{F}_n$ be the sigma-algebra generated by the first n coordinates of $G^{\mathbb{N}}$. Then

$$\begin{aligned}\mathbb{E}[\varphi_{n+1}|\mathcal{F}_n](w_1,\ldots) &= \int_G \varphi(w_1 \cdots w_n w_{n+1})\ d\mu(w_{n+1}) \\ &= \varphi(w_1 \cdots w_n) = \varphi_n(w_1,\ldots)\end{aligned}$$

and $|\varphi_n| \leq |\varphi|$ so the φ_n form a martingale. We can then define $\overline{\varphi}(w_1,\ldots) := \lim_n \varphi_n(w_1,\ldots)$ which exists by the martingale convergence theorem for $\mu^{\mathbb{N}}$ almost every path $(w_1,\ldots)$. Now $\overline{\varphi} : \mathbb{G}^{\mathbb{N}} \to \mathbb{R}$ is T-invariant so $\overline{\varphi}$ descends to a function, also denoted $\overline{\varphi}$, in $L^\infty(PB(G,\mu))$.

Moreover, using Dominated Convergence and that φ is μ-harmonic,

$$\begin{aligned}\widehat{\overline{\varphi}}(g) &= \int_{PB(G,\mu)} \overline{\varphi}(gx)\ d\nu(x) = \int_{G^{\mathbb{N}}} \overline{\varphi}(gw_1, w_2, \ldots)\ d\mu^{\mathbb{N}}(w_1,\ldots) \\ &= \int_{G^{\mathbb{N}}} \lim_n \varphi(gw_1 \cdots w_n)\ d\mu^{\mathbb{N}}(w_1,\ldots) = \lim_n \int_{G^n} \varphi(gw_1 \cdots w_n)\ d\mu^n(w_1,\ldots,w_n) \\ &= \lim_n \varphi(g) = \varphi(g)\end{aligned}$$

so in fact the maps $\mathrm{Har}(G,\mu) \to L^\infty(PB(G,\mu)) \to \mathrm{Har}(G,\mu)$ form an isomorphism.

Likewise, for $f \in L^\infty(PB(G,\mu))$ we have that $f \circ \phi \in L^\infty(G^{\mathbb{N}}, \mu^{\mathbb{N}})$ is T-invariant where $\phi : G^{\mathbb{N}} \to PB(G)$ is the T-component map. By the definition of the topology and measure structure on $(G^{\mathbb{N}}, \mu^{\mathbb{N}})$ for $\mu^{\mathbb{N}}$-almost every $\omega \in G^{\mathbb{N}}$ we have

$$f \circ \phi(\omega) = \lim_n f \circ \phi(\omega_1, \omega_2, \ldots, \omega_n, \omega')$$

for any $\omega' \in G^{\mathbb{N}}$. This follows since the sequence of functions above necessarily converges almost surely by the previous work to some limit, and necessarily converges in L^2 to $f \circ \phi(\omega)$ since the $f \circ \phi(\omega_1, \ldots, \omega_n, \omega')$ are defined on increasing cylinder sets. Then

$$\begin{aligned}\overline{\widehat{f}}\,(\omega) &= \lim_n \widehat{f}(\omega_1 \cdots \omega_n) = \lim_n \int_{G^{\mathbb{N}}} f(\omega_1 \cdots \omega_n \phi(\omega'))d\mu^{\mathbb{N}}(\omega') \\ &= \lim_n \int_{G^{\mathbb{N}}} f(\phi(T^{n+1}(\omega_1,\ldots,\omega_n,\omega')))d\mu^{\mathbb{N}}(\omega') = \lim_n \int_{G^{\mathbb{N}}} f(\phi(\omega_1,\ldots,\omega_n,\omega'))d\mu^{\mathbb{N}}(\omega') \\ &= \int_{G^{\mathbb{N}}} \lim_n f(\phi(\omega_1,\ldots,\omega_n,\omega'))d\mu^{\mathbb{N}}(\omega') = \int_{G^{\mathbb{N}}} f(\omega)d\mu^{\mathbb{N}}(\omega') = f(\omega)\end{aligned}$$

by Dominated Convergence. Hence $\overline{\widehat{f}} = f$ almost surely.

Therefore the map $\varphi \mapsto \overline{\varphi}$ from $\mathrm{Har}(G,\mu) \to L^\infty(PB(G,\mu))$ is the inversion of the Poisson Transform.

2.2.6 Definition of Boundary

The formal definition of a (G, μ) boundary is now clear:

Definition 2.6. Any (G, μ)-space which is a quotient of the Poisson Boundary for (G, μ) is called a (G, μ)**-boundary**. A boundary is also sometimes referred to as a **proximal** (G, μ)-space.

Boundaries correspond to G-invariant sub-σ-algebras of $L^\infty(PB(G, \mu))$ which in turn correspond to sub-σ-"algebras" of $\mathrm{Har}(G, \mu)$. Keep note of the fact that $\widehat{f_1 f_2} \neq \widehat{f_1}\widehat{f_2}$ but that for boundaries $\overline{\varphi_1 \varphi_2} = \overline{\varphi_1}\ \overline{\varphi_2}$ (in fact this characterizes boundaries).

2.3 The Limit Measures

Let (X, ν) be a (G, μ)-space. For any $f \in L^\infty(X, \nu)$ the sequence of numbers $w_1 w_2 \cdots w_n \nu(f)$ converges for $\mu^{\mathbb{N}}$-almost every path $(w_1, \ldots)$. This means that $w_1 \cdots w_n \nu$ converge in the weak-* topology almost surely (take a countable dense subset of $C(X)$; for each function in that set there is a measure one set of convergence hence there is a measure one set that works for all functions in the countable dense set which in turn works for all continuous functions by continuity).

Definition 2.7. The probability measure $\nu_\omega = \lim_n w_1 \cdots w_n \nu$ is the **limit measure** for ω (it is also referred to as the **conditional measure**).

The **barycenter equation** states that

$$\nu = \int_{G^{\mathbb{N}}} \nu_\omega \ d\mu^{\mathbb{N}}(\omega)$$

or, in other words, that ν is the barycenter of the limit measures. The proof is an easy consequence of Dominated Convergence and stationarity:

$$\begin{aligned} \int_{G^{\mathbb{N}}} \nu_\omega(f) \ d\mu^{\mathbb{N}}(\omega) &= \int_{G^{\mathbb{N}}} \lim_n \omega_1 \cdots \omega_n \nu(f) \ d\mu^{\mathbb{N}}(\omega) \\ &= \lim_n \int_{G^{\mathbb{N}}} \omega_1 \cdots \omega_n \nu(f) \ d\mu^{\mathbb{N}}(\omega) = \lim_n \mu^{(n)} * \nu(f) = \nu(f) \end{aligned}$$

where $\mu^{(n)}$ is the n-fold convolution of μ with itself.

2.3.1 Joinings

An unfortunate truism about stationary systems is that the usual product system will not be stationary. In fact it is easily checked that if (X, ν) and (Y, η) are (G, μ)-spaces then $(X \times Y, \nu \times \eta)$ with the diagonal action of G has

$$\mu * (\nu \times \eta) = \int_G g\nu \times g\eta \ d\mu(g)$$

and so $\mu * (\nu \times \eta) = \nu \times \eta$ would imply that

$$\int_G g\nu \times g\eta \; d\mu(g) = \nu \times \eta = \int_{G\times G} g\nu \times h\eta \; d\mu \times \mu(g,h)$$

which does not always happen. In fact, when that is the case for the join of a system with itself we see that for any $f \in L^\infty(X,\nu)$ with $\nu(f) = 0$ we would have that

$$\mu * (\nu \times \nu)(f \times f) = \int_G \left|g\nu(f)\right|^2 \; d\mu(g) = \int_G \left|\widehat{f}(g)\right|^2 \; d\mu(g)$$

and that, since $\nu(f) = 0$,

$$(\mu * \nu) \times \nu(f \times f) = \mu * \nu(f)\nu(f) = 0$$

and therefore these being equal would imply that $\widehat{f} = 0$ for all g hence $g\nu(f) = \nu(f)$ for all f and therefore (X,ν) is a measure-preserving system. The product system is therefore not in general stationary (and in fact when it is stationary one of the systems is measure-preserving).

To rectify this problem, Furstenberg and Glasner have introduced the concept of a joining of stationary systems:

Definition 2.8. Let (X,ν) and (Y,η) be (G,μ)-spaces. Define the measure $\lambda \in P(X \times Y)$ by

$$\lambda = \int_{G^{\mathbb{N}}} \nu_\omega \times \eta_\omega \; d\mu^{\mathbb{N}}(\omega)$$

Then $(X \times Y, \lambda)$ is the **join** of (X,ν) and (Y,η).

The join is the closest stationary system to the product system available. Note that it is stationary:

Proposition 2.3.1. *The join of two (G,μ)-spaces is a (G,μ)-space.*

Proof. Let (X,ν) and (Y,η) be (G,μ)-spaces. Let $\lambda \in P(X \times Y)$ be the join measure. Then

$$\begin{aligned}\mu * \lambda = \int_G g\lambda \; d\mu(g) &= \int_G \int_{G^{\mathbb{N}}} g\nu_\omega \times g\eta_\omega \; d\mu^{\mathbb{N}}(\omega) \; d\mu(g) \\ &= \int_G \int_{G^{\mathbb{N}}} \nu_{g\omega} \times \eta_{g\omega} \; d\mu^{\mathbb{N}}(\omega) \; d\mu(g) = \int_{G^{\mathbb{N}}} \nu_\omega \times \eta_\omega \; d\mu * \mu^{\mathbb{N}}(\omega)\end{aligned}$$

where we have used that $g\nu_\omega = \nu_{g\omega}$ which follows from that fact that for $f \in C(X)$ we also have $f_g(x) = f(gx)$ is continuous (taking a compact model where the G-action is continuous) and therefore

$$g\nu_\omega(f) = \nu_\omega(f_g) = \lim_n \omega_1 \cdots \omega_n \nu(f_g) = \lim_n g\omega_1 \cdots \omega_n \nu(f) = \nu_{g\omega}(f)$$

and therefore, writing $T : G^{\mathbb{N}} \to G^{\mathbb{N}}$ for $T(\omega_1, \omega_2, \omega_3, \ldots) = (\omega_1\omega_2, \omega_3, \ldots)$,

$$\begin{aligned}\mu * \lambda &= \int_{G^{\mathbb{N}}} \nu_\omega \times \eta_\omega \; d\mu * \mu^{\mathbb{N}}(\omega) \\ &= \int_{G^{\mathbb{N}}} \nu_\omega \times \eta_\omega \; dT_*\mu^{\mathbb{N}}(\omega) \\ &= \int_{G^{\mathbb{N}}} \nu_{T(\omega)} \times \eta_{T(\omega)} \; d\mu^{\mathbb{N}}(\omega) \\ &= \int_{G^{\mathbb{N}}} \nu_\omega \times \eta_\omega \; d\mu^{\mathbb{N}}(\omega) = \lambda\end{aligned}$$

since $T_*\mu^{\mathbb{N}} = \mu * \mu^{\mathbb{N}}$ and ν_ω is T-invariant by construction. □

Note that if λ is the join of ν and η then $\lambda_\omega = \nu_\omega \times \eta_\omega$, that is the limit measure of the join are the products of the limit measures.

We also remark that if (X, ν) is a (G, μ)-space and (Y, η) is a measure-preserving G-space then the join of (X, ν) with (Y, η) is simply $(X \times Y, \nu \times \eta)$.

Proposition 2.3.2. *The join of two proximal (G, μ)-spaces, i.e. boundaries, is a proximal (G, μ)-space.*

Proof. The limit measures are point masses for each proximal space hence the same holds for the join. Below we will show that in fact a (G, μ)-space is a boundary if and only if the limit measures are point masses (Theorem 2.11). □

This allows us to define the maximal proximal space by taking the join of all proximal spaces. This turns out to be an equivalent way to define the Poisson Boundary.

2.3.2 The Boundary Map

Let (X, ν) be a compact model for a (G, μ) space (meaning X is a compact G space and $\mu * \nu = \nu$). Consider the map $G^{\mathbb{N}} \to P(X)$ given by $\omega \mapsto \nu_\omega$. This map is defined $\mu^{\mathbb{N}}$ almost everywhere (as above). The map is obviously T invariant so it descends to the **boundary map** $\beta : PB(G, \mu) \to (P(X), \eta)$ where η is the pushforward of $\mu^{\mathbb{N}}$. Clearly this is a measurable G-map between G spaces.

Theorem 2.9 (Naturality of Limit Measures)**.** *Let $\pi : (X, \nu) \to (Y, \eta)$ be a Borel G-map of G-spaces. Then for $\mu^{\mathbb{N}}$-a.e. ω: $\pi_*\nu_\omega = \eta_\omega$.*

Proof. Fix $f \in C(Y)$ and ω such that the limit measures exist. Then

$$\begin{aligned}\eta_\omega(f) &= \lim_n \omega_1 \cdots \omega_n \eta(f) = \lim_n \int_Y f(\omega_1 \cdots \omega_n y) \; d\eta(y) \\ &= \lim_n \int_X f(\omega_1 \cdots \omega_n \pi(x)) \; d\nu(x) = \lim_n \int_X f(\pi(\omega_1 \cdots \omega_n x)) \; d\nu(x) \\ &= \lim_n \omega_1 \cdots \omega_n \nu(f \circ \pi) = \nu_\omega(f \circ \pi) = \pi_*\nu_\omega(f)\end{aligned}$$

□

Theorem 2.10. *Any group acts amenably (in the sense of Zimmer [Zim84]) on its Poisson Boundary.*

Proof. The boundary map is a G-equivariant measurable map from PB to $P(X)$ for any G-space X. □

2.3.3 Boundary Limit Measures Are Point Masses

Theorem 2.11. *Let (X, ν) be a compact model for a (G, μ)-space. Then (X, ν) is a (G, μ)-boundary if and only if the limit measures ν_ω are point masses $\mu^{\mathbb{N}}$-almost surely.*

Proof. Let $\pi : (G^{\mathbb{N}}, \mu^{\mathbb{N}}) \to (P(X), \alpha)$ be the boundary map and $\phi : (G^{\mathbb{N}}, \mu^{\mathbb{N}}) \to (X, \nu)$ the map witnessing that (X, ν) is a boundary.

Take $f \in L^\infty(X, \nu)$ and observe that since $\overline{\widehat{f}} = f \circ \phi$ almost surely

$$\pi(\omega)(f) = \nu_\omega(f) = \overline{\widehat{f}}\,(\omega) = f(\phi(\omega)) = \delta_{\phi(\omega)}(f)$$

where $\delta.$ represents the point mass. Since this is true for every $f \in L^\infty(X, \nu)$ and since X is compact, we have that $\nu_\omega = \delta_{\phi(\omega)}$ almost surely.

In fact, letting $\delta : (X, \nu) \to (P(X), \delta_*\nu)$ be the map $x \mapsto \delta_x$, we have that $\pi = \delta \circ \phi$ (measurably) and therefore that (X, ν) is isomorphic to $(P(X), \pi_*\mu^{\mathbb{N}})$ as (G, μ)-spaces. In particular,

$$\delta_*\nu = \int_X \delta_x \, d\nu(x) = \int_X \delta_x \, d\phi_*\mu^{\mathbb{N}}(x) = \int_{G^{\mathbb{N}}} \delta_{\phi(\omega)} \, d\mu^{\mathbb{N}}(\omega) = \int_{G^{\mathbb{N}}} \pi(\omega) \, d\mu^{\mathbb{N}}(\omega) = \pi_*\mu^{\mathbb{N}}$$

Conversely, if almost every limit measure is a point mass then there is a measurable map $\omega \mapsto x(\omega)$ from $G^{\mathbb{N}}$ to X such that $\nu_\omega = \delta_{x(\omega)}$ (which is obviously a shift-invariant G-map). Consider the push-forward of $\mu^{\mathbb{N}}$ under this map: call it η. Let $f \in L^\infty(X, \nu)$. Then

$$\eta(f) = \int_{G^{\mathbb{N}}} f(x(\omega)) \, d\mu^{\mathbb{N}}(\omega) = \int_{G^{\mathbb{N}}} \delta_{x(\omega)}(f) \, d\mu^{\mathbb{N}}(\omega) = \int_{G^{\mathbb{N}}} \nu_\omega(f) \, d\mu^{\mathbb{N}}(\omega) = \nu(f)$$

where the last equality is the barycenter equation. Therefore the push-forward of $\mu^{\mathbb{N}}$ is in fact ν so the map $\omega \mapsto x(\omega)$ is a (G, μ)-map witnessing that (X, ν) is a (G, μ)-boundary. □

This gives the consequence that if any compact model of a (G, μ) space has the property that almost every limit measure is a point mass then in fact every compact model has that property. That is to say, having point masses as limit measures is in fact a measurable property (that does not depend on the model).

2.3.4 Uniqueness of the Boundary Map

As noted above, (X, ν) is a (G, μ) boundary (i.e. is a factor of $PB(G, \mu)$ if and only if the image of the boundary map is contained in the set of point masses). In particular, $\nu_\omega = \delta_{[\omega]}$ where $[\omega]$ is the image of ω in $PB(G, \mu)$ then mapped to X.

Also, we have the barycenter equation. Let bar : $P(P(X)) \to P(X)$ be the barycenter map: $\text{bar}(\alpha) = \int_{P(X)} p \, d\alpha(p)$. Then $\text{bar}(\eta) = \nu$, i.e. $\int_{G^{\mathbb{N}}} \nu_\omega \, d\mu^{\mathbb{N}}(\omega) = \nu$.

Proposition 2.3.3. *The boundary map is essentially unique.*

Proof. Let $\beta : G^{\mathbb{N}} \to P(X)$ be a T-invariant G-map such that the barycenter $\text{bar}((\beta)_* \mu^{\mathbb{N}}) = \nu$. Write $\eta = \beta_* \mu^{\mathbb{N}}$ and then $(P(X), \eta)$ is a (G, μ)-boundary. Now β factors through the Poisson Boundary (being T-invariant) and so by the naturality of limit measures we have that

$$\eta_\omega = (\beta_* \mu^{\mathbb{N}})_\omega = \beta_*((\mu^{\mathbb{N}})_\omega) = \beta_* \delta_\omega = \delta_{\beta(\omega)}$$

which agrees with the fact that for a boundary the limit measures are point masses. Now

$$\begin{aligned} \beta(\omega) &= \text{bar}(\delta_{\beta(\omega)}) = \text{bar}(\eta_\omega) \\ &= \text{bar}(\lim_n \omega_1 \cdots \omega_n \eta) = \lim_n \omega_1 \cdots \omega_n \text{bar}(\eta) \\ &= \lim_n \omega_1 \cdots \omega_n \nu = \nu_\omega \end{aligned}$$

But this means that β is the boundary map as claimed. □

2.3.5 Limit Measures are not Absolutely Continuous

We remark that Furstenberg and Glasner have shown that the limit measures being absolutely continuous can happen only when the action is measure-preserving:

Theorem 2.12 (Furstenberg-Glasner [FG10])**.** *Let (X, ν) be a (G, μ)-space. Then (X, ν) is a measure-preserving G-space if and only if the limit measures ν_ω are absolutely continuous with respect to ν for almost every $\omega \in G^{\mathbb{N}}$.*

Therefore if (X, ν) is a stationary dynamical systems which is not measure-preserving then in fact there is a positive measure set of random paths leading to limit measures which are not absolutely continuous with respect to ν.

2.4 Amenability and the Poisson Boundary

Amenability is intricately connected with the Poisson Boundary and the corresponding (non)existence of bounded harmonic functions on a group. We will use this characterization of amenability in our results in later chapters.

2.4.1 Boundaries of Amenable Groups

A result of Kaimanovich and Vershik [KV83] states that amenability is equivalent to the existence of a probability measure on the group yielding a trivial Poisson Boundary:

Theorem 2.13 ([KV83]). *Let G be a locally compact second countable or countable discrete group. Then G is amenable if and only if there exists $\mu \in P(G)$ with support generating G such that the Poisson Boundary $PB(G, \mu)$ is measurably isomorphic to the one point system.*

Proof. By Theorem 1.37 the group G is amenable if and only if every compact metric G-space admits a G-invariant (Borel) probability measure. Assume that there exists $\mu \in P(G)$ with support generating G such that $PB(G, \mu)$ is the trivial (one point) system. Let X be any compact metric space on which G acts. Let $\nu \in P(X)$ such that $\mu * \nu = \nu$ (which we know always exists). Now the boundary map $G^{\mathbb{N}} \to P(X)$ maps $\omega \mapsto \nu_\omega$ but factors through the Poisson Boundary (which is a single point) so $\nu_\omega = \nu$ for every ω. But then $g\nu = g\nu_\omega = \nu_{g\omega} = \nu$ so ν is in fact G-invariant (actually ν is invariant for the support of μ hence for all of G). So in fact G is amenable.

We will not actually make use of the converse in our work so we refer the reader to [KV83] Theorem 4.3 for a complete proof that G being amenable implies the existence of such a measure on G making the Poisson Boundary trivial. $\square$

2.4.2 The "Minimal" Amenable Space

On the one hand, the Poisson Boundary is an amenable space for the group and is therefore quite large from the point of view of group dynamics. On the other hand, the limit measures are almost surely point masses so in some sense the Poisson Boundary is quite small in that there is no "extra room" beyond that which is needed to encompass all of the group actions. This unique position of the Poisson Boundary as the "minimal" amenable space for the group makes it quite useful in many contexts.

2.5 Measures on Subgroups

A key aspect of Furstenberg's boundary theory is that lattices "inherit" boundaries of the ambient group they sit inside. The major component in proving this is the following Theorem that essentially says that there is a measure on the lattice such that any harmonic function on the ambient group, when restricted to the lattice, is harmonic for this measure. This Theorem is also crucial in Margulis' proof of the Normal Subgroup Theorem for lattices in higher-rank Lie groups.

We will also make use of this idea in relation to cocycles as part of an effort to bring together the two halves of the arguments involved in results like the Normal Subgroup Theorem. Recall that:

Definition 2.14. Let G be a group (either countable discrete or locally compact and metrizable). A probability measure μ on G is **admissible** when the support of μ generates G (generates as a group) and when some convolution power of μ is not singular with respect to the Haar measure on G.

Theorem 2.15 (Furstenberg, [Mar91]). *Let G be a locally compact second countable group and α an admissible probability measure on it. Let Γ be a lattice. Then there exists a probability measure μ on Γ with full support such that for any closed convex subset of a Banach space V on which G acts isometrically and any $v \in V$ such that $\alpha * v = v$ it holds that $\mu * v = v$. In particular, if ν is an α-stationary measure on a G-space then ν is μ-stationary.*

The main example of such a convex G-space is $P(X)$ where X is a compact metric G-space. In this case, the Theorem states that if $\nu \in P(X)$ and $\alpha * \nu = \nu$ then there is a measure on the lattice μ such that also $\mu * \nu = \nu$.

2.5.1 Density Lemma

Lemma 2.5.1. *Let G be a locally compact second countable group and α an admissible probability measure on G. Let $g_1, g_2 \in G$. Then there exist $n, n' \in \mathbb{N}^+$ and $\delta > 0$ such that $g_1\alpha^{(n)} > \delta g_2\alpha^{(n')}$ where $\alpha^{(n)}$ denotes the n-fold convolution of α with itself.*

The reader is referred to [Mar91] for more information about this fact; we prove it only in the case when α is the restriction of Haar measure to a compact set since that will be enough for our purposes.

Proof. Let n' be such that the density of $\alpha^{(n')}$ is positive on a neighborhood of the identity. Note that n' is independent of g_1 and g_2. We will write α in place of $\alpha^{(n')}$ from here on (convolution powers of $\alpha^{(n')}$ are convolution powers of α after all).

We will assume that $\alpha = m|_K$ where m is Haar measure on G and $K = \overline{U}$ and $\langle U \rangle = G$ where U is an open set containing the identity and U is symmetric: $U^{-1} = U$. Of course we normalize the Haar measure so that $\alpha(G) = \alpha(K) = 1$. This is justified since (some convolution power of) our original measure strictly dominates (a multiple of) this measure. Define the function for $g, h \in G$ by

$$\delta(g,h) = \sup\left\{\delta > 0 : \exists \text{ open } U' \ni e\ \exists n \in \mathbb{N}\ \forall x \in U' \quad \frac{dh^{-1}g\alpha^{(n)}}{d\alpha}(x) \geq \delta\right\}$$

Since $\langle K \rangle = G$ there is some n such that $h^{-1}g \in U^n$. Then $h^{-1}g\alpha^n \geq \delta\alpha$ for some $\delta > 0$ (since α is constant density on U there is some lower bound on the density of $\alpha * \alpha$ on U). Hence $\delta(g,h) > 0$ for all g, h.

□

2.5.2 Density of Translations

Lemma 2.5.2. *Let G be a locally compact second countable group and α an admissible probability measure on G and let v be an α-stationary vector in some Banach G-space with isometric action. For any $g, h \in G$ there exists $\delta(g,h) > 0$ and an admissible $\omega \in P(G)$, both independent of v, such that*

$$gv = \delta(g,h)hv + (1 - \delta(g,h))\omega * v$$

Proof. By the previous Lemma, there exists n, n' and δ such that $g\alpha^{(n)} > \delta h\alpha^{(n')}$. Since $\alpha * \nu = \nu$ we have $\alpha^{(n)} * \nu = \nu$ and therefore

$$gv = g\alpha^{(n)} * v = \delta h\alpha^{(n')} * v + (g\alpha^{(n)} - \delta h\alpha^{(n')}) * v = \delta hv + (1-\delta)\omega * v$$

where $\omega = (1-\delta)^{-1}(g\alpha^{(n)} - \delta h\alpha^{(n')})$. This is a positive measure by the preceding Lemma.

□

2.5.3 Proof of Measures for Subgroups

Proof. (of Theorem) Let $\rho_e \in P(\Gamma)$ be any symmetric fully supported probability measure on Γ. Define the set

$$\Theta' = \{\mu' \in P(\Gamma) : \ \mu' \text{ is fully supported } \}$$

and let $\mathcal{V}$ denote the class of all α-stationary vectors in closed convex subsets of Banach spaces on which G acts isometrically. Define the function

$$L(g) = \sup\{0 \le \epsilon \le 1 : (\exists \mu' \in \Theta')(\exists \mu'' \in P(G))(\forall v \in \mathcal{V}) \quad gv = \epsilon\mu' * v + (1-\epsilon)\mu'' * v\}$$

Fix $g \in G$. By the previous Lemma, for any $\gamma \in \Gamma$ we have that

$$gv = \delta(g,\gamma)\gamma v + (1 - \delta(g,\gamma))\omega(g,\gamma) * v$$

Define $\rho_g \in P(\Gamma)$ by

$$\rho_g(\gamma) = \frac{\delta(g,\gamma^{-1})}{\delta(g,\gamma) + \delta(g,\gamma^{-1})} 2\rho_e(\gamma)$$

Observe that

$$\delta(g,\gamma^{-1})\rho_g(\gamma^{-1}) = \frac{\delta(g,\gamma^{-1})\delta(g,\gamma)}{\delta(g,\gamma) + \delta(g,\gamma^{-1})} 2\rho_e(\gamma) = \delta(g,\gamma)\rho_g(\gamma)$$

by the symmetry of ρ_e and that

$$2\sum_\gamma \rho_g(\gamma) = \sum_\gamma \rho_g(\gamma) + \rho_g(\gamma^{-1}) = \sum_\gamma \frac{\delta(g,\gamma) + \delta(g,\gamma^{-1})}{\delta(g,\gamma) + \delta(g,\gamma^{-1})} 2\rho_e(\gamma) = 2\sum_\gamma \rho_e(\gamma) = 2$$

so $\rho_g \in P(\Gamma)$. Now set $\epsilon = \sum_\gamma \delta(g,\gamma)\rho_g(\gamma)$ and

$$\mu'(\gamma) = \epsilon^{-1}\delta(g,\gamma)\rho_g(\gamma) \quad \text{and} \quad \omega = (1-\epsilon)^{-1} \int_\Gamma (1 - \delta(g,\gamma))\omega(g,\gamma) \, d\rho_g(\gamma)$$

which are probability measures. This then means that

$$\begin{aligned}\epsilon\mu' * v + (1-\epsilon)\omega * v &= \sum_{\Gamma} \epsilon\mu'(\gamma)\gamma v + \sum_{\Gamma} \rho_g(\gamma)(1-\delta(g,\gamma))\omega(g,\gamma) * v \\ &= \sum_{\Gamma} \rho_g(\gamma)\Big(\delta(g,\gamma)\gamma v + (1-\delta(g,\gamma))\omega(g,\gamma) * v\Big) \\ &= \sum_{\Gamma} \rho_g(\gamma) g v = gv\end{aligned}$$

and of course $0 < \epsilon \leq 1$ since ρ_g is a probability measure and $1 \geq \delta(g,\gamma) > 0$. Since ρ_g is fully supported, so is μ'. Hence μ' witnesses that fact that $L(g) \geq \epsilon$. Therefore $L(g) > 0$ for all $g \in G$.

For $\gamma \in \Gamma$ and $g \in G$, write μ'_g and μ''_g to be the measures witnessing that $L(g) > \epsilon$ for some fixed $\epsilon > 0$. Then for any $v \in \mathcal{V}$

$$\gamma g v = \epsilon\gamma\mu'_g * v + (1-\epsilon)\gamma\mu'_g * v$$

and so taking $\mu'_{\gamma g} = \gamma\mu'_g$ and likewise for μ'' we obtain that $L(\gamma g) \geq \epsilon$. Note that

$$\mu'_{\gamma g}(\gamma') = \gamma\mu'_g(\gamma') = \mu'_g(\gamma^{-1}\gamma') < \delta(g, \gamma^{-1}\gamma') = \delta(\gamma g, \gamma')$$

so $\mu'_{\gamma g}$ satisfies the requirements for L. We therefore conclude that $L(\gamma g) = L(g)$. So L is left-Γ-invariant. In particular, $L(\gamma) = L(e)$ for all $\gamma \in \Gamma$ so L is constant on Γ.

Assume for the moment that from this we can deduce that L is constant on G (or at least uniformly bounded above zero). Take $\epsilon > 0$ to be less than a uniform lower bound on L (when L is constant any $\epsilon < L(e)$ is fine). For each g there is then $\mu'_g \in P(\Gamma)$ such that

$$gv = \epsilon\mu'_g * v + (1-\epsilon)\mu''_g * v$$

for some $\mu''_g \in P(G)$ and every $v \in \mathcal{V}$. Hence for any $\sigma \in P(G)$

$$\sigma * v = \epsilon\mu'_\sigma * v + (1-\epsilon)\mu''_\sigma * v$$

where $v \in \mathcal{V}$ is arbitrary and

$$\mu'_\sigma = \int_G \mu'_g \, d\sigma(g)$$

is a probability measure on Γ and likewise for μ''.

Set $\sigma_0 = \delta_e$. Given σ_m choose $\mu_{m+1} \in P(\Gamma)$ and $\sigma_{m+1} \in P(G)$ such that for every $v \in \mathcal{V}$

$$\sigma_m * v = \epsilon\mu_{m+1} * v + (1-\epsilon)\sigma_{m+1} * v$$

By induction (using that V is a closed convex subset of a Banach space to ensure convergence and that the G-action is isometric so $\|\eta * v\| \leq \|v\|$ for any $\eta \in P(G)$ or $P(\Gamma)$)

$$v = \sigma_0 * v = \epsilon\mu_1 * v + (1-\epsilon)\epsilon\mu_2 * v + \cdots$$

and so setting

$$\mu = \epsilon \sum_{m=0}^{\infty} (1-\epsilon)^m \mu_{m+1}$$

we get that (using that the G-action is isometric)

$$\mu * v = v$$

and of course $\sum(1-\epsilon)^m = 1/\epsilon$ so this is a probability measure.

It remains to show that L being constant on Γ in fact implies that L is uniformly bounded above zero on G. This follows from the Cauchy-Schwarz-Buniakowski inequality (as in Margulis): observe that

$$L(g) \geq \int_G L(gg')\ d\alpha(g')$$

since

$$gv = g\alpha * v = \int_G gg'v\ d\alpha(g')$$

and set $L'(\Gamma g) = 1 - L(g)$ which is a well-defined function on $\Gamma\backslash G$ since L is left-Γ-invariant. For $x \in \Gamma\backslash G$,

$$L'(x) \leq \int_G L'(xg)\ d\alpha(g)$$

from the inequality for L and therefore, using the Cauchy-Schwarz-Buniakowski inequality, letting m be the Haar measure on G (which is finite on $\Gamma\backslash G$ as it is a lattice),

$$\begin{aligned}
\int_{\Gamma\backslash G} \left|L'(x)\right|^2\ dm(x) &= \int_{\Gamma\backslash G} \Big| \int_G L'(xg)\ d\alpha(g)\Big|^2\ dm(x) \\
&\leq \int_{\Gamma\backslash G}\int_G \left|L'(xg)\right|^2\ d\alpha(g)\ dm(x) \\
&= \int_G \int_{\Gamma\backslash G} \left|L'(x)\right|^2\ dg^{-1}m(x)\ d\alpha(g) \\
&= \int_{\Gamma\backslash G} \left|L'(x)\right|^2\ dm(x)
\end{aligned}$$

by the G-invariance of the Haar measure. The inequality is therefore an equality $L(xg) = L(x)$ for $\alpha \times m$-almost every (g, x). Now in fact we may replace α by any of its convolution powers since the inequality still holds and since α is admissible there is some convolution power which is nonsingular with respect to m. As the support of α generates G, and each sufficiently large convolution power is nonsingular, we obtain that $L(xg) = L(x)$ for m-almost every g and x. Hence L is constant m-almost surely.

□

2.5.4 Moments

The above construction of a measure on the lattice leaves almost no information about the measure constructed. In particular, nothing is known about the moments (in terms of word length), information which can be useful to have (as we will see).

The proof of this involves Brownian motion and stopping times, combined with a general form of Harnack's Inequality. The reader is referred to [Fur71] for the original idea, [LS84] for the general construction and [Kai88] and [Kai92] for general exposition.

Theorem 2.16 (Furstenberg, Lyons-Sullivan, Kaimanovich)**.** *The measure obtained on a finitely generated lattice may be assumed to be symmetric and to have finite first and second moments when the ambient group G is a Lie.*

The proof of this involves the idea of "discretizing" Brownian motion and we will not go into details, the reader is referred to the works referenced above for the specific construction.

2.6 Boundaries of Specific Groups

We collect here some results giving explicit descriptions of the Poisson Boundary of certain classes of groups, including Lie groups, lattices in Lie groups, and almost connected groups.

2.6.1 Boundaries of Lie Groups

The original motivation for Furstenberg's boundary theory was to generalize the classical Poisson Transform on the unit disc to Lie groups. Semisimple Lie groups have the most well-developed and complete boundary theory and appear to be the largest class of groups where the boundary is (or perhaps can be) well-understood.

We also remark that boundaries were used by Zimmer as the first examples of amenable actions of nonamenable groups and by Jaworski as the first examples of SAT spaces (to be discussed in the following chapter).

Theorem 2.17 (Furstenberg [Fur63])**.** *The Poisson boundary of a semisimple Lie group G with finite center (relative to an admissible μ) is isomorphic to G/P (with the image of μ) where P is a minimal parabolic subgroup of G (such subgroups are all conjugate).*

In fact, Furstenberg showed that if G is a semisimple Lie group and K is a maximal compact subgroup of G then for every admissible $\alpha \in P(G)$ which is K-invariant (that is, for any $\alpha_0 \in P(G)$ take $\alpha = m_K * \alpha_0 * m_K$ where m_K is Haar measure restricted to K and normalized to $m_K(K) = 1$) the Poisson Boundary $PB(G, \alpha)$ is the same. That is, the parabolic group P above depends only on K and not on the measure.

In particular, the Lie group $\mathrm{PSL}_2(\mathbb{R})$ has the unit disc as its boundary which shows that in fact the general construction of boundary theory generalizes the original motivating example of the classical Poisson formula as was intended.

2.6.2 Boundaries of Lattices in Lie Groups

Shortly after developing the Poisson Transform for Lie groups, Furstenberg studied the boundaries of lattices in such groups:

Theorem 2.18 (Furstenberg [Fur67]). *Let G be a semisimple Lie group and Γ a lattice in G. Let $\alpha \in P(G)$ be a K-invariant admissible measure on G (where K is a maximal compact subgroup of G). Then there exists $\mu \in P(\Gamma)$ such that $PB(\Gamma, \mu) = PB(G, \alpha) = G/P$.*

The reader is referred to Furstenberg's work [Fur71] section 5.1 in particular for details. The measure μ is precisely the measure constructed using "discretized Brownian motion" as discussed previously. A consequence of this fact, and one of the motivations for the initial study of boundaries of lattices, is that while $\mathrm{PSL}_2(\mathbb{Z})$ is (isomorphic to) the free product $\mathbb{Z}_2 * \mathbb{Z}_3$, in fact $\mathrm{PSL}_n(\mathbb{Z})$ for $n \geq 3$ cannot have a finite index free subgroup.

Margulis showed (the Factor Theorem) that all Γ-quotients of G-boundaries are in fact G-boundaries and that therefore the Γ-boundaries are the same as the G-boundaries (using the measure on the lattice constructed above). Zimmer later improved on this and Bader and Shalom extended the result to lattices in products of groups. The reader is referred to [Mar91], [Zim84] and [BS05] for more information.

2.6.3 Boundaries of Almost Connected Groups

While Lie groups are an important class of groups, we should also mention that a key aspect of Furstenberg's result has been shown to hold more generally for the class of almost connected groups.

Definition 2.19. Let G be a locally compact group and G^0 the connected component of G. Then G is **almost connected** when G/G^0 is compact (or finite).

Theorem 2.20 (Raugi [Rau77]). *Let G be an almost connected locally compact group and $\mu \in P(G)$ an admissible probability measure on G (the support of μ generates G and some convolution power of μ is nonsingular with respect to Haar measure) with finite first moment. Then the Poisson Boundary $PB(G, \mu)$ is a homogenous G-space (that is, G acts transitively on it, i.e. it is of the form G/P for some P).*

Jaworski later improved this by removing the moment restriction:

Theorem 2.21 (Jaworski [Jaw98]). *Let G be an almost connected locally compact group and $\mu \in P(G)$ an admissible probability measure on G. Then the Poisson Boundary $PB(G, \mu)$ is a homogenous G-space.*

Jaworski also showed that in general (the not almost connected case) this can fail: there are groups where the Poisson Boundary is not a transitive space. The free group is an easy example of this phenomenon.

2.6.4 Boundaries and Normal Subgroups

We conclude the chapter by remarking on the interaction between boundaries and quotient groups.

Theorem 2.22. *Let G be a locally compact group and $N \lhd G$ be a normal subgroup. Let $\mu \in P(G)$ and write μ^* for the pushforward of μ to G/N. Then*

$$PB(G/N, \mu^*) \simeq PB(G, \mu)//N$$

where $PB(G, \mu)//N$ is the N-ergodic components of $PB(G, \mu)$.

The reader is referred to [BS05] for details; this fact is easy and appears in various places in the literature. The key idea is that the boundaries for G/N are precisely the boundaries of G on which N acts trivially.

Chapter 3

SAT Actions

While boundary theory is very useful in understanding quasi-invariant dynamics, it suffers from two defects: one, it imposes a measure on the group so the results obtained then generally hold for the group and measure together but (except in the special case of Lie groups) changing the measure generally changes the boundary; and two, determining whether a given space is a boundary is not easy to do.

Introduced in the mid 1990s by Jaworski in [Jaw94] (with ideas going back to [Jaw91]), SAT (strongly approximately transitive) is a dynamical property of a group acting on a measure space (no measure on the group) that is the natural opposite of measure-preserving. Boundaries are SAT spaces, so such spaces certainly exist, but SAT is defined purely in terms of dynamical (as opposed to algebraic) properties.

3.1 Strong Approximate Transitivity

The definition of SAT makes it clear that it is a dynamical property that is "opposite" measure-preserving:

Definition 3.1 (Jaworski)**.** Let (X, ν) be a G-space. We say (X, ν) is **SAT** when for any measurable $B \subseteq X$ such that $\nu(B) > 0$ and any $\epsilon > 0$ there exists $g \in G$ such that $\nu(gB) > 1 - \epsilon$.

Clearly SAT is equivalent to saying that for every measurable set A with $\nu(A) < 1$ there is a sequence $g_n \in G$ such that $\lim_n \nu(gA_n) = 0$ and we will use the two interchangeably.

An important fact about SAT to keep in mind is that it precludes the existence of invariant measures (even σ-finite measures):

Theorem 3.2 (Jaworski [Jaw94])**.** *If (X, ν) is a SAT G-space and λ is a σ-finite G-invariant measure on X then X is atomic.*

Jaworski also showed that almost nilpotent groups (equivalently, groups of polynomial growth by Gromov's theorem) do not admit nontrivial SAT actions.

3.2 An Example

We present now an example of a SAT action to help the reader gain some intuition. Necessarily the group involved cannot be the integers or anything like them.

Consider the natural action of $\mathrm{PSL}_2(\mathbb{R})$ on the unit circle S with the Lebesgue measure. That is, $\mathrm{PSL}_2(\mathbb{R})$ is the collection of fractional linear transformations preserving the upper half-place (and preserving oriented area) and we can then translate (via Riemann Mapping

Theorem) this to $\mathrm{PSL}_2(\mathbb{R})$ acting on the unit disc preserving area. In particular, $\mathrm{PSL}_2(\mathbb{R})$ under this mapping must preserve the unit circle.

However, the Lebesgue measure on the circle is decidedly not invariant under these maps. In fact, the image of Lebesgue measure under elements of $\mathrm{PSL}_2(\mathbb{R})$ will correspond to solutions to the Dirichlet problem on boundary values of harmonic functions (details are left to the reader; see Chapter 2: Stationary Dynamical Systems). In particular, the action will be SAT since we will be able to obtain any bounded measurable function on the circle as the boundary values of a harmonic function.

More generally, the action of a locally compact group on its Poisson Boundary (corresponding to any admissible measure on the group) will be a SAT action. Since the Poisson Boundary is characterized by the limit measures being point masses, it is easy to see that any positive measure "becomes" a measure one set as an appropriate limit measure is approached.

3.3 Characterizations

We now present some known characterizations of SAT actions. The first is in terms of an associated functional being an isometry; the second is in terms of the total variation metric on measures; and the third is topological.

3.3.1 Isometry Characterization

Theorem 3.3 (Jaworski). *Let (X,ν) be a G-space and consider the map $L^\infty(X,\nu) \to L^\infty(G,\mathrm{Haar})$ given by $f \mapsto g^{-1}\nu(f)$ ($g^{-1}\nu(f)$ being a function in g). Then (X,ν) is a SAT space if and only if this map is an isometry, that is:*

$$\sup_{g\in G} \left|g\nu(f)\right| = \|f\|_{L^\infty(X,\nu)}$$

Proof. It suffices to show that (X,ν) is SAT if and only if for any measurable $B \subseteq X$ with $\nu(B) > 0$ we have that $\|\nu(\cdot B)\|_{L^\infty(G)} = 1 = \|\mathbb{1}_B\|_{L^\infty(X)}$ since characteristic functions form a basis for a dense subset of L^∞.

Assume (X,ν) is SAT. Then for any $\epsilon > 0$ there exists $g \in G$ such that $\nu(gB) > 1-\epsilon$. Hence $\|\nu(\cdot B)\|_{L^\infty(G)} = \sup_{g\in G}\nu(gB) > 1-\epsilon$. So the map is an isometry as it is continuous in f and the characteristic functions form a basis for the L^∞ functions.

Conversely, if the map is an isometry then for any measurable $B \subseteq X$ with $\nu(B) > 0$ we have that $\sup_{g\in G}\nu(gB) = \|\mathbb{1}_B\|_{L^\infty(X)} = 1$ and so for any $\epsilon > 0$ there exists g such that $\nu(gB) > 1-\epsilon$. □

3.3.2 Total Variation Characterization

Theorem 3.4 (Jaworksi). *Let (X,ν) be a G-space. Then (X,ν) is SAT if and only if for any $\eta \in P(X)$ in the same measure class as ν there exists $\nu_n \in P(X)$ which are convex combinations of $g\nu$ ranging over $g \in G$ such that*

$$\|\eta - \nu_n\|_{TV} \to 0$$

where $\|\cdot\|_{TV}$ *is the total variation norm:* $\|\alpha\|_{TV} = \sup\{|\alpha(B)| : B \text{ measurable}\}$.

This can in turn be phrased as:

Theorem 3.5 (Jaworski)**.** *A* G*-space* (X, ν) *is SAT if and only if for any compact model the (convex) closure of the* G*-orbit of* ν *in* $P(X)$ *contains the point masses, that is* $\overline{\text{conv}}\ G\nu = P(X)$.

Here, and throughout, $\overline{\text{conv}}\ C$ denotes the convex closure of a subset C of a convex space. Jaworksi also proved a "zero-two" law for SAT actions:

Theorem 3.6 (Jaworski)**.** *Let* (X, ν) *be a SAT* G*-space. Then for any fixed* $g \in G$

$$\sup_{h \in G} \|h^{-1}gh\nu - \nu\|_{TV} \in \{0, 2\}$$

3.3.3 Topological Characterization

Strong approximate transitivity implies strong topological properties of any compact model. Furstenberg and Glasner recently showed:

Theorem 3.7 (Furstenberg-Glasner 2009)**.** *The action of a group* G *on a measure space* (X, ν) *is SAT if and only if every compact model of the action is contractible.*

3.3.4 Proximal Characterization

We now prove an easy characterization of SAT that makes the connection between SAT and proximal (boundary) spaces more explicit. Recall that a proximal (G, μ)-space is defined by saying that almost every "limit measure" is a point mass. We show now that SAT is characterized by the existence of a limit measure that is a point mass:

Theorem 3.8. *Let* (X, ν) *be a* G*-space. Then* (X, ν) *is SAT if and only if for any compact model there exists a sequence* $g_n \in G$ *such that* $g_n\nu \to \delta_x$ *in* $P(X)$ *(weak-*) for some topologically generic point* x *(meaning the orbit is dense).*

Proof. The topological characterization makes one direction trivial: if (X, ν) is (a compact model for) a SAT G-space then for every $x \in X$ there exists a sequence g_n such that $g_n\nu \to \delta_x$ in weak-* (and SAT implies ergodicity so there is a generic point). Assume now that for any compact model X there is a sequence $g_n \in G$ and a topologically generic point x such that $g_n\nu \to \delta_x$ and $\overline{Gx} = X$. Take a compact model on which the G-action is continuous. Fix $y \in X$. Let $f \in C(X)$ and $\epsilon > 0$. Choose h such that $|f(hx) - f(y)| < \epsilon$ which exists since $y \in \overline{Gx}$ and f is continuous. Write $f_h(z) = f(hz)$ so $f_h \in C(X)$ and choose n such that $|g_n\nu(f_h) - \delta_x(f_h)| < \epsilon$ (possible since $g_n\nu \to \delta_x$). Then

$$|hg_n\nu(f) - f(y)| = |g_n\nu(f_h) - f(y)| \leq |g_n\nu(f_h) - f_h(x)| + |f_h(x) - f(y)| < 2\epsilon$$

and therefore, taking $\epsilon \to 0$, we have that $\delta_y \in \overline{G\nu}$. Therefore (X, ν) is SAT as this holds for every point y so the model is contractible. □

The requirement that the point be topologically generic appears to be necessary to account for examples such as the source-sink dynamical system on $[0,1]$ where points flow to the left and 0 is a stable fixed point and 1 an unstable fixed point. The group $\mathbb{R}$ contracts Lebesgue measure on $[0,1]$ to δ_0 under this flow but this is obviously not a SAT action.

3.4 Properties of SAT Actions

In this section we study the behavior of SAT spaces for locally compact second countable groups and their lattices and we mainly establish results that we use in the following chapters.

Lemma 3.4.1. *Let G be a group and $\varphi : (X,\nu) \to (Y,\eta)$ a G-map between G-spaces. If (X,ν) is a SAT G-space then so is (Y,η).*

Proof. Let A be a measurable subset of Y with $\eta(A) > 0$. Then $\varphi^{-1}(A)$ is a measurable subset of X and $\nu(\varphi^{-1}(A)) = \varphi_*\nu(A) = \eta(A) > 0$. Since (X,ν) is SAT there is a sequence g_n such that $\nu(g_n\varphi^{-1}(A)) \to 1$. Since φ is G-equivariant,

$$\eta(g_nA) = \varphi_*\nu(g_nA) = \nu(\varphi^{-1}(g_nA)) = \nu(g_n\varphi^{-1}(A)) \to 1$$

and therefore (Y,η) is SAT. □

3.4.1 SAT Implies Ergodic

Lemma 3.4.2 (Jaworski)**.** *Let (X,ν) be a SAT G-space. Then the G-action on (X,ν) is ergodic.*

Proof. Let $B \subseteq X$ be a G-invariant Borel set. Suppose that $\nu(B) > 0$. Since the action is SAT there exists $g_n \in G$ such that $\nu(g_nB) \to 1$. But $g_nB = B$ since B is invariant hence $\nu(B) = 1$. Therefore any G-invariant set has measure zero or measure one. □

3.4.2 SAT As A Measure Class Property

We prove the following fact not previously appearing in the literature:

Lemma 3.4.3. *SAT is a property of the measure class, not the measure: if (X,ν) is a SAT G-space and $\nu' \in P(X)$ is in the same measure class as ν (meaning they are mutually absolutely continuous) then (X,ν') is also a SAT G-space. More is true: for each B the same sequence $\{g_n\}$ works for both measures.*

Proof. Let $B \subseteq X$ with $\nu(B) < 1$. Then there exists $g_n \in G$ such that $\nu(g_nB) \to 0$. Suppose that $\limsup \nu'(g_nB) = \delta > 0$. Let $\{n_j\}$ be the sequence attaining this limit. Then $\nu(g_{n_j}B) \to 0$ and $\nu'(g_{n_j}B) \to \delta$. Pick a further subsequence $\{n_{j_t}\}$ such that $\nu(g_{n_{j_t}}B) < 2^{-t}$. Now define the sets

$$B_k = \bigcup_{t=k}^{\infty} g_{n_{j_t}}B$$

and then

$$\nu(B_k) \leq \sum_{t=k}^{\infty} 2^{-t} = 2^{-k+1} \to 0$$

and

$$\nu'(B_k) \geq \nu'(g_{n_{j_k}} B) \to \delta$$

Note that the B_k are a decreasing sequence of sets and therefore, by Lemma 3.4.5,

$$\nu(\bigcap_k B_k) = 0 \qquad \text{but} \qquad \nu'(\bigcap_k B_k) \geq \delta$$

which contradicts that the measures are in the same class. Therefore no such δ and subsequence exists meaning that $\nu'(g_n B) \to 0$ as claimed. □

Lemma 3.4.4. *Let G be a locally compact second countable group acting on a measure space (X, ν) quasi-invariantly. Let A_j be a sequence of measurable sets such that $\nu(A_j) \to 0$ and let $g_j \to g_\infty$ be a convergent sequence in G. Then $\nu(g_j A_j) \to 0$.*

Proof. Choose a subsequence j_t such that $\nu(A_{j_t}) < 2^{-t}$. Let $B_n = \bigcup_{t=n+1}^{\infty} A_{j_t}$. Then $\nu(B_n) \leq 2^{-n}$ and the B_n are a decreasing sequence of sets. Let $B = \cap_n B_n$. Then $\nu(B_n) \to 0$ and B_n are decreasing (as sets) so $\nu(B) = 0$ by Lemma 3.4.5.

Let K be any compact neighborhood of g_∞ (which exists since G is locally compact). There there is some J such that $g_j \in K$ for $j \geq J$. Then $\nu(g_j A_j) \leq \nu(KB_n)$ for j in the subsequence $\{j_t\}$ sufficiently large (KB_n is the set $\{kx | k \in K, x \in B_n\}$).

Hence $\limsup_{t\to\infty} \nu(g_{j_t} A_{j_t}) \leq \nu(KB_n)$ for all n and therefore by Lemma 3.4.5,

$$\limsup_{t\to\infty} \nu(g_{j_t} A_{j_t}) \leq \nu(KB)$$

but K was an arbitrary compact neighborhood of g_∞ and therefore, again by Lemma 3.4.5 (applied to KB as $K \downarrow \{g_\infty\}$, K compact, using the second countability of G),

$$\limsup_{t\to\infty} \nu(g_{j_t} A_{j_t}) \leq \nu(\bigcap_K KB) = \nu(g_\infty B)$$

and $\nu(B) = 0$ so by quasi-invariance $\nu(g_{j_t} A_{j_t}) \to 0$. There is therefore a subsequence of j where the conclusion holds.

Now suppose that $\nu(g_j A_j) \geq \delta$ for infinitely many j. Applying the above to that sequence of j's we obtain a further subsequence where $\nu(g_j A_j) \to 0$ which is a contradiction. □

The following basic measure-theoretic fact was used above:

Lemma 3.4.5. *Let (X, ν) be a probability space and $\delta > 0$. Suppose B_n is a decreasing sequence of measurable sets (meaning $B_{n+1} \subseteq B_n$). Then $\nu(\bigcap_n B_n) = \lim_n \nu(B_n)$.*

Proof. First note that $\nu(B_n)$ is a decreasing sequence of nonnegative numbers since B_n is a decreasing sequence of sets. Hence it has a limit $\delta \geq 0$. Write $A_n = X \setminus B_n$ so that $A_n \subseteq A_{n+1}$. By countable additivity,

$$\begin{aligned} 1 - \nu(\bigcap_{n=1}^{\infty} B_n) &= \nu(\bigcup_{n=1}^{\infty} A_n) = \nu(\bigcup_{n=1}^{\infty} A_{n+1} \setminus A_n) = \sum_{n=1}^{\infty} \nu(A_{n+1} \setminus A_n) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N-1} \nu(A_{n+1} \setminus A_n) = \lim_{N\to\infty} \nu(\bigcup_{n=1}^{N-1} (A_{n+1} \setminus A_n)) = \lim_{N\to\infty} \nu(A_N) = 1 - \delta \end{aligned}$$

□

3.4.3 Lattices Inherit SAT

The next result, new to our work, is the first indication that SAT actions, like boundaries, have a place in rigidity theory for lattices in locally compact gorups.

Lemma 3.4.6. *Let G be a locally compact second countable group and Γ a cocompact lattice in G. Let (X, ν) be a SAT G-space. Then (X, ν) is a SAT Γ-space.*

Proof. Let $K \subseteq G$ be a compact set such that $K\Gamma = G$. Let $B \subseteq X$ such that $\nu(B) < 1$. Then there exists $g_n \in G$ such that $\nu(g_n B) \to 0$ since the G action is SAT. Write $g_n = k_n \gamma_n$ for $k_n \in K$ and $\gamma_n \in \Gamma$. Since K is compact there is a convergent subsequence $k_{n_j} \to k_\infty$ (and so $k_{n_j}^{-1} \to k_\infty^{-1}$ since inverse is continuous). By Lemma 3.4.4 we then have that $\nu(k_{n_j}^{-1} g_{n_j} B) \to 0$. Hence $\nu(\gamma_{n_j} B) \to 0$. Therefore the Γ action is SAT. □

3.4.4 SAT Is Geometric

An immediate consequence of the previous fact is that SAT is a geometric property:

Lemma 3.4.7. *Let Γ be a group and (X, ν) a SAT Γ-space. Let Γ_0 be a finite index subgroup of Γ. Then (X, ν) is a SAT Γ_0-space.*

Proof. This is a special case of Lemma 3.4.6. For completeness: let $\ell_1, \ldots, \ell_m$ be a system of representatives for Γ/Γ_0. Let $B \subseteq X$ be a measurable set with $\nu(B) > 0$. Then there exists a sequence $\gamma_n \in \Gamma$ such that $\nu(\gamma_n B) \to 0$ since (X, ν) is Γ-SAT. For each γ_n write

$$\gamma_n = \ell_{j_n} \gamma_{0,n}$$

where $j_n \in \{1, \ldots, m\}$ and $\gamma_{0,n} \in \Gamma_0$. Since there are only finitely many choices for j_n there exists a subsequence $\{n_t\}$ such that j_{n_t} is constant. Along that sequence, $\gamma_{n_t} = \ell_j \gamma_{0,n_t}$ so

$$\nu(\ell_{j_{n_t}} \gamma_{0,n_t} B) = \nu(\gamma_{n_t} B) \to 0$$

By Lemma 3.4.4, since $\ell_{j_{n_t}} = \ell_j \to \ell_j$ is a convergent sequence, this means that

$$\nu(\ell_{j_{n_t}}^{-1} \ell_{j_{n_t}} \gamma_{0,n_t} B) \to 0$$

and therefore $\nu(\gamma_{0,n_t}B) \to 0$ so (X, ν) is Γ_0-SAT. □

3.5 Actions of Subgroups

The previous section showed that SAT actions are in some sense rigid for cocompact lattices. Ideally one would like to get away from the restriction of cocompactness and we now show that one can, in the special case of boundaries.

3.5.1 Rigidity of SAT on Boundaries

We now prove a basic rigidity statement for SAT in the special case of Poisson Boundaries of the ambient group. This result is new to our work and will be crucial in to the proof of our Normal Subgroup Theorem for Commensurators of Lattices.

Lemma 3.5.1. *Let G be a locally compact second countable group and Γ a lattice in G. Then the action of Γ on any Poisson Boundary of G is SAT.*

Proof. Let (X, ν) be the Poisson Boundary for (G, μ) where μ is a symmetric probability measure on G. Let m be the Haar measure on G restricted to G/Γ and normalized to be a probability measure (possible since Γ is a lattice). The **Random Ergodic Theorem** (see [Fur02]) states that for any $f \in L^\infty(G/\Gamma)$ and m-almost every $z \in G/\Gamma$ and $\mu^{\mathbb{N}}$-almost every $\omega \in G^{\mathbb{N}}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\omega_n \cdots \omega_1 z) = \int f \; d\nu$$

Let K_0 be a compact subset of G such that $m(K_0 \cap F) > 0$ where F is a closed fundamental domain for Γ in G.

Write K for the compact set in G/Γ which is the image of K_0 in G/Γ and let $\mathbb{1}_K$ be the characteristic function of K on G/Γ.

By the Random Ergodic Theorem for m-almost every z and $\mu^{\mathbb{N}}$-almost every $\omega \in G^{\mathbb{N}}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_K(\omega_n \cdots \omega_1 z) = m(K) > 0$$

Pick $z \in G/\Gamma$ such that the above holds for $\mu^{\mathbb{N}}$-almost every ω. Then $\omega_n \cdots \omega_1 z \in K$ infinitely often ($\mu^{\mathbb{N}}$-a.s.). By the symmetry of μ also $\omega_n^{-1} \cdots \omega_1^{-1} z \in K$ infinitely often ($\mu^{\mathbb{N}}$-a.s.).

Fix a measurable set $B \subseteq X$ with $\nu(B) < 1$. Let $z_0 \in G$ be a representative of $z \in G/\Gamma$. Set $A = z_0 B$ and note that $\nu(A) < 1$ (by quasi-invariance).

Since (X, ν) is the Poisson Boundary the conditional measures ν_ω are point masses almost surely and in fact

$$\mu^{\mathbb{N}}\{\omega : \nu_\omega(A) = 0\} = 1 - \nu(A) > 0$$

Pick ω such that $\nu_\omega(A) = 0$ and $\omega_1 \cdots \omega_n z \in K$ infinitely often (such ω must exist since a positive measure set intersects a full measure set nontrivially).

Let $\{n_j\}$ be the times where $\omega_{n_j}^{-1} \cdots \omega_1^{-1} z \in K$ (which is happening in G/Γ). Then

$$0 = \nu_\omega(A) = \lim_{j\to\infty} \nu(\omega_{n_j}^{-1} \cdots \omega_1^{-1} A) = \lim_{j\to\infty} \nu(\omega_{n_j}^{-1} \cdots \omega_1^{-1} z_0 B)$$

and $\omega_{n_j}^{-1} \cdots \omega_1^{-1} z_0 \in K_0\Gamma$ for each j (since $z_0 \in z\Gamma$ and $\omega_{n_j}^{-1} \cdots \omega_1^{-1} z \in K$).

Write $\omega_{n_j}^{-1} \cdots \omega_1^{-1} z_0 = k_j \gamma_j$ for $k_j \in K_0$ and $\gamma_j \in \Gamma$. Then $\lim_{j\to\infty} \nu(k_j\gamma_j B) = 0$.

Choose a subsequence j_ℓ such that $k_{j_\ell} \to k_\infty$ for some $k_\infty \in K$ (possible since K is compact). Set $B_\ell = k_{j_\ell}\gamma_{j_\ell} B$. Then $\nu(B_\ell) \to 0$ and $k_{j_\ell}^{-1} \to k_\infty^{-1}$ in K. By Lemma 3.4.4 this means that $\nu(k_{j_\ell}^{-1} B_\ell) \to 0$. Hence

$$\lim_{\ell\to\infty} \nu(\gamma_{j_\ell} B) = 0$$

Since B was an arbitrary measurable set with $\nu(B) < 1$ and $\gamma_{j_\ell} \in \Gamma$ for all ℓ this means the Γ-action is SAT. □

3.5.2 Uniqueness of SAT Maps

We prove now that SAT maps are unique, a fact new to our work and key to the proof of the Factor Theorem:

Lemma 3.5.2. *Let G be a locally compact second countable group and (X, ν) and (Y, η) and be SAT G-spaces. Let $\varphi : (X, \nu) \to (Y, \eta)$ and $\varphi' : (X, \nu) \to (Y, \eta')$ be G-maps such that η and η' are in the same measure class. Then $\varphi = \varphi'$ almost everywhere.*

Proof. By Theorem 1.35 we may take X and Y to be compact metric spaces where G acts continuously and such that $\varphi, \varphi' : X \to Y$ are continuous maps.

By Theorem 3.7 since (X, ν) is a SAT G-space the model is contractible. Let $x_0 \in X$ be arbitrary. Then there exists a sequence $g_n \in G$ such that $g_n\nu \to \delta_{x_0}$ in weak-* since the model is contractible.

Since φ is continuous so is the pushforward map φ_* (in the weak-* topology) and therefore $\varphi_*(g_n\nu) \to \varphi_*(\delta_{x_0})$. By the G-equivariance of φ this means that $g_n\eta = g_n\varphi_*\nu \to \varphi_*(\delta_{x_0})$. Clearly $\varphi_*(\delta_{x_0}) = \delta_{\varphi(x_0)}$ and so $g_n\eta \to \delta_{\varphi(x_0)}$.

Of course the same reasoning gives that $g_n\eta' \to \delta_{\varphi'(x_0)}$. By Lemma 3.4.3, for any measurable set B, since $\eta(g_nB) \to \delta_{\varphi(x_0)}(B)$ and also $\eta'(g_nB) \to \delta_{\varphi'(x_0)}(B)$ we have that $\delta_{\varphi(x_0)}(B) = \delta_{\varphi'(x_0)}(B)$ (the Lemma says that the same sequence that sends one of them to zero would do the same for the other, so they are either both zero or both one). This means that $\delta_{\varphi(x_0)} = \delta_{\varphi'(x_0)}$ and so $\varphi(x_0) = \varphi'(x_0)$. Since x_0 was arbitrary this means that $\varphi = \varphi'$ as maps between the compact models. So $\varphi = \varphi'$ measurably. □

3.6 The SAT Factor Theorem

Factor Theorems play a key role in using dynamics to study the structure of groups, particularly lattices. Previous factor theorems, including those of Margulis [Mar91], Zimmer and Bader-Shalom [BS05], have always applied only to boundary actions. Our SAT Factor Theorem applies to general SAT actions and is therefore more general and more dynamical in nature.

The reader unfamiliar with the notion of commensuration, commensurable groups, and in particular the commensurator of a subgroup of an ambient group, should consult Chapter 8: Commensuration for definitions and an introduction to the concept. We write $\mathrm{Comm}(\Gamma)$ to mean the commensurator of Γ in G in what follows.

Theorem 3.9. *Let G be a locally compact second countable group and $\Gamma < G$ a lattice in G. Let $\Lambda < \mathrm{Comm}(\Gamma)$ be a subgroup of the commensurator of Γ such that Λ is dense in G and $\Gamma < \Lambda$.*

Let (X, ν) be a G-space where the Γ-action on (X, ν) is SAT and let (Y, η) be a Λ-space such that there exists a Γ-map $\varphi : (X, \nu) \to (Y, \eta)$.

Then φ extends to a G-map in the sense that there is some G-space (Y', η') which is measurably Λ-isomorphic to (Y, η) (meaning the isomorphism is a Λ-map) such that φ extends to a G-map from (X, ν) to (Y', η') (in the sense that the various Λ-maps commute).

The factor theorem roughly says that if we have a G-space on which Γ acts SAT and a Γ-map from it (meaning the map only respects the lattice as far as equivariance) to a space where the commensurator of the lattice acts then in fact the target space is measurably a G-space and the map is a G-map. This means that merely knowing that the lattice acts SAT is enough to guarantee that lattice factors are in fact coming from the ambient group. Note also that unlike in the measure-preserving case it is nontrivial that the action of a dense subgroup extends measurably to a quasi-invariant action of the group.

Proof. Let (X, ν) be a G-space where the Γ-action on (X, ν) is SAT and let (Y, η) be a Λ-space such that there exists a Γ-map $\varphi : (X, \nu) \to (Y, \eta)$. Note that Γ acts ergodically on (Y, η) since it does on (X, ν) because it acts SAT on (X, ν) (Lemma 3.4.2).

Fix $\lambda \in \Lambda$. Since Λ commensurates Γ the subgroup $\Gamma_0 = \Gamma \cap \lambda^{-1}\Gamma\lambda$ has finite index in Γ. As Γ is a lattice in G this means that Γ_0 is also a lattice in G.

Consider the map $\varphi_\lambda : X \to Y$ defined by

$$\varphi_\lambda(x) := \lambda^{-1}\varphi(\lambda x)$$

Since φ is Γ-equivariant we have that φ_λ is Γ_0-equivariant: for $\gamma_0 \in \Gamma_0$ we have $\lambda\gamma_0\lambda^{-1} \in \Gamma$ and so by the Γ-equivariance

$$\varphi_\lambda(\gamma_0 x) = \lambda^{-1}\varphi(\lambda\gamma_0 x) = \lambda^{-1}\varphi(\lambda\gamma_0\lambda^{-1}\lambda x) = \lambda^{-1}\lambda\gamma_0\lambda^{-1}\varphi(\lambda x) = \gamma_0\varphi_\lambda(x)$$

Let $\eta = \varphi_*\nu$ be the pushforward of ν to Y over φ and $\eta' = (\varphi_\lambda)_*\nu$ be the pushforward over φ_λ. Note that η and η' are in the same measure class: if $\eta(N) = 0$ then $\eta(\lambda N) = 0$ by the

Λ-quasi-invariance of η and therefore $\nu(\varphi^{-1}(\lambda N)) = 0$ but

$$\begin{aligned}\eta'(N) &= \int \mathbb{1}_N(\varphi_\lambda(x))\ d\nu(x) = \int \mathbb{1}_N(\lambda^{-1}\varphi(\lambda x))\ d\nu(x) \\ &= \int \mathbb{1}_{\lambda N}(\varphi(x))\ d\lambda\nu(x) = \lambda\nu(\varphi^{-1}(\lambda N))\end{aligned}$$

so by the Λ-quasi-invariance of ν this is zero meaning the measures are in the same class.

By Lemma 3.4.7 the action of Γ_0 on (X, ν) is SAT since the Γ-action is (and Γ_0 has finite index in Γ). Then the action of Γ_0 is SAT on (Y, η) by Lemma 3.4.1 since it is a quotient of a SAT space and is therefore ergodic by Lemma 3.4.2.

Since φ and φ_λ are both Γ_0-equivariant maps, one to a SAT Γ_0-space and one to another measure in the class of the SAT measure, by Lemma 3.5.2 this then means that $\varphi_\lambda = \varphi$ a.e. Hence for each λ we have that

$$\lambda^{-1}\varphi(\lambda x) = \varphi(x)$$

for almost every x meaning that φ is in fact a Λ-map (when Λ is countable the union of countably many a.e. sets is still an a.e. set and in the case Λ is not countable, by the second countability of G, we can take a countable subset of Λ which is dense in Λ in the relative topology from G).

Since Λ is dense in G, treating (Y, η) as a Λ-invariant sub-σ-algebra of (X, ν) means that as a σ-algebra (Y, η) is in fact G-invariant. Then by Mackey's work there exists a measurably Λ-isomorphic G-space (Y', η') as claimed–(Y, η) corresponds to a Λ-invariant sub-σ-algebra of (X, ν) and therefore to a G-invariant sub-σ-algebra by density (which requires knowing that π is Λ-equivariant) and hence has a compact G-model and G-map. □

Chapter 4

Factors and Extensions

We now have established two extremal types of actions, namely measure-preserving actions and SAT actions (or boundary actions). We will now discuss quotients and extensions of G-spaces. This chapter focuses on the well-studied theory of relatively measure-preserving G-maps and various properties they enjoy. Results in this chapter on G-maps and G-factors are classical and well-known except where stated.

4.1 G-Space Quotients

Recall that a G-space is a probability measure space (X, ν) such that G acts on X and ν is quasi-invariant (null sets are preserved) under the action and that a G-map of G-spaces is an equivariant map (commutes with the action) pushing one probability measure to the other (see Chapter 1: Quasi-Invariant Group Actions).

Definition 4.1. Let $\pi : (X, \nu) \to (Y, \eta)$ be a G-map between G-spaces. Then (Y, η) is a G**-quotient** or G**-factor** of (X, ν) and likewise (X, ν) is a G**-extension** of (Y, η).

This chapter will be concerned with classification and structure of G-maps and G-quotients. Our first observation is that quotients of G-spaces correspond to G-invariant sub-σ-algebras:

Proposition 4.1.1. *Let (X, ν) be a G-space. If $\pi : (X, \nu) \to (Y, \eta)$ is a G-map of G-spaces then the pullback of η-measurable sets on Y to X is a sub-σ-algebra of the ν-measurable sets on X which is invariant under G.*

Conversely, if $\mathcal{F}$ is a sub-σ-algebra of ν-measurable sets that is G-invariant (that is, for $B \in \mathcal{F}$ and $g \in G$ also $gB \in \mathcal{F}$) then there is a G-quotient (Y, η) with pullback of measurable sets being $\mathcal{F}$.

Proof. Write $\mathcal{B}$ to denote the Borel sets. We can and will assume that π is a Borel map and that X and Y are compact Borel models of the actions.

Given $\pi : (X, \nu) \to (Y, \eta)$ define $\mathcal{F} = \{\pi^{-1}(B) : B \in \mathcal{B}(Y)\}$. Observe that if $A \in \mathcal{F}$ then $A = \pi^{-1}(B)$ for some Borel set $B \subseteq Y$. Then

$$\begin{aligned} X \setminus A &= X \setminus \pi^{-1}(B) = \{x \in X : x \notin \pi^{-1}(B)\} \\ &= \{x \in X : \pi(x) \notin B\} = \{x \in X : \pi(x) \in Y \setminus B\} = \pi^{-1}(Y \setminus B) \end{aligned}$$

so $\mathcal{F}$ is closed under complements (as the Borel sets are). If $A_n \in \mathcal{F}$ for $n \in \mathbb{N}$ then, writing

$A_n = \pi^{-1}(B_n)$,

$$\bigcup_n A_n = \bigcup_n \pi^{-1}(B_n) = \{x \in X : \exists n \quad \pi(x) \in B_n\}$$
$$= \{x \in X : \pi(x) \in \bigcup_n B_n\} = \pi^{-1}(\bigcup_n B_n)$$

so $\mathcal{F}$ is closed under countable union as well. Hence $\mathcal{F}$ is a sub-σ-algebra. Finally, if $A \in \mathcal{F}$ and $g \in G$ then, since $\pi(gx) = g\pi(x)$,

$$gA = g\pi^{-1}(B) = \{x \in X : g^{-1}x \in \pi^{-1}(B)\} = \{x \in X : \pi(g^{-1}x) \in B\}$$
$$= \{x \in X : g^{-1}\pi(x) \in B\} = \{x \in X : \pi(x) \in gB\} = \pi^{-1}(gB)$$

and since $G \curvearrowright Y$ in a Borel manner, gB is Borel so gA is also.

For the converse, observe that $(X, \mathcal{F}, \nu)$ is a measure algebra with a quasi-invariant G-action and therefore there is a compact model realizing this algebra as its Borel sets and the G-map is given by conditional expectation. ☐

The following standard fact is also useful to keep in mind:

Proposition 4.1.2. *Let (X, ν) and (Y, η) be G-spaces. Then (Y, η) is a G-factor of (X, ν) if and only if there is an equivariant unital weak-* continuous map from $L^\infty(Y, \eta)$ onto a weak-* closed *-subalgebra of $L^\infty(X, \nu)$.*

Proof. If Y is a factor then there is an equivariant map $\pi : X \to Y$ and for $f \in L^\infty(Y, \eta)$ the function $f \circ \pi$ is in $L^\infty(X, \nu)$ and this mapping has the desired properties.

Conversely, the fact that the map is unital weak-* continuous and equivariant means it takes characteristic functions to characteristic functions in an equivariant way. Since the image is closed, the image of characteristic functions is also and therefore defines a sub-σ-algebra which is G-invariant so the claim follows by the previous proposition. ☐

4.2 The Disintegration Map

Given a G-map between G-spaces $\pi : (X, \nu) \to (Y, \eta)$ the key notion in understanding the map is the disintegration of ν over η:

Proposition 4.2.1. *Let $\pi : (X, \nu) \to (Y, \eta)$ be a G-map of G-spaces. For almost every y there exists a unique measure $D_\pi(y)$ supported on $\pi^{-1}(y)$ and such that*

$$\int_Y D_\pi(y) \; d\eta(y) = \nu$$

Proof. Let $\mathcal{F}$ be any G-invariant sub-σ-algebra of the Borel sets of X (assuming X is a Borel model for (X, ν)). We will in fact take $\mathcal{F} = \{\pi^{-1}(B) : B \in \mathcal{B}(Y)\}$ in what follows. For

$f \in L^\infty(X, \nu)$ and $A \in \mathcal{F}$ define

$$\eta_f(A) = \int_A f \, d\nu$$

Then η_f is a probability measure on X whose measurable sets are $\mathcal{F}$. Clearly η_f is absolutely continuous with respect to ν since if $\nu(A) = 0$ then $\eta_f(A) = \int_A f \, d\nu = 0$. Therefore the Radon-Nikodym derivative ${d\eta_f}/{d\nu}$ exists and is in $L^1(X, \nu)$. Now ${d\eta_f}/{d\nu}$ is $\mathcal{F}$-measurable by construction so when $\mathcal{F}$ is the pullbacks of the Borel sets of Y over π we know that ${d\eta_f}/{d\nu}$ is π-invariant. Hence it descends to an $L^1(Y, \eta)$ function:

$$F_f(y) = \frac{d\eta_f}{d\nu}(\pi^{-1}(y))$$

is well-defined since the derivative is constant on fibers over y. Define the map $D_\pi : Y \to P(X)$ by

$$D_\pi(y)(f) = F_f(y)$$

This indeed defines a measure since if $f_n \to f$ in $C(X)$ then

$$\left|\eta_{f_n}(A) - \eta_f(A)\right| \leq \int_A \left|f_n(x) - f(x)\right| \, d\nu(x) \leq \|f_n - f\| \to 0$$

so $F_{f_n} \to F_f$ pointwise and therefore $D_\pi(y)$ is a continuous functional on $C(X)$. Now for positive $f_n \in C(X)$ we have that, by Fubini,

$$\eta_{\sum f_n}(A) = \int_A \sum_n f_n \, d\nu = \sum_n \int_A f_n \, d\nu = \sum_n \eta_{f_n}(A)$$

and therefore

$$D_\pi(y)(\sum_n f_n) = \sum_n D_\pi(y)(f_n)$$

hence $D_\pi(y)$ is a measure. Now for $f \geq 0$ clearly

$$\eta_f(A) = \int_A f \, d\nu \geq 0$$

so $D_\pi(y)$ is positive and also $D_\pi(y)(1) = F_1(y) = {d\eta_1}/{d\nu}(y) = 1$ so $D_\pi(y) \in P(X)$.

Observe that for $f \in C(X)$ and $y \in Y$ such that $f(x) = 0$ for all x such that $\pi(x) = y$ and for $B \subseteq \pi^{-1}(y)$ measurable

$$\eta_f(B) = \int_B f \, d\nu = 0$$

since $f = 0$ on B. Hence $\eta_f\big|_{\pi^{-1}(y)} = 0$ so

$$D_\pi(y)(f) = F_f(y) = \frac{d\eta_f}{d\nu}(x) = 0$$

for x such that $\pi(x) = y$. Therefore $D_\pi(y)$ is supported on $\pi^{-1}(y)$. We also have that

$$\begin{aligned}\int_Y D_\pi(y)(f)\ d\eta(y) &= \int_X F_f(\pi(x))\ d\nu(x) = \int_X \frac{d\eta_f}{d\nu}(x)\ d\nu(x)\\ &= \int_X d\eta_f = \eta_f(X) = \int_X f\ d\nu = \nu(f)\end{aligned}$$

meaning that $\int_Y D_\pi(y)\ d\eta(y) = \nu$ as required.

For uniqueness, observe that the Radon-Nikodym derivative can be defined from the disintegration map by reversing the previous construction and therefore the uniqueness of Radon-Nikodym derivatives implies the uniqueness of disintegration. □

Definition 4.2. The measures $D_\pi(y)$ above are the **disintegration measures**. The map $D_\pi : Y \to P(X)$ which is defined η-almost everywhere is the **disintegration** of ν over η. D_π is also referred to as the **disintegration map**.

Definition 4.3. Let $\pi : (X,\nu) \to (Y,\rho)$ be a G-map of G-spaces. Fix $y \in Y$. The **fiber of** π **over** y is simply the set $\pi^{-1}(y) = \{x \in X : \pi(x) = y\}$ (where we are of course implicitly referring to specific compact models of X and Y).

The disintegration map "disintegrates" ν over η by splitting ν into measures on each fiber whose "average" is ν. In essence, this means that $D_\pi(y)$ is the "value" of ν on the points mapping to y.

4.2.1 Disintegration of Composition Maps

The disintegration map of a composition of G-maps can, of course, be written in terms of the disintegrations of its pieces. Specifically:

Proposition 4.2.2. *Let $\pi : (X,\nu) \to (Y,\eta)$ and $\phi : (Y,\eta) \to (Z,\rho)$ be G-maps. Then the composition $\phi\pi : (X,\nu) \to (Z,\rho)$ is also a G-map and for ρ-almost every z*

$$D_\phi(z) = \pi_* D_{\phi\pi}(z)$$

(where $\pi_* : P(X) \to P(Y)$ is given by $\pi_*\alpha(B) = \alpha(\pi^{-1}(B))$) and

$$D_{\phi\pi}(z) = \int_Y D_\pi(y)\ dD_\phi(z)(y)$$

Proof. Observe that since $D_{\phi\pi}(z)$ is supported on $(\phi\pi)^{-1}(z) = \pi^{-1}(\phi^{-1}(z))$ we have that

$\pi_* D_{\phi\pi}(z)$ is supported on $\pi(\pi^{-1}(\phi^{-1}(z))) = \phi^{-1}(z)$. Also

$$\int_Z \pi_* D_{\phi\pi}(z)\ d\rho(z) = \pi_* \int_Z D_{\phi\pi}(z)\ d\rho(z) = \pi_*\nu = \eta$$

from the barycenter equation for $D_{\phi\pi}$ and that G-maps push measures to one another. This then means that $\pi_* D_{\phi\pi}(z)$ satisfies both conditions for being the disintegration of ϕ and so by uniqueness the first claim is proven.

Similarly, since $D_\pi(y)$ is supported on a subset of $\pi^{-1}(y)$ the measure $\int_Y D_\pi(y)\ dD_\phi(z)(y)$ is supported on a subset of $\cup_{y\in\phi^{-1}(z)}\pi^{-1}(y) = (\phi\pi)^{-1}(z)$ and we have that

$$\int_Z \int_Y D_\pi(y)\ dD_\phi(z)(y)\ d\rho(z) = \int_Y D_\pi(y)\ d\eta(y) = \nu$$

from the barycenter equations for D_ϕ and D_π. Then by uniqueness of disintegration the second claim holds as well. □

4.2.2 Disintegration is Measurable

We point out now the fairly obvious fact that the disintegration map is a measurable property and does not depend on the compact models for the spaces or map.

Proposition 4.2.3. *Let $\pi : (X,\nu) \to (Y,\eta)$ be a G-map. Let (X',ν') and (Y',η') be compact models for (X,ν) and (Y,η) and let $\pi' : X' \to Y'$ be a Borel representative of π for these models. Then $D_\pi = D_{\pi'}$ almost surely (identifying Y and Y' over the measurable isomorphism).*

Proof. Let $\phi : X \to X'$ be the measurable G-isomorphism (so ϕ is a G-map defined ν-almost everywhere and $\nu'(\phi(B)) = \nu(B)$ for every measurable set B) and $\rho : Y \to Y'$ the other measurable G-isomorphism. Then $\phi^{-1} : X' \to X$ and $\rho^{-1} : Y' \to Y$ are also G-isomorphisms. Of course the diagram of G-maps

$$\begin{array}{ccc} (X,\nu) & \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} & (X',\nu') \\ \pi\downarrow & & \downarrow\pi' \\ (Y,\eta) & \underset{\rho^{-1}}{\overset{\rho}{\rightleftarrows}} & (Y',\eta') \end{array}$$

commutes and therefore $\pi'(\phi(x)) = \rho(\pi(x))$, that is $\pi'\phi = \rho\pi$. Therefore, by the properties of disintegration of composition, for (almost every) $y' \in Y'$

$$D_{\pi'}(y') = \phi_* D_{\pi'\phi}(y') = \phi_* D_{\rho\pi}(y')(y') = \phi_* \int_Y D_\pi(y)\ dD_\rho(y')(y)$$

Now ϕ is an isomorphism so ϕ_* is also an isomorphism and ρ is an isomorphism so $D_\rho(y') =$

$\delta_{\rho^{-1}(y')}$ is the point mass at the preimage of y'. So

$$D_{\pi'}(y') = \phi_* D_\pi(\rho^{-1}(y'))$$

meaning they are isomorphic, i.e. equal, almost everywhere as claimed. □

4.2.3 Ergodic Decomposition

When (Y, η) is taken to be $(X, \mathcal{I}, \nu)$ where $\mathcal{I}$ denotes the G-invariant Borel subsets of X (which is of course a G-invariant sub-σ-algebra) the disintegration map yields the **ergodic decomposition**.

It is termed this since $(X, D_\pi(y))$ is an ergodic G-space for almost every y in this case. The ergodic decomposition plays a key role in ergodic theory, being the cornerstone of the idea that understanding ergodic transformations is "enough" to understand all transformations. It also plays a key role in the abstract definition of the Poisson Boundary (though we did not elaborate on it at that time).

4.2.4 Conditional Expectation

Given a G-map $\pi : (X, \nu) \to (Y, \eta)$ between G-spaces, let $\mathcal{F}$ be the pullback of the Borel sets of Y over π, that is $\mathcal{F} = \{\pi^{-1}(B) : B \in \mathcal{B}(Y)\}$. The conditional expectation is defined as follows: let f be a Borel function on X. Then the conditional expectation of f over $\mathcal{F}$, written $\mathbb{E}[f|\mathcal{F}]$, is the unique $\mathcal{F}$-measurable function such that $\nu(\mathbb{E}[f|\mathcal{F}]) = \nu(f)$.

The connection between conditional expectation and disintegration is fairly clear. Specifically, given $f \in L^\infty(X, \nu)$ define

$$F(y) = D_\pi(y)(f)$$

and observe that $F \circ \pi$ is an $\mathcal{F}$-measurable function on X and that

$$\nu(F \circ \pi) = \pi_* \nu(F) = \eta(F) = \int_Y D_\pi(y)(f)\ d\eta(y) = \nu(f)$$

by the definition of disintegration. Hence $F \circ \pi$ is the conditional expectation (by uniqueness of conditional expectation). That is

$$D_\pi(\pi(x))(f) = \mathbb{E}[f|\mathcal{F}](x)$$

4.3 Measure-Preserving Extensions

The notion of a measure-preserving extension is crucial in any structure theory of quasi-invariant group actions. Recall that:

Definition 4.4. A G-space (X, ν) is **measure-preserving** when $g\nu = \nu$ for all $g \in G$.

To lift this notion to extensions or factors we define:

Definition 4.5. A G-map $\pi : (X, \nu) \to (Y, \rho)$ is called **relatively measure-preserving** when G commutes with D_π: for all $g \in G$ and almost every $y \in Y$ we have $gD_\pi(y) = D_\pi(gy)$.

This will also be stated as saying that (X, ν) is a **measure-preserving extension** of (Y, η) or that (Y, η) is a **measure-preserving factor** of (X, ν).

4.3.1 Measure-Preserving Extensions of a Point

A basic fact about relatively measure-preserving extensions is that a system is measure-preserving if and only if it is a measure-preserving extension of a point, which in some sense justifies the terminology.

Proposition 4.3.1. *A G-space is measure-preserving if and only if the canonical G-map from it to a point is relatively measure-preserving.*

Proof. The map $\pi : (X, \nu) \to (\{p\}, \delta)$ given by $\pi(x) = p$ clearly has $D_\pi(p) = \nu$. Thus $gD_\pi(p) = g\nu$ and $D_\pi(gp) = D_\pi(p) = \nu$. So $gD_\pi(p) = D_\pi(gp)$ if and only if $g\nu = \nu$. □

4.3.2 Composition of Measure-Preserving Extensions

The most useful structural fact about measure-preserving extensions is that they are extremal in the space of possible maps in the following sense:

Proposition 4.3.2. *Let $\pi : (X, \nu) \to (Y, \rho)$ and $\phi : (Y, \rho) \to (Z, \gamma)$ be G-maps between G-spaces. Then $\phi\pi$ is relatively measure-preserving if and only if both π and ϕ are relatively measure-preserving.*

Proof. Observe by Proposition 4.2.2 that

$$D_\phi(z) = \pi_* D_{\phi\pi}(z)$$

and also that

$$D_{\phi\pi}(z) = \int_Y D_\pi(y)\; dD_\phi(z)(y)$$

Assume now that $\phi\pi$ is relatively measure-preserving. For $z \in Z$ and $g \in G$ we then have

$$gD_\phi(z) = g\pi_* D_{\phi\pi}(z) = \pi_* gD_{\phi\pi}(z) = \pi_* D_{\phi\pi}(gz) = D_\phi(gz)$$

so ϕ is relatively measure-preserving. Then for $z \in Z$ and $y \in Y$ such that $\phi(y) = z$ and any $g \in G$

$$\begin{aligned}\int_Y D_\pi(gy)\; dD_\phi(z)(y) &= \int_Y D_\pi(y)\; dgD_\phi(z)(y) = \int_Y D_\pi(y)\; dD_\phi(gz)(y) \\ &= D_{\phi\pi}(gz) = gD_{\phi\pi}(z) = \int_Y gD_\pi(y)\; dD_\phi(z)(y)\end{aligned}$$

meaning that $D_\pi(gy) = gD_\pi(y)$ for $D_\phi(z)$-almost every y. Hence π is relatively measure-preserving also.

Conversely, if π and ϕ are relatively measure-preserving then

$$\begin{aligned} gD_{\phi\pi}(z) &= \int_Y gD_\pi(y)\ dD_\phi(z)(y) = \int_Y D_\pi(gy)\ dD_\phi(z)(y) \\ &= \int_Y D_\pi(y)\ dgD_\phi(z)(y) = \int_Y D_\pi(y)\ dD_\phi(gz)(y) = D_{\phi\pi}(gz) \end{aligned}$$

so $\phi\pi$ is relatively measure-preserving. □

4.3.3 The Maximal Measure-Preserving Factor

An ingredient of any potential structure theory for G-spaces is the idea of a maximal measure-preserving factor, that is, for any G-space (X, ν) a factor (Y, η) such that (X, ν) is a relatively measure-preserving extension of (Y, η) and such that every relatively measure-preserving factor is "between" them.

Definition 4.6 (Kaimanovich-Vershik [KV83]). Let (X, ν) be a G-space. The **Radon-Nikodym factor** of this space is obtained by shrinking the measure algebra as follows: let $\mathcal{RN}$ be the smallest σ-algebra (contained in that of (X, ν)) such that all the Radon-Nikodym derivatives $dg\nu/d\nu$ are measurable. The Radon-Nikodym factor is then $(X, \mathcal{RN}, \nu)$ and the factor map is conditional expectation. The Radon-Nikodyn factor will be denoted $RN(X, \nu)$.

Note that the Radon-Nikodym factor is a G-space since $\mathcal{RN}$ is necessarily G-invariant (due to its minimality).

Theorem 4.7 (Kaimanovich-Vershik [KV83]). *The G-map $(X, \nu) \to (X, \mathcal{RN}, \nu)$ is relatively measure-preserving. Moreover, it is the maximal relatively measure-preserving factor of (X, ν) in that if $(X, \nu) \to (Y, \rho)$ is relatively measure-preserving then the Radon-Nikodym factor for Y is isomorphic to that of X and so $(X, \nu) \to (Y, \rho) \to RN(Y, \rho)$ is isomorphic to $(X, \nu) \to RN(X, \nu)$. In particular, if $RN(X, \nu) \to (Z, \gamma)$ is relatively measure-preserving then it is an isomorphism.*

We will provide a proof of this later.

4.4 Boundaries of Products

We now prove that the Poisson Boundary of a direct product is the product of the Poisson Boundaries. We will not need this result in what follows but it illustrates the usefulness of relatively measure-preserving maps and factors in general:

Theorem 4.8 (Bader-Shalom). *Let G_1 and G_2 be locally compact second countable groups and $\mu_1 \in P(G_1)$ and $\mu_2 \in P(G_2)$ be probability measures on them with supports generating* *the groups. Then the Poisson Boundary of the product is isomorphic to the product of the Poisson Boundaries:*

$$PB(G_1 \times G_2, \mu_1 \times \mu_2) \simeq PB(G_1, \mu_1) \times PB(G_2, \mu_2)$$

and in fact any $(G_1 \times G_2, \mu_1 \times \mu_2)$-boundary is a product of boundaries.

Specifically, let G_1 and G_2 be locally compact groups and $\mu_1 \in P(G_1)$ and $\mu_2 \in P(G_2)$ be probability measures on them (with generating supports). We aim to show that

$$PB(G_1 \times G_2, \mu_1 \times \mu_2) \simeq PB(G_1, \mu_1) \times PB(G_2, \mu_2)$$

The key fact in proving this is:

Theorem 4.9 (Bader-Shalom)**.** *Let G_1 and G_2 be locally compact groups and (X, ν) be an ergodic $G_1 \times G_2$-space. Then there exists a G_1-space (X_1, ν_1) and a G_2-space (X_2, ν_2) and a $G_1 \times G_2$-map $\pi : (X, \nu) \to (X_1 \times X_2, \nu_1 \times \nu_2)$ (with the product action).*

When $\mu_1 \in P(G_1)$ and $\mu_2 \in P(G_2)$ and (X, ν) is a $(G_1 \times G_2, \mu_1 \times \mu_2)$-space then each (X_i, ν_i) is a (G_i, μ_i)-space and π is relatively measure-preserving.

Proof. Let $\mathcal{F}$ denote the sub-σ-algebra of (X, ν) sets which are invariant under G_1. That is, $\mathcal{F}$ is the G_1-ergodic components of (X, ν). Clearly $\mathcal{F}$ is a G_1-invariant algebra and it is also G_2-invariant since the actions commute. Let (X_2, ν_2) be a compact model for $(X, \mathcal{F}, \nu)$ and observe that $G_1 \times G_2 \curvearrowright (X_2, \nu_2)$ with trivial G_1-action. Hence (X_2, ν_2) is in fact a G_2-space. Likewise, construct (X_1, ν_1) as a compact model of the σ-algebra of G_2-invariant sets.

Let $\pi_1 : (X, \nu) \to (X_1, \nu_1)$ be the $G_1 \times G_2$-map. Write $\mu = \mu_1 \times \mu_2$ and observe that since the actions of the two groups commute we have that

$$\mu * \mu_1 * \nu = \mu_1 * \mu * \nu = \mu_1 * \nu$$

meaning that $\mu_1 * \nu$ is μ-stationary. But it is in the same measure class as ν and ν is ergodic so $\mu_1 * \nu = \nu$ because ergodic stationary measures are extremal (Theorem 2.3). Then (X, ν) is a (G_1, μ_1)-space and therefore so is (X_1, ν_1).

Furthermore, treating (X, ν) as a (G_1, μ_1)-space we see that $\mu_1 * (g_2\nu) = g_2\mu_1 * \nu = g_2\nu$ for any $g_2 \in G_2$ and so $g_2\nu$ is μ_1-stationary and in the same measure class as ν. Therefore, as shown in [BS05], ${}^{dg_2\nu}/_{d\nu}$ is G_1-invariant making it (X_2, ν_2)-measurable. So $(X, \nu) \to (X_2, \nu_2)$ is a relatively measure-preserving G_2-map. Since G_2 acts trivially on (X_1, ν_1) then also $(X, \nu) \to (X_1 \times X_2, \nu_1 \times \nu_2)$ is a relatively measure-preserving G_2-map. Likewise $(X, \nu) \to (X_1 \times X_2, \nu_1 \times \nu_2)$ is a relatively measure-preserving G_1-map. Therefore π is a relatively measure-preserving $G_1 \times G_2$-map. □

Returning to the proof that products of boundaries are boundaries of products, let (B, ν) be the Poisson Boundary of $G_1 \times G_2$ under $\mu_1 \times \mu_2$ and apply the previous result to obtain that (B, ν) is a relatively measure-preserving extension of $(B_1 \times B_2, \nu_1 \times \nu_2)$. But being a Poisson Boundary (B, ν) is a SAT space hence the only relatively measure-preserving factor of it is trivial, an easy fact which nonetheless has not appeared in the literature:

Proposition 4.4.1. *Let (X, ν) be a SAT G-space and $\pi : (X, \nu) \to (Y, \eta)$ be a relatively measure-preserving G-map. Then π is an isomorphism.*

Proof. For $f \in L^\infty(X, \nu)$ define

$$F_f(x) = f(x) - D_\pi(\pi(x))(f)$$

and since (X, ν) is a SAT G-space

$$\sup_{g \in G} \left| g\nu(F_f) \right| = \|F_f\|$$

But we see that for η-almost every $y \in Y$

$$D_\pi(y)(F_f) = D_\pi(y)(f) - D_\pi(y)(D_\pi(\pi(\cdot))(f)) = D_\pi(y)(f) - D_\pi(y)(f) = 0$$

and we observe that for any $g \in G$, using that π is relatively measure-preserving,

$$g\nu(F_f) = \int_Y gD_\pi(y)(F_f)\ d\eta(y) = \int_Y D_\pi(gy)(F_f)\ d\eta(y) = \int_Y D_\pi(y)(F_f)\ dg\eta(y)$$

and therefore

$$g\nu(F_f) = \int_Y 0\ dg\eta(y) = 0$$

since $g\eta$ is in the same measure class as η. But this means that

$$\|F_f\| = \sup_{g \in G} \left| g\nu(F_f) \right| = \sup_{g \in G} 0 = 0$$

and therefore for ν-almost every x

$$0 = F_f(x) = f(x) - D_\pi(\pi(x))(f)$$

so $\mathbb{E}[f|\mathcal{F}(Y)] = f$ almost everywhere for every $f \in L^\infty(X, \nu)$ (where $\mathcal{F}(Y)$ is the Y-measurable functions treated as a sub-σ-algebra of the X-measurable functions). This means that every measurable function on (X, ν) is (Y, η)-measurable. Hence π is an isomorphism (of measure algebras hence models). □

So we now have that $(B, \nu) \simeq (B_1 \times B_2, \nu_1 \times \nu_2)$. We claim that (B_1, ν_1) is in fact the Poisson Boundary of (G_1, μ_1) (and likewise for G_2). To see this, observe that any bounded μ_1-harmonic function φ lifts to a $\mu_1 \times \mu_2$-harmonic function $\varphi'(g_1, g_2) = \varphi(g_1)$ and therefore for any $f \in L^\infty(PB(G_1, \mu_1))$ there is a corresponding $f^* \in L^\infty(B, \nu)$ obtained by, for $\omega \in (G_1 \times G_2)^{\mathbb{N}}$,

$$f^*([\omega]) = \overline{\varphi'}(\omega) = \overline{\varphi}(\omega_1) = f([\omega_1])$$

where ω_1 is the $G_1^{\mathbb{N}}$ coordinates (the actions commute so this is well-defined). Therefore $L^\infty(PB(G_1, \mu_1))$ forms a G_1-invariant (and of course also G_2-invariant) sub-σ-algebra of

$L^\infty(B,\nu)$ which sits inside $L^\infty(B_1,\nu_1)$. Hence $(B_1,\nu_1) \to PB(G_1,\mu_1)$ is a G_1-map. But this is necessarily a relatively measure-preserving extension since G_1 acts amenably on its Poisson Boundary and we have already seen that (B,ν) does admit nontrivial measure-preserving factors so, as above, $(B_1,\nu_1) \simeq PB(G_1,\mu_1)$ (treating them both as $G_1 \times G_2$-spaces with trivial G_2-action). Of course the same holds for G_2 and therefore we have shown that

$$PB(G_1 \times G_2, \mu_1 \times \mu_2) \simeq PB(G_1,\mu_1) \times PB(G_2,\mu_2)$$

with the product action.

4.5 Proximal Extensions

The usual counterpart to relative measure-preserving extensions are the so-called proximal extensions. Unfortunately, defining them requires moving to the category of (G,μ)-spaces, and therefore introducing a measure on the group.

Definition 4.10. Let $\pi : (X,\nu) \to (Y,\eta)$ be a G-map between (G,μ)-spaces. Then π is a **proximal map** and (X,ν) is a **proximal extension** of (Y,η) when $\pi : (X,\nu_\omega) \to (Y,\eta_\omega)$ is an isomorphism for $\mu^{\mathbb{N}}$-almost every $\omega \in \mathbb{G}^N$ (where ν_ω are the conditional measures).

Proximal extensions are also sometimes referred to as boundary extensions. It is not difficult to see that proximal extensions satisfy that: the composition of proximal extensions is proximal; if a composition of two extensions is proximal then each of the extensions is proximal; and a space is a boundary if and only if it is a proximal extension of a point. We will not prove these facts here as we will not make use of proximal extensions.

Proposition 4.5.1. *Let $\pi : (X,\nu) \to (Y,\eta)$ be an extension of (G,μ)-stationary dynamical systems. If π is both measure-preserving and proximal then π is an isomorphism.*

Proof. Let $D_\pi : Y \to P(X)$ be the disintegration map which is G-equivariant since π is measure-preserving. For $\mu^{\mathbb{N}}$-a.e. ω,

$$\begin{aligned}\int_Y D_\pi(y)\ d\eta_\omega(y) &= \lim_n \int_Y D_\pi(\omega_1 \cdots \omega_n y)\ d\eta(y) = \lim_n \int_Y \omega_1 \cdots \omega_n D_\pi(y)\ d\eta(y) \\ &= \lim_n \omega_1 \cdots \omega_n \int_Y D_\pi(y)\ d\eta(y) = \lim_n \omega_1 \cdots \omega_n \nu = \nu_\omega\end{aligned}$$

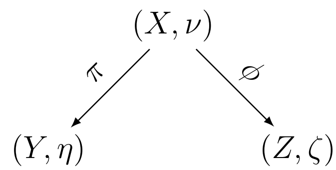
That is, measure-preserving on π implies that the disintegration map for $(X,\nu_\omega) \to (Y,\eta_\omega)$ is the same map as the disintegration map for $(X,\nu) \to (Y,\eta)$.

On the other hand, for $\mu^{\mathbb{N}}$-a.e. ω, the map $\pi : (X,\nu_\omega) \to (Y,\eta_\omega)$ is one-one since π is proximal. Therefore the disintegration map for $\pi : (X,\nu_\omega) \to (Y,\eta_\omega)$ sends points to point masses almost surely.

But the disintegration map from ν_ω to η_ω agrees almost surely with D_π. Therefore for $\mu^{\mathbb{N}}$-a.e. ω and η_ω-a.e. y we have that $D_\pi(y)$ is a point mass. Since $\int \eta_\omega\ d\mu^{\mathbb{N}}(\omega) = \eta$ this means that for η-a.e. y we have $D_\pi(y)$ is a point mass. So π is an isomorphism. ∎

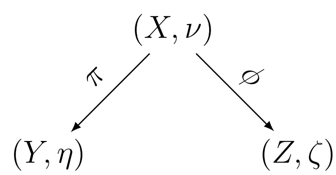
4.6 Common Factors

Let (X, ν) be a G-space. Suppose we have two G-factors (Y, η) and (Z, ζ) of (X, ν) in the sense that $\pi : (X, \nu) \to (Y, \eta)$ and $\phi : (X, \nu) \to (Z, \zeta)$ are both G-maps. That is,

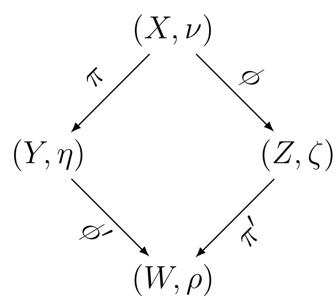


The aim of this section is to show that there then exists a G-space (W, ρ) which is a **common factor** of both (Y, η) and (Z, ν) in the sense that there is a commutative diagram of G-spaces and G-maps:

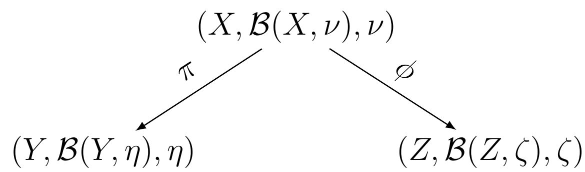
Theorem 4.11. *Let $(X, \nu), (Y, \eta), (Z, \zeta)$ be G-spaces and let $\pi : (X, \nu) \to (Y, \eta)$ and $\phi : (X, \nu) \to (Z, \zeta)$ be G-maps. That is, given the following diagram of G-spaces and G-maps:*



Then there exists a G-space (W, ρ) and G-maps $\phi' : (Y, \eta) \to (W, \rho)$ and $\pi' : (Z, \zeta) \to (W, \rho)$ such that the following diagram commutes:



In order to construct this common factor, it will be easier to work with the factors as invariant sub-σ-algebras (Proposition 4.1.1). To that end, we will now explicitly write out the σ-algebra of measurable sets. We will use $\mathcal{B}$ to denote the measure algebra of a G-space (which is isomorphic to the Borel sets of any compact model for that space). So our diagram becomes:



Let $\mathcal{F}(X)$ be the measurable sets of (X, ν), that is $\mathcal{F}(X) = \mathcal{B}(X, \nu)$. Let $\mathcal{F}(Y) \subseteq \mathcal{F}(X)$ be defined by

$$\mathcal{F}(Y) = \{\pi^{-1}(B) : B \in \mathcal{B}(Y, \eta)\}$$

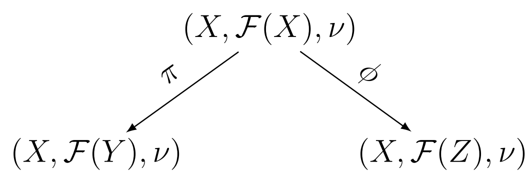
and likewise let $\mathcal{F}(Z) \subseteq \mathcal{F}(X)$ by

$$\mathcal{F}(Z) = \{\phi^{-1}(B) : B \in \mathcal{B}(Z, \zeta)\}$$

Then according to Proposition 4.1.1 we know that

$$(Y, \mathcal{B}(Y, \eta), \eta) \simeq (X, \mathcal{F}(Y), \nu) \qquad \text{and} \qquad (Z, \mathcal{B}(Z, \zeta), \zeta) \simeq (X, \mathcal{F}(Z), \nu)$$

where the isomorphism is a G-map of measure algebras. Identifying π and ϕ over these isomorphisms our diagram has now become:



Note that a compact model for $(X, \mathcal{F}(X), \nu)$ is not strictly speaking a compact model for $(X, \mathcal{F}(Y), \nu)$ since $\mathcal{F}(Y)$ will not separate points in X. However, a compact model for (X, ν) will be a compact space on which the measure algebra $\mathcal{F}(Y)$ is realized as a sub-algebra of the Borel sets.

Now we are ready to define the common factor. Let $\mathcal{F} \subseteq \mathcal{F}(X)$ be defined by

$$\mathcal{F} = \mathcal{F}(Y) \cap \mathcal{F}(Z)$$

This is easily seen to be a G-invariant sub-σ-algebra since each of $\mathcal{F}(Y)$ and $\mathcal{F}(Z)$ are (for example, if $B \in \mathcal{F}$ then $B \in \mathcal{F}(Y)$ and $B \in \mathcal{F}(Z)$ so $gB \in \mathcal{F}(Y)$ and $gB \in \mathcal{F}(Z)$ for all $g \in G$ by the invariance of those algebras and therefore $gB \in \mathcal{F}$).

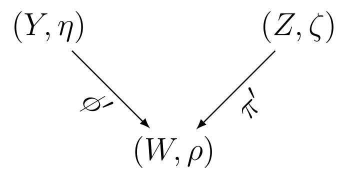
We have already identified the maps over the isomorphisms, so in fact $\pi : (X, \mathcal{F}(X), \nu) \to (X, \mathcal{F}(Y), \nu)$ is a homomorphism of measure algebras $\pi^* : \mathcal{F}(X) \to \mathcal{F}(Y)$ (meaning that for an $\mathcal{F}(X)$-measurable function f that $\pi^*(f)$ is $\mathcal{F}(Y)$-measurable where $\pi^*(f) = \mathbb{E}[f|\mathcal{F}(Y)] = D_\pi(\pi(\cdot))(f))$ which is G-equivariant such that $\pi\big|_{\mathcal{F}(Y)}$ is the identity. Likewise $\phi : \mathcal{F}(X) \to \mathcal{F}(Z)$ is G-equivariant and $\phi\big|_{\mathcal{F}(Z)}$ is the identity.

Define the map $\phi' : \mathcal{F}(Y) \to \mathcal{F}(Y) \cap \mathcal{F}(Z)$ by

$$\phi' = \phi\big|_{\mathcal{F}(Y)}$$

that is, ϕ' is the same map as ϕ on measure algebras but is only defined on the $\mathcal{F}(Y)$ measurable sets. Clearly this is is a G-equivariant map and its restriction to $\mathcal{F} = \mathcal{F}(Y) \cap \mathcal{F}(Z)$ is the identity (we are of course relying on the G-invariance of all these sigma-algebras for what we are writing to make sense). Likewise, define $\pi' : \mathcal{F}(Z) \to \mathcal{F}$ by $\pi' = \pi\big|_{\mathcal{F}(Z)}$.

Now define (W, ρ) to be a compact model for $(X, \mathcal{F}, \nu)$, which exists by Proposition 4.1.1 since $\mathcal{F}$ is a G-invariant sub-σ-algebra. This is a G-factor and (as G is locally compact) there is then a compact model. Therefore we have constructed the diagram:



as intended.

The only remaining property to show is that the composition maps commute. To see this, we make use of the uniqueness of the disintegration maps. Specifically, if $\pi'\phi \neq \phi'\pi$ then they must have distinct disintegration maps since then for some $x \in X$ it must be that $\pi'(\phi(x)) \neq \phi'(\pi(x))$ and so the disintegrations over those two points must have distinct supports.

Let f be an $\mathcal{F}(X)$-measurable function on X. Consider the function

$$F(x) = D_{\phi'\pi}(\phi'(\pi(x)))(f)$$

In terms of conditional expectation this is simply

$$F(x) = \mathbb{E}[\mathbb{E}[f|\mathcal{F}(Y)]|\mathcal{F}(Z)]$$

and also consider the function

$$F'(x) = D_{\pi'\phi}(\pi'(\phi(x)))(f)$$

which has the property that

$$F(x) = \mathbb{E}[\mathbb{E}[f|\mathcal{F}(Z)]|\mathcal{F}(Y)]$$

Now the uniqueness of conditional expectation tells us that

$$\mathbb{E}[f|\mathcal{F}(Y) \cap \mathcal{F}(Z)]$$

is the unique $\mathcal{F}$-measurable function such that $\nu(f) = \nu(\mathbb{E}[f|\mathcal{F}(Y) \cap \mathcal{F}(Z)])$. Of course, $F(x)$ is $\mathcal{F}$-measurable since the inner conditional expectation forces it to be $\mathcal{F}(Y)$-measurable and the outer forces it to be $\mathcal{F}(Z)$-measurable. Using the definition of conditional expectation (twice)

$$\nu(F) = \nu(\mathbb{E}[\mathbb{E}[f|\mathcal{F}(Y)]|\mathcal{F}(Z)]) = \nu(\mathbb{E}[f|\mathcal{F}(Y)]) = \nu(f)$$

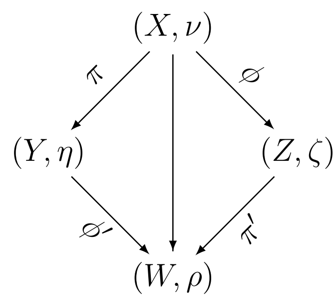
and therefore by the uniqueness of conditional expectation (which is of course equivalent to that of disintegration maps)

$$F(x) = \mathbb{E}[f|\mathcal{F}(Y) \cap \mathcal{F}(Z)]$$

but then by the same reasoning

$$F'(x) = \mathbb{E}[f|\mathcal{F}(Y) \cap \mathcal{F}(Z)] = F(x)$$

and hence $F = F'$ which in turn means that $\pi'\phi = \phi'\pi$ at the level of invariant σ-algebras on X. Therefore $\pi'\phi = \phi'\pi$ at the level of measure algebras hence almost everywhere on compact models, that is, as G-maps. Thus the diagram commutes as claimed and we can write without issue



meaning that (W, ρ) is the common factor of (Y, η) and (Z, ζ) we were aiming for.

4.7 Maximal Relative Factors

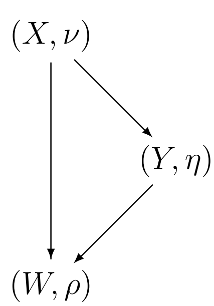
We return now to the Radon-Nikodym factor, which is the maximal factor for relatively measure-preserving extensions.

4.7.1 The Maximal Measure-Preserving Factor

Let (X, ν) be a G-space and let $\pi : (X, \nu) \to (Y, \eta)$ and $\phi : (X, \nu) \to (Z, \zeta)$ be *relatively measure-preserving* G-maps between G-spaces. Construct, as in the previous section, the common factor of (Y, η) and (Z, ζ) and denote it by (W, ρ).

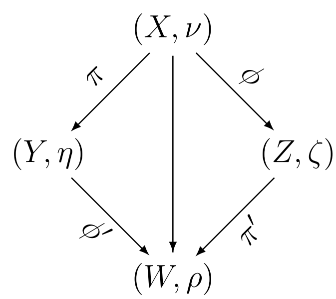
We can rephrase our results above on relatively measure-preserving factors by saying that:

Theorem 4.12. *Let $(X, \nu), (Y, \eta), (W, \rho)$ be G-spaces and $\pi : (X, \nu) \to (Y, \eta)$ and $\phi : (Y, \eta) \to (W, \rho)$ and define the composition map $\phi\pi : (X, \nu) \to (W, \rho)$ by composition. That is, form the commutative diagram:*



Then the morphism on the left is relatively measure-preserving if and only if both on the right are.

Therefore given the diagram



such that π and ϕ are relatively measure-preserving, it is enough to show that π' (equivalently ϕ') is also relatively measure-preserving. We will defer this for now. Then in fact all the morphisms in the above diagram are relatively measure-preserving by the preceding Theorem so in particular (W, ρ) is a relatively measure-preserving factor of (X, ν).

Since then any two relatively measure-preserving factors have a common factor which is also measure-preserving, there is necessarily, by abstract considerations, a maximal measure-preserving factor in the sense that it is a relatively measure-preserving factor of all the measure-preserving factors of (X, ν) (including (X, ν) itself). Precisely speaking, this is a type of direct limit construction:

Theorem 4.13. *Let (X, ν) be a G-space. There exists a G-factor (Y, η) such that (Y, η) is a relatively measure-preserving factor of (X, ν) and such that any relatively measure-preserving factor (Z, ζ) of (X, ν) necessarily has (Y, η) as a relatively measure-preserving factor.*

Proof. Let (Z_q, ζ_q) be an enumeration of all relatively measure-preserving G-factors of (X, ν) (where q ranges over some ordinal). We will omit the measures for the rest of the proof in the interests of clarity. Note that since the measure algebra for X can be realized as the Borel sets of a compact metric space there is a set-theoretic bound on the number of such factors since each factor corresponds to an invariant sub-σ-algebra and since a compact metric space is second countable there is a countable collection of open sets which generate the Borel sets.

This set is naturally partially ordered by saying that $Z_q \leq Z_{q'}$ when Z_q is a G-factor of $Z_{q'}$. Since the Z_q are all relatively measure-preserving factors of (X, ν) the factor map $Z_{q'} \to Z_q$ is relatively measure-preserving as well. Now let Z_{q_n} be a chain in this set (that is $Z_{q_n} \to Z_{q_{n+1}}$ ranging over n in some ordinal). Let $\mathcal{F}_{q_n}$ be the corresponding invariant algebra and observe that $\bigcap_n \mathcal{F}_{n_q}$ is then an invariant sub-σ-algebra which corresponds to some Z_q (by the previous work Z_q exists since the composition of all these maps is still measure-preserving). Hence any chain has a maximal element and therefore by Zorn's Lemma there is a maximal element in the partially ordered set. □

It remains only to show that ϕ' is relatively measure-preserving: observe that $D_{\phi'}$ is G-equivariant (that is $g^{-1} D_{\phi'}(gw) = D_{\phi'}(w)$ for all g) if and only if the conditional expectation is G-equivariant, that is for f an $\mathcal{F}(Y)$-measurable function on X we require that

$$\mathbb{E}[g \cdot f|\mathcal{F}(W)](gx) = \mathbb{E}[f|\mathcal{F}(W)](x)$$

Since f is already $\mathcal{F}(Y)$-measurable we know that

$$\mathbb{E}[f|\mathcal{F}(Z)] = \mathbb{E}[f|\mathcal{F}]$$

and since ϕ is relatively measure-preserving we know that

$$\mathbb{E}[g \cdot f|\mathcal{F}(Z)](gx) = \mathbb{E}[f|\mathcal{F}(Z)](x)$$

but this just means that

$$\mathbb{E}[g \cdot f|\mathcal{F}(W)](gx) = \mathbb{E}[f|\mathcal{F}(W)](x)$$

that is, ϕ' is relatively measure-preserving (of course this is not surprising as it is simply the restriction of ϕ to an invariant sub-algebra).

4.8 Radon-Nikdoym Derivatives

In order to understand quasi-invariant actions, the Radon-Nikodym derivatives of the translates of the measure are an obvious object of study:

Definition 4.14. Let (X, ν) be a G-space. The functions

$$\frac{dg\nu}{d\nu}(x)$$

are the **Radon-Nikodym derivatives** for the G-action on (X, ν).

Since $G \curvearrowright (X, \nu)$ quasi-invariantly, $g\nu << \nu$ and $\nu << g\nu$. The Radon-Nikodym Theorem then states that ${}^{dg\nu}/_{d\nu} \in L^1(X, \nu)$ exists almost everywhere.

4.8.1 Radon-Nikodym Derivatives of Stationary Measures

We pause briefly to point out a useful fact about the Radon-Nikodym derivatives of stationary measures, though we will not need this in what follows.

Proposition 4.8.1 (Kaimanovich)**.** *Let G be a countable group and $\mu \in P(G)$ a probability measure on it. Let (X, ν) be a (G, μ)-space, that is $\mu * \nu = \nu$. Then for all $g \in G$ and almost every $x \in X$*

$$\mu(g^{-1}) \leq \frac{dg\nu}{d\nu}(x) \leq \frac{1}{\mu(g)}$$

Proof. For almost every $x \in X$ we have that

$$1 = \frac{d\mu * \nu}{d\nu} = \int_G \frac{dg\nu}{d\nu}(x) \; d\mu(g) \geq \mu(g) \frac{dg\nu}{d\nu}(x)$$

and therefore ${}^{dg\nu}/_{d\nu}(x) \leq {}^{1}/_{\mu(g)}$. Then, since this holds for any x,

$$\mu(g) \leq \frac{d\nu}{dg\nu}(g^{-1}x) = \frac{dg^{-1}\nu}{d\nu}(x)$$

□

When G is locally compact we can obtain a similar result using the same proof (replacing $\mu(g)$ by $\mu(K)$ where K is a compact set in G. We also remark that since $\mu * \mu * \nu = \nu$ as well one can in fact get a bound on all g not just on those in the support of μ.

4.8.2 The Radon-Nikodym Factor

As remarked previously, the maximal measure-preserving factor, called the Radon-Nikodym factor, is known to be the factor corresponding to the smallest sub-σ-algebra of $\mathcal{F}(X)$ such that ${}^{dg\nu}/_{d\nu}$ are measurable (for each g):

Theorem 4.15 (Kaimanovich-Vershik [KV83])**.** *Let (X, ν) be a G-space. The G-factor $(X, \mathcal{RN}, \nu)$, defined by taking $\mathcal{RN}$ to be the σ-algebra generated by the Radon-Nikodym derivatives ${}^{dg\nu}/_{d\nu}$ for all g, is the maximal measure-preserving factor of (X, ν) in the sense that if (Y, η) is a relatively measure-preserving factor of (X, ν) then $(X, \mathcal{RN}, \nu)$ is a relatively measure-preserving factor of (Y, η). In particular, the only relatively measure-preserving factor of $(X, \mathcal{RN}, \nu)$ is itself.*

Proof. We have already shown the existence of a maximal factor. Let $\pi : (X, \nu) \to (Y, \eta)$ be a relatively measure-preserving G-map of G-spaces. Then for any $f \in L^\infty(X, \nu)$ we have that

$$\begin{aligned}
g\nu(f) &= \int_X f(gx)\ d\nu(x) = \int_Y \int_X f(gx)\ dD_\pi(y)(x)\ d\eta(y) \\
&= \int_Y \int_X f(x)\ dgD_\pi(y)(x)\ d\eta(y) = \int_Y \int_X f(x)\ dD_\pi(gy)(x)\ d\eta(y) \\
&= \int_Y \int_X f(x)\ dD_\pi(y)(x)\ dg\eta(y) = \int_Y \int_X f(x)\ dD_\pi(y)(x) \frac{dg\eta}{d\eta}(y)\ d\eta(y) \\
&= \int_Y \int_X f(x) \frac{dg\eta}{d\eta}(\pi(x))\ dD_\pi(y)(x)\ d\eta(y) = \int_Y D_\pi(y)(f \frac{dg\eta}{d\eta} \circ \pi)\ d\eta(y)
\end{aligned}$$

and also that

$$g\nu(f) = \nu(f \frac{dg\nu}{d\nu}) = \int_Y D_\pi(y)(f \frac{dg\nu}{d\nu})\ d\eta(y)$$

and therefore, since this holds for all f, we have that

$$\frac{dg\nu}{d\nu} = \frac{dg\eta}{d\eta} \circ \pi$$

for almost every x. In particular this means that ${dg\nu}/{d\nu}$ is $\mathcal{F}(Y)$-measurable since it is π-invariant.

Therefore, the Radon-Nikodym derivatives are measurable with respect to any invariant sub-σ-algebra arising from a relatively measure-preserving map. In particular, they are all measurable with respect to the maximal measure-preserving factor.

Conversely, define $\mathcal{RN}$ to be the σ-algebra generated by the Radon-Nikodym derivatives. Clearly $\mathcal{RN}$ is G-invariant since

$$g \cdot \frac{dh\nu}{d\nu} = \frac{dgh\nu}{dg\nu} = \frac{dgh\nu}{d\nu}\frac{d\nu}{dg\nu}$$

and each of the two functions on the right is in $\mathcal{RN}$. Hence $\mathcal{RN}$ defines a G-factor of (X, ν). Let π be the factor map to a compact model (Y, η) for this algebra and observe that

$$\frac{dg\nu}{d\nu}(x) = \frac{dg\eta}{d\eta}(\pi(x))$$

since the Radon-Nikodym derivatives are measurable. By reversing the above argument we see that π is relatively measure-preserving and therefore $(X, \mathcal{RN}, \nu)$ is a relatively measure-preserving factor of (X, ν).

Since $(X, \mathcal{RN}, \nu)$ would then be a relatively measure-preserving factor of every relatively measure-preserving factor, but on the other hand the maximal measure-preserving factor must map to it, we have shown that the factor corresponding to the algebra of Radon-Nikodym derivatives is the maximal factor as claimed. □

4.9 Structure Theory for Stationary Systems

We conclude our discussion of factors and extensions by considering a possible structure theory for stationary dynamical systems. Furstenberg's structure theory for measure-preserving systems, developed in connection with multiple recurrence, gives a canonical structure for any measure-preserving system as a weakly mixing extension of a (transfinite) tower of compact extensions of a point. Ideally one would like a similar statement for quasi-invariant systems and we return to this question in the next chapter.

4.9.1 Furstenberg-Glasner Structure Theorem

Furstenberg and Glasner in [FG10] have established a partial structure theorem for stationary dynamical systems. Specifically, they have shown that for an arbitrary (G, μ)-space (X, ν) there exists canonically defined (G, μ)-spaces (X^*, ν^*) and (Y, η) such that the following diagram of G-maps exists

$$\begin{array}{ccc} & (X^*, \nu^*) & \\ \swarrow & & \searrow \\ (X, \nu) & & (Y, \eta) \end{array}$$

such that

- $(X^*, \nu^*) \to (X, \nu)$ is a proximal factor map
- $(X^*, \nu^*) \to (Y, \eta)$ is a relatively measure-preserving factor map
- (Y, η) is a proximal (G, μ)-space (boundary)

That is to say, any stationary dynamical system can be written as a proximal factor of a stationary system that is a measure-preserving extension of a proximal space.

4.9.2 Structure Conjecture

The defect in the above theorem is that one must take an extension of the original space first and then one has a canonical structure theory. Ideally, a structure theory involving only factors would exist. There is a well-known and long-standing question (see e.g. Furman [Fur02]) as to whether measure-preserving extensions and boundaries can somehow be used as the building blocks of a structure theory for stationary systems.

II

Representation Theory

Chapter 5

Unitary Representations

Linear representations play a key role in the study of groups, forming the foundation of quantum mechanics and harmonic analysis. Fourier analysis can be viewed as a special case of representation theory. We will focus on the most well-developed part of this theory: representations of locally compact groups as (strongly) continuous unitary operators on Hilbert spaces.

The material in this chapter is all well-known and classical or due to Shalom, except where indicated. This chapter is in essence the background preparation for the next chapters where we present our new results on cohomology of group actions.

5.1 Representations on Hilbert Spaces

Here, and throughout, $\mathcal{H}$ will denote a Hilbert space (which we will assume separable when convenient though this is generally not necessary). $\mathcal{B}(\mathcal{H})$ denotes the bounded operators on the Hilbert space and $\mathcal{U}(\mathcal{H})$ the unitary operators: $\|u\| = 1$.

Definition 5.1. Let G be a topological group. A homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ is called a **unitary representation** of G on the Hilbert space $\mathcal{H}$. The representation is (strongly) continuous when π is a continuous map from the topology of G to the (strong) topology on $\mathcal{U}(\mathcal{H})$, that is for all $x \in \mathcal{H}$ the map $g \mapsto \pi(g)x$ is norm continuous.

That the map be a homomorphism means that $\pi(gh) = \pi(g)\pi(h)$ where multiplication of unitary operators is by composition and further that $\pi(g^{-1}) = \pi(g)^{-1}$ so in particular $\pi(e)$ is the identity operator. That π maps G into $\mathcal{U}(\mathcal{H})$ means that $\|\pi(g)v\| = \|v\|$ for all $v \in \mathcal{H}$ and $g \in G$.

5.1.1 Notational Conventions

We will use the phrase "let π be a **representation** of a group G" to always mean that G is locally compact and π is a strongly continuous unitary representation on a Hilbert space. Of course, countable groups with the discrete topology fall into the category of locally compact groups so these are included.

5.1.2 Irreducible Representations

Definition 5.2. Let π be a representation of a group G. Then π is **irreducible** when the only invariant subspaces are trivial.

That is, if $\pi : G \to \mathcal{U}(\mathcal{H})$ and $V \subseteq \mathcal{H}$ is a subspace such that $\pi(g)V \subseteq V$ for all g then $V = \{0\}$ or $V = \mathcal{H}$.

A representation is **finite-dimensional** when the underlying Hilbert space is finite-dimensional (i.e. is matrices).

It is easy to see that if a representation admits an invariant subspace then the complement of that subspace is also invariant. This in turn leads to the fact that any representation can be decomposed as the direct sum or direct integral of irreducible representations. In particular, any finite-dimensional representation is the finite direct sum of irreducible representations.

5.1.3 Faithful Representations

Occasionally we will want to know that a unitary representation "shows the whole group" in the sense that it is not trivial on some subgroup. Of course the kernel of a representation would necessarily be a normal subgroup so in the case of simple groups this is not much of an issue. A representation is called faithful when the kernel is trivial:

Definition 5.3. A representation π of G is called **faithful** when the kernel of π is trivial. That is, if $\pi(g)$ is the identity operator then g is the identity element.

5.1.4 The Howe-Moore Theorem

A useful fact, which we will make use of in our results later, is that representations of Lie groups which do not have almost invariant vectors (see Section 5.5) are mixing ($\pi(g) \to 0$ weakly as $g \to \infty$ in the sense of leaving compact sets). The "Howe-Moore Theorem" (see e.g. Zimmer [Zim84] Chapter 2) states:

Theorem 5.4 (Howe-Moore)**.** *Let G be a (finite) product of semisimple Lie (or p-adic Lie) groups, none of which are compact, and let π be a unitary representation of G such that π restricted to each semisimple factor does not have invariant vectors. Then the matrix coefficients vanish at ∞, that is, for any vectors v and w we have $\langle \pi(g)v, w\rangle \to 0$ as $g \to \infty$ (meaning as g leaves compact subsets of G).*

In fact Zimmer states the Lie case in Chapter 2 of [Zim84] and the general p-adic result in Chapter 10. The reader is referred there for some background and a proof.

5.2 Cocycles and Cohomology

Cohomology is the main tool in representation theory, especially in the realm of rigidity results. We present the basic ideas and results that we will use later. We adopt the approach of treating cocycles as maps satisfying a certain identity, the cocycle identity, rather than the more traditional cohomology and cochain approach. This simplifies our exposition greatly, and we remark only for the reader familiar with the usual cohomology that by cocycle we strictly speaking mean 1-cocycle.

5.2.1 Cocycles

Cocycles play a major role in rigidity theory and will play a crucial role in our results in the following chapters.

Definition 5.5. Let π be a representation of a group G on a Hilbert space $\mathcal{H}$. A continuous map $\beta : G \to \mathcal{H}$ is called a **cocycle** when it satisfies the **cocycle identity**:

$$\beta(gh) = \pi(g)\beta(h) + \beta(g)$$

for all $g, h \in G$.

An immediate consequence of this identity is that

$$\beta(e) = \beta(e^2) = \pi(e)\beta(e) + \beta(e) = \beta(e) + \beta(e) = 2\beta(e)$$

and so $\beta(e) = 0$ for any cocycle. Therefore

$$\pi(g)\beta(g^{-1}) = \beta(gg^{-1}) - \beta(g) = -\beta(g)$$

for all $g \in G$.

5.2.2 Coboundaries

The first obvious method for creating cocycles is to form coboundaries:

Definition 5.6. Let π be a representation of a group G on a Hilbert space $\mathcal{H}$. Let $v \in \mathcal{H}$ be a fixed vector. Define

$$\beta_v(g) = \pi(g)v - v$$

Then β_v is a **coboundary**.

Observe that

$$\begin{aligned}\beta_v(gh) &= \pi(gh)v - v = \pi(g)\pi(h)v - v = \pi(g)\pi(h)v - \pi(g)v + \pi(g)v - v \\ &= \pi(g)\big(\pi(h)v - v\big) + \pi(g) - v = \pi(g)\beta_v(h) + \beta_v(g)\end{aligned}$$

so coboundaries are in fact cocycles.

5.2.3 Cohomology

Definition 5.7. Let π be a representation of a group G on a Hilbert space $\mathcal{H}$. Denote by

$$Z^1(G, \pi) = \{\beta : G \to \mathcal{H} \mid \beta \text{ is a cocycle}\}$$

and

$$B^1(G, \pi) = \{\beta \in Z^1(G, \pi) \mid \beta \text{ is a coboundary}\}$$

The **cohomology** of π is then defined as

$$H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi)$$

the equivalence classes of cocycles modulo coboundaries.

The superscript 1 denotes that what we have defined here is the first cohomology. We will not need higher cohomologies for our results so we will not go into details about them.

Definition 5.8. Two cocycles $\beta_1, \beta_2 \in Z^1(G, \pi)$ are **cohomologous** when their difference is a cocycle: $\beta_1 - \beta_2 \in B^1(G, \pi)$. That is, when they are in the same cohomology class.

5.2.4 Cocycles on Compact Groups

We will make use of some standard and easy facts about cohomology on compact groups:

Lemma 5.2.1. *Let K be a finite (or compact) group and $b : K \to \mathcal{H}$ be a π-cocycle. Then b is a coboundary.*

Proof. Let m be the Haar measure on K (which we may take to have total mass one since K is finite or compact). Set

$$v = \int_K b(k)\ dm(k)$$

and observe that for any $k \in K$

$$\int_K b(kk')\ dm(k') = \int_K b(k) + \pi(k)b(k')\ dm(k') = b_k + \pi(k) \int_K b(k')\ dm(k') = b(k) + \pi(k)v$$

and on the other hand, by the invariance of m,

$$\int_K b(kk')\ dm(k') = \int_K b(k')\ dkm(k') = \int_K b(k')\ dm(k') = v$$

and therefore

$$b(k) = v - \pi(k)v$$

meaning b is a coboundary. □

Lemma 5.2.2. *Let K be a compact group and $\varphi : K \to \mathbb{R}$ an additive homomorphism. Then φ is trivial.*

Proof. Write m for the Haar measure on K (which is a finite measure since K is compact, hence all integrals below are finite). Then for any $k_0 \in K$,

$$\int_K \varphi(k_0 k)\ dm(k) = \int_K \varphi(k)\ dk_0 m(k) = \int_K \varphi(k)\ dm(k)$$

by the left invariance of Haar measure. But also

$$\int_K \varphi(k_0 k)\ dm(k) = \varphi(k_0) + \int_K \varphi(k)\ dm(k)$$

since φ is a homomorphism. Therefore $\varphi(k_0) = 0$. □

5.3 Reduced Cohomology

Cohomology is an algebraic structure that is quite useful in the study of groups and representations. The notion of reduced cohomology brings analysis and analytic techniques to bear on cohomology by introducing a topology on cocycles.

Definition 5.9. Let π be a representation of a group G. Denote by $Z^1(G,\pi)$ the space of cocycles and $B^1(G,\pi)$ the subspace of coboundaries. These are vector spaces over $\mathbb{R}$ or $\mathbb{C}$ (according to whether the Hilbert space is real or complex).

Topologize $Z^1(G,\pi)$ with uniform convergence on compact sets: $\beta_n \to \beta$ means that $\beta_n(g) \to \beta(g)$ in the strong topology on the Hilbert space uniformly over compact sets in G.

Let $\overline{B^1(G,\pi)}$ be the closure in this topology of the coboundaries. The **reduced cohomology** is defined as

$$\overline{H^1}(G,\pi) = Z^1(G,\pi)/\overline{B^1(G,\pi)}$$

Definition 5.10. Two cocycles are **almost cohomologous** when they are in the same reduced cohomology class. That is, β_1 and β_2 are almost cohomologous when there exists vectors v_n such that

$$\beta_1(g) + \pi(g)v_n - v_n \to \beta_2(g)$$

as $n \to \infty$ in the strong topology uniformly over compact sets in G.

Definition 5.11. A cocycle $\beta \in \overline{B^1(G,\pi)}$ is called an **almost coboundary**: β is almost cohomologous to zero.

Reduced cohomology is much more useful for our results than ordinary cohomology and in fact we will not make (much) use of ordinary cohomology in what follows.

5.4 Affine Actions

A unitary representation of a group G on a Hilbert space $\mathcal{H}$ can be thought of as defining a linear action of the group G on the Hilbert space. Concretely, $g \cdot v = \pi(g)v$ is readily seen to define a group action. It is linear in the sense that $g \cdot 0 = 0$ (meaning the identity element is preserved) and $g \cdot (v+w) = g \cdot v + g \cdot w$ (meaning the addition operation on the vector space is preserved).

5.4.1 Actions Associated to Cocycles

Definition 5.12. Let π be a representation of a group G on a Hilbert space $\mathcal{H}$ and let $\beta : G \to \mathcal{H}$ be a cocycle. The **affine isometric action associated to** β is given by

$$g \cdot v := \pi(g)v + \beta(g)$$

To see that this is an action, observe that

$$\begin{aligned}(gh) \cdot v &= \pi(gh)v + \beta(gh) = \pi(g)\pi(h)v + \pi(g)\beta(h) + \beta(g) \\ &= \pi(g)\big(\pi(h)v + \beta(h)\big) + \beta(g) = \pi(g)(h \cdot v) + \beta(g) = g \cdot (h \cdot v)\end{aligned}$$

as required. This action is not linear since

$$g \cdot v + g \cdot w - g \cdot (v + w) = \beta(g) \neq 0$$

for a general cocycle β. It is affine in the sense that it is linear up to a "constant". Furthermore

$$g \cdot v - g \cdot w = \pi(g)v + \beta(g) - \pi(g)w - \beta(g) = \pi(g)(v - w)$$

and therefore, since $\pi(g) \in \mathcal{U}(\mathcal{H})$,

$$\|g \cdot v - g \cdot w\| = \|\pi(g)(v - w)\| = \|v - w\|$$

meaning the action is isometric.

5.4.2 Cocycles Associated to Actions

Definition 5.13. Let G be a group and X a topological vector space such that G acts continuously on X. The action is said to be **affine** when

$$g \cdot (x + y) = g \cdot x + g \cdot y - c(g)$$

for some (continuous) map $c : G \to X$.

Let $\cdot$ denote an arbitrary (continuous) affine action of G on $\mathcal{H}$. Then the associated map c must satisfy that

$$c(g) = g \cdot v + g \cdot w - g \cdot (v + w)$$

for all $v, w \in \mathcal{H}$. Therefore

$$c(g) = g \cdot v + g \cdot 0 - g \cdot (v + 0) = g \cdot v + g \cdot 0 - g \cdot v = g \cdot 0$$

Then we can define a map $\pi : G \to \mathcal{B}(\mathcal{H})$ by

$$\pi(g)v := g \cdot v - c(g)$$

and obtain that $\pi(g)0 = g \cdot 0 - c(g) = 0$ and

$$\pi(g)(v + w) = g \cdot (v + w) - c(g) = g \cdot v + g \cdot w - 2c(g) = \pi(g)v + \pi(g)w$$

so this is a linear map. Moreover,

$$c(gh) - \pi(g)c(h) - c(g) = (gh) \cdot 0 - g \cdot (h \cdot 0) + g \cdot 0 - g \cdot 0 = 0$$

so c is a π-cocycle. Then we see that

$$\begin{aligned}\pi(g)\pi(h)v &= \pi(g)(h \cdot v - c(h)) = \pi(g)(h \cdot v) - \pi(g)c(h) \\ &= g \cdot (h \cdot v) - c(g) - \pi(g)c(h) = (gh) \cdot v - c(gh) = \pi(gh)v\end{aligned}$$

so π defines an action of G and therefore π is a representation of G. Now

$$\|\pi(g)(v - w)\| = \|\pi(g)v - \pi(g)w\| = \|(g \cdot v - c(g)) - (g \cdot w - c(g))\| = \|g \cdot v - g \cdot w\|$$

and therefore the affine action is isometric if and only if π maps G into the unitary operators.

The correspondence between affine isometric actions and cocycles of unitary representations is an indication of how cocycles come into play when studying group actions (both those on Hilbert spaces and those induced by a quasi-invariant action to the space of probability measures).

5.5 Almost Invariant Vectors

Almost invariant vectors, which play a defining role for property (T), turn out to have a decisive role in the theory of reduced cohomology. We describe now how fixed and almost fixed points relate to affine actions and how almost invariant vectors influence cohomology.

5.5.1 Fixed Points

We first explore what happens when there is a fixed point for an action associated to a cocycle:

Proposition 5.5.1. *Let π be a unitary representation of a group G on a Hilbert space $\mathcal{H}$ and $\beta : G \to \mathcal{H}$ a cocycle. Then β is a coboundary if and only if the associated action has a fixed point.*

Proof. Assume β is a coboundary: $\beta(g) = \pi(g)v_0 - v_0$ for some $v_0 \in \mathcal{H}$. Then the associated action is given by

$$g \cdot v = \pi(g)v + \beta(g) = \pi(g)v + \pi(g)v_0 - v_0 = \pi(g)(v + v_0) - v_0$$

and therefore

$$g \cdot (-v_0) = \pi(g)(-v_0 + v_0) - v_0 = -v_0$$

so $-v_0$ is a fixed point for the action.

Conversely, let β be an arbitrary cocycle assume that v_0 is a fixed point for the associated affine action. Then the associated action to β satisfies

$$v_0 = g \cdot v_0 = \pi(g)v_0 + \beta(g)$$

and therefore

$$\beta(g) = \pi(g)(-v_0) - (-v_0)$$

so β is a cocycle. □

5.5.2 Almost Fixed Points

Weakening slightly the idea of a fixed point for an action, we consider:

Definition 5.14. Let (V, d) be a metric space and $G \curvearrowright V$ be an affine action. The action **does not admit almost fixed points** when there exists a compact set $Q \subseteq G$ and $\epsilon > 0$ such that for all $v \in V$ there exists $g \in Q$ such that $d(gv, v) \geq \epsilon$.

That is, "an" almost fixed point for a compact set $Q \subseteq G$ is a sequence $\{v_n\}$ in V such that $d(gv_n, v_n) \to 0$ for all $g \in Q$.

Definition 5.15. An action of a locally compact group G on a metric space is called **uniform** when there are no almost fixed points.

5.5.3 Almost Invariant Vectors

Related to the notion of almost fixed points is the analogous concept for vector spaces:

Definition 5.16. Let π be a unitary representation of a locally compact group G on a Hilbert space $\mathcal{H}$. The action **does not admit almost invariant vectors** when there exists a compact $Q \subseteq G$ and $\epsilon > 0$ such that for all $v \in \mathcal{H}$ there exists $g \in Q$ such that $\|\pi(g)v - v\| \geq \epsilon \|v\|$.

One reason almost invariant vectors are of interest is that:

Proposition 5.5.2. *Let π be a unitary representation of a locally compact group G on a Hilbert space. If π does not admit almost invariant vectors then $H^1(G, \pi) = \overline{H^1}(G, \pi)$.*

That is, the reduced cohomology differs from the ordinary cohomology only if there are almost invariant vectors for the representation. Equivalently, the subspace of coboundaries is closed in the space of cocycles (under strong convergence uniformly on compact subsets of the group) if there are no almost invariant vectors.

This is a well-known (and fairly easy) result that appears in the work of Guichardet [Gui80] and Delorme [Del77] (Delorme in fact shows the converse which is somewhat more involved).

Proof. First observe that the only way that $H^1(G, \pi) \neq \overline{H^1}(G, \pi)$ is when there exists two cocycles β and β' which are almost cohomologous but not cohomologous. Then $\beta - \beta'$ is a cocycle which is almost cohomologous to zero but not cohomologous to zero. That is, $\beta - \beta' \in \overline{B^1}(G, \pi)$ but is not a coboundary. Hence all we need to show is that $B^1(G, \pi) = \overline{B^1}(G, \pi)$.

Let $\beta \in \overline{B^1}(G, \pi)$. Then there exists coboundaries β_n such that $\beta_n \to \beta$ (strongly, uniformly over compact sets) by the definition of the topology on cocycles. Write $\beta_n(g) = \pi(g)v_n - v_n$ for some vectors v_n. Then we have that

$$\pi(g)v_n - v_n \to \beta(g)$$

uniformly over g in compact sets. Since π does not admit almost invariant vectors there is some compact set $Q \subseteq G$ and $\epsilon > 0$ and there exists $q_n \in Q$ such that

$$\|\pi(q_n)v_n - v_n\| \geq \epsilon \|v_n\|$$

Since Q is compact we can set $\alpha = \max_{q \in Q} \|\beta(q)\|$ to be some finite number. Then, since $\pi(q)v_n - v_n \to \beta(q)$ uniformly over $q \in Q$ (as Q is compact),

$$\limsup_{n\to\infty} \|v_n\| \leq \epsilon^{-1} \limsup_{n\to\infty} \|\pi(q_n)v_n - v_n\| \leq \epsilon^{-1} \max_{q\in Q} \|\beta(q)\| = \epsilon^{-1}\alpha$$

Then for any $g \in G$ we have that

$$\|\beta(g)\| = \lim_{n\to\infty} \|\pi(g)v_n - v_n\| \leq 2 \limsup_{n\to\infty} \|v_n\| \leq 2\epsilon^{-1}\alpha$$

so β is uniformly bounded. The associated affine action $g \cdot v = \pi(g)v + \beta(g)$ then has the property that

$$\|g \cdot v\| \leq \|\pi(g)v\| + \|\beta(g)\| \leq \|v\| + 2\epsilon^{-1}\alpha$$

and therefore the affine isometric action has bounded orbits.

Since Hilbert space is uniformly convex and the action is isometric, having a bounded orbit implies that there is a global fixed point for the isometric action. But this then means that β is a coboundary by Proposition 5.5.1. Hence any almost coboundary is necessarily a coboundary as claimed. □

5.6 Characterizing Property (T)

Akin to the characterization of amenability as being equivalent to every action on a metric space admitting an invariant probability measure (Theorem 1.37), the main result we will need about cohomology is due to Shalom and characterizes property (T) in terms of reduced cohomology.

The reader unfamiliar with property (T) should consult Appendix B: Amenability and Property (T) for the definition and some basic facts and properties of groups having property (T). The following is Theorem 6.1 in [Sha00a] (see also Theorem 4.2 in [Sha06]):

Theorem 5.17 (Shalom)**.** *Let G be a locally compact, second countable, compactly generated group. If G does not have property (T) then there exists a continuous unitary representation π of G such that $\overline{H^1}(G, \pi) \neq 0$. Moreover one may assume that this representation is irreducible.*

That is, property (T) is equivalent to the fact that every (irreducible) unitary representation on a Hilbert space does not admit any cocycles except those almost cohomologous to the trivial cocycle. Recall that property (T) implies compact generation so the requirement that the group be compactly generated in the above is not truly a restriction.

5.7 Cohomology and Subgroups

Relating the cohomology of a subgroup, such as a lattice or a commensurator, to the cohomology of the ambient group is a key feature of rigidity theory. One of our main results, the injectivity of reduced cohomology for dense subgroups of totally disconnected groups (Theorem 7.1), is such a statement. We now state some known results about this phenomenon that we will need in the following chapters.

5.7.1 Superrigidity for Reduced Cohomology

The "Superrigidity for Reduced Cohomology" is Theorem 4.1 of the work of Shalom [Sha00a] (combined with the discussion in section 10.4 of that paper on non-cocompact lattices):

Theorem 5.18 (Shalom)**.** *Let Γ be an irreducible integrable lattice in a (finite) product of at least two locally compact, second countable, compactly generated groups $G = \prod G_j$. Let $(\pi, \mathcal{H})$ be a unitary Γ-representation and $b : \Gamma \to \mathcal{H}$ be a π-cocycle. Then b is almost cohomologous to a a cocycle of the form $b_0 + b_1 + \cdots + b_k$ where b_0 takes values in the space of Γ-invariant vectors and each b_j takes values in a Γ-invariant subspace $\mathcal{H}_j$ on which the Γ-representation extends continuously to a G-representation which factors through a representation of G_j. Each b_j extends continuously to a cocycle depending only on G_j and b_0 extends continuously to a cocycle of G.*

Corollary 5.19. *Let Γ be an irreducible integrable lattice in $G = \prod G_j$ where each G_j is a locally compact, second countable, compactly generated group and let $(\pi, \mathcal{H})$ be a nontrivial irreducible unitary representation of Γ. Then there exists a representation σ of one of the factors G_j such that $\overline{H^1}(\Gamma, \pi) = \overline{H^1}(G_j, \sigma)$. In fact, $\sigma|_\Gamma = \pi$.*

Proof. Let $b : \Gamma \to \mathcal{H}$ be a cocycle. By the above Theorem b is almost cohomologous to a cocycle of the form $b_0 + b_1 + \cdots + b_k$ where each b_j takes values in distinct Γ-invariant subspaces. Since π is irreducible there can only be one such subspace and therefore b is almost cohomologous to b_j which is a cocycle for a representation of G that factors through G_j for some specific j (we cannot end up with b_0 since the representation is nontrivial). Moreover, another such cocycle b' cannot lead to a different j since that would also contradict the irreducibility of π. Call this G_j-representation σ. So $\sigma|_\Gamma = \pi$ and σ is a continuous extension of π to G_j.

Therefore we have a map from $\overline{H^1}(\Gamma, \pi) \to \overline{H^1}(G_j, \sigma)$. Now let b and b' be (Γ, π)-cocycles almost cohomologous to b_j and b'_j, respectively. Let B and B' be the continuous extensions of b_j and b'_j, respectively, to (G_j, σ)-cocycles. Suppose that B and B' are almost cohomologous. Then there is some sequence of vectors v_n such that $B(g) + \pi(g)v_n - v_n \to B'(g)$ uniformly over compact sets in G_j. But then $b_j(\gamma)+\pi(\gamma)v_n - v_n \to b'_j(\gamma)$ for each $\gamma \in \Gamma$ meaning that b_j and b'_j are almost cohomologous. This in turn means that b and b' are almost cohomologous since they are both in the equivalence class of b_j. Hence the map is injective.

To see the map is surjective, take any σ-cocycle and restrict it to the projection of Γ in G_j. Then treat this restriction as a $(\Gamma, \pi|_\Gamma)$-cocycle. Since the restriction of σ to the projection of Γ is π, this new cocycle is a π-cocycle. This new cocycle's reduced cohomology class must map to the original reduced cohomology class since it extends continuously to the original cocycle and we cannot have almost cohomologous G_j-cocycles with restrictions not almost cohomologous. □

The reader wishing to know more is referred to Shalom's work [Sha98], [Sha00a], [Sha00b] and [Sha06]. In particular, superrigidity can be used to show strong rigidity: that any isomorphism of lattices in products of groups extends to an isomorphism of the ambient groups. We also remark that Shalom has shown analogous statements of superrigidity and strong rigidity for commensurators of lattices though we will not make use of them here.

Chapter 6

Harmonic Cocycles

The main new idea in our work on representations is that of a harmonic cocycle. Our main contribution in this area, Theorem 6.2, states that each reduced cohomology class contains a unique harmonic representative. Material presented in this chapter is all new to this work except where stated.

6.1 Harmonicity

Harmonic maps have been studied in many contexts, and the idea of a harmonic cocycle appears in the work of Mok in the 1990s and in various places from then on, notably in Kleiner's work on Gromov's Theorem [Kle10].

Definition 6.1. Let $\mu \in P(G)$. A cocycle $\beta : G \to \mathcal{H}$ is **μ-harmonic** when for all $g \in G$,

$$\beta(g) = \int_G \beta(gg')\ d\mu(g')$$

Lemma 6.1.1. *A cocycle β is harmonic if and only if it is harmonic at the identity:*

$$\int_G \beta(g)\ d\mu(g) = 0 \quad \textit{if and only if} \quad \int_G \beta(gg')\ d\mu(g') = \beta(g) \textit{ for all } g \in G$$

Proof. By the cocycle identity:

$$\int_G \beta(gg')\ d\mu(g') = \int_G \beta(g) + \pi(g)\beta(g')\ d\mu(g') = \beta(g) + \pi(g) \int_G \beta(g')\ d\mu(g')$$

□

6.2 Harmonic Representatives Theorem

The main theorem in this chapter characterizes harmonic cocycles as the unique representatives of reduced cohomology classes. The consequences of this result appear later in this chapter and ultimately lead to progress on the Margulis-Zimmer conjecture (see Chapter 12: The Margulis-Zimmer Conjecture).

Theorem 6.2. *Let G be a locally compact second countable compactly generated group and $\pi : G \to B(\mathcal{H})$ a unitary representation of G on a Hilbert space and μ a compactly supported symmetric probability measure on G with support generating G. In any reduced cohomology class $[\beta] \in \overline{H^1}(G, \pi)$ there exists a unique μ-harmonic representative.*

This is in fact a special case of:

Theorem 6.3. *Let G be a locally compact second countable compactly generated group and $\pi : G \to B(\mathcal{H})$ a unitary representation of G on a Hilbert space and μ a symmetric probability measure on G with support generating G. In any reduced cohomology class $[\beta] \in \overline{H^1}(G, \pi)$ that contains an $L^2(\mu)$ representative there exists a unique μ-harmonic representative.*

6.3 Characterizing Property (T)

Before proving the Theorem, we state a consequence of it–a characterization of property (T) in terms of harmonic cocycles (the reader is referred to Appendix B: Amenability and Property (T) for definitions and facts about property (T)):

Theorem 6.4. *A locally compact second countable compactly generated group has property (T) if and only if the only harmonic cocycle (for every compactly supported measure generating the group and any representation) is zero.*

Specifically, property (T) is equivalent to: let π be any unitary representation of G on a Hilbert space $\mathcal{H}$ and let $\mu \in P(G)$ be a compactly supported probability measure on G with support generating G. Let $\beta : G \to \mathcal{H}$ be a π-cocycle such that β is μ-harmonic. Then $\beta = 0$.

Proof. Let G, π, μ and β as above. Assume that G has property (T). Then every cocycle is almost cohomologous to zero since the reduced cohomology is trivial (Theorem 5.17). Hence β is almost cohomologous to zero. Clearly zero is μ-harmonic. Hence by uniqueness (the Theorem in the previous section) $\beta = 0$.

Conversely, assume that G does not have property (T). Then there is some unitary representation π with nontrivial reduced cohomology. Let φ be a cocycle that is not almost cohomologous to zero. By the Theorem φ is almost cohomologous to a μ-harmonic cocycle β. But β cannot be zero since φ is not almost cohomologous to zero. □

6.4 Fixed Points

We first handle the case of $\mathcal{H}$ having fixed points under $\pi(G)$. Decompose $\mathcal{H} = \mathcal{H}_{fixed} \oplus \mathcal{H}_{unfixed}$ where $\mathcal{H}_{fixed}$ is the set of fixed points for π. These are clearly G-invariant subspaces of $\mathcal{H}$.

Let $\beta : G \to \mathcal{H}$ be a cocycle. Write $\beta(g) = \beta_{fixed}(g) + \beta_{unfixed}(g)$ where $\beta_{fixed}(g)$ is the projection of $\beta(g)$ to $\mathcal{H}_{fixed}$ (and likewise for $\beta_{unfixed}$). By the cocycle identity,

$$\beta_{fixed}(gh) + \beta_{unfixed}(gh) = \pi(g)\beta_{fixed}(h) + \pi(g)\beta_{unfixed}(h) + \beta_{fixed}(g) + \beta_{unfixed}(g)$$

and therefore, since $\pi(g)\beta_{fixed}(h) = \beta_{fixed}(h)$,

$$\beta_{fixed}(gh) - \beta_{fixed}(g) - \beta_{fixed}(h) = -\beta_{unfixed}(gh) + \pi(g)\beta_{unfixed}(h) + \beta_{unfixed}(g)$$

The vector on the left is in $\mathcal{H}_{fixed}$ while that on the right is in $\mathcal{H}_{unfixed}$. Therefore both are in fact 0. So $\beta_{fixed} : G \to \mathcal{H}_{fixed}$ and $\beta_{unfixed} : G \to \mathcal{H}_{unfixed}$ are both cocycles.

Now $\beta_{fixed}(g^{-1}) = -\pi(g^{-1})\beta_{fixed}(g) = -\beta_{fixed}(g)$ by the cocycle identity and that $\beta_{fixed}(g)$ is a fixed point so

$$\int_G \beta_{fixed}(g)\ d\mu(g) = 0$$

since μ is symmetric. Since π is the trivial representation on $\mathcal{H}_{fixed}$ and there are no nontrivial coboundaries for the trivial representation, β_{fixed} is the sole representative of its (reduced) cohomology class and is also μ-harmonic.

Now assume that the Theorem holds for $\beta_{unfixed}$. So there is a unique μ-harmonic representative for $[\beta_{unfixed}]$. Call this representative $\varphi_{unfixed}$. Define the cocycle on $\mathcal{H}$

$$\varphi := \beta_{fixed} + \varphi_{unfixed}$$

Since β_{fixed} and $\varphi_{unfixed}$ are μ-harmonic, so is φ. Since $\varphi_{unfixed} \in [\beta_{unfixed}]$ we clearly have $\varphi \in [\beta]$. So $[\beta]$ has a μ-harmonic representative.

Suppose ϕ were also a μ-harmonic representative of $[\beta]$. Decompose $\phi = \phi_{fixed} + \phi_{unfixed}$. Observe that

$$\phi(g) - \beta(g) = \lim \pi(g)v_n - v_n$$

for some v_n which we may take to be in $\mathcal{H}_{unfixed}$ (if $v_n = v_{n,fixed} + v_{n,unfixed}$ then $\pi(g)v_n - v_n = \pi(g)v_{n,unfixed} - v_{n,unfixed}$). Then

$$\phi_{fixed}(g) - \beta_{fixed}(g) = \beta_{unfixed}(g) - \phi_{unfixed}(g) + \lim \pi(g)v_n - v_n$$

and the term on the left is in $\mathcal{H}_{fixed}$ while that on the right is in $\mathcal{H}_{unfixed}$. So they are both 0. Then $\phi_{fixed} = \beta_{fixed}$ and $\phi_{unfixed} \in [\beta_{unfixed}]$. So, by the uniqueness of $\varphi_{unfixed}$ in $[\beta_{unfixed}]$ we have that $\phi = \beta_{fixed} + \varphi_{unfixed} = \varphi$. So φ is unique.

Therefore to prove the Theorem we need only to show that it holds when we assume that the Hilbert space $\mathcal{H}$ has no fixed points for the linear G action.

6.5 Energy

The energy of a cocycle, called that due to its similarity to the definition of energy in physics, measures how far the cocycle is from being trivial. Energy is a natural sort of "norm" on cocycles once one has placed a measure on the group.

6.5.1 The Energy Function

We define now the energy and the energy function, which play a central role in the study of harmonic cocycles.

Definition 6.5. Let G be a group and μ a symmetric probability measure on G. Let $\pi : G \to B(\mathcal{H})$ be a unitary representation of G on a Hilbert space $\mathcal{H}$. Let $\beta : G \to \mathcal{H}$ be a cocycle. The **μ-energy of** β is defined to be

$$E^{\mu}_{\beta} := \int_G \|\beta(g)\|^2 \; d\mu(g)$$

and the **μ-energy function for** β is $E^{\mu}_{\beta} : \mathcal{H} \to [0, \infty]$ given by

$$E^{\mu}_{\beta}(v) := \int_G \|\pi(g)v - v + \beta(g)\|^2 \; d\mu(g)$$

The energy of a cocycle is the value of its energy function at 0, that is $E^{\mu}_{\beta} = E^{\mu}_{\beta}(0)$, and in fact for $v \in \mathcal{H}$ the cocycle $\beta_v(g) = \beta(g) + \pi(g)v - v$ (which is cohomologous to β) has energy

$$E^{\mu}_{\beta_v} = E^{\mu}_{\beta}(v)$$

In order that the energy be finite, it is sufficient that μ be compactly supported (since $\|\pi(g)v - v + \beta(g)\| < \infty$ for each g). More generally, since

$$\|\pi(g)v - v + \beta(g)\| \leq 2\|v\| + \|\beta(g)\|$$

it is sufficient that $\|\beta(g)\| \in L^2(G, \mu)$. The hypotheses of our Theorem then ensure that E^{μ}_{β} is finite (which we will implicitly use throughout what follows).

6.5.2 Properties of the Energy Function

Recall that $\check{\mu}$ is the symmetric opposite of μ: $d\check{\mu}(g) = d\mu(g^{-1})$.

Lemma 6.5.1.

$$E^{\mu}_{\beta}(v + w) - E^{\mu}_{\beta}(v) = \int_G \|\pi(g)w - w\|^2 \; d\mu(g) - 2\int_G \langle \pi(g)v - v + \beta(g), w\rangle \; d(\mu + \check{\mu})(g)$$

Proof.

$$\begin{aligned} E^{\mu}_{\beta}(v + w) - E^{\mu}_{\beta}(v) &= \int_G \|\pi(g)v - v + \beta(g) + \pi(g)w - w\|^2 - \|\pi(g)v - v + \beta(g)\|^2 \; d\mu(g) \\ &= \int_G \|\pi(g)w - w\|^2 + 2\langle \pi(g)v - v + \beta(g), \pi(g)w - w\rangle \; d\mu(g) \end{aligned}$$

and using the cocycle identity (and that $\pi(g)$ is unitary)

$$\begin{aligned} \langle \pi(g)v - v + \beta(g), \pi(g)w\rangle &= \langle v - \pi(g^{-1})v + \pi(g^{-1})\beta(g), w\rangle \\ &= \langle v - \pi(g^{-1})v + \beta(g^{-1}g) - \beta(g^{-1}), w\rangle \\ &= \langle v - \pi(g^{-1})v - \beta(g^{-1}), w\rangle \end{aligned}$$

because $\beta(e) = 0$. Now since

$$\int_G \langle \pi(g)v - v + \beta(g), \pi(g)w \rangle \; d\mu(g) = \int_G \langle v - \pi(g^{-1})v - \beta(g^{-1}), w \rangle \; d\mu(g)$$
$$= -\int_G \langle \pi(g)v - v + \beta(g), w \rangle \; d\check{\mu}(g)$$

we have that

$$E^\mu_\beta(v + w) - E^\mu_\beta(v) = \int_G \|\pi(g)w - w\|^2 \; d\mu(g)$$
$$- 2\int_G \langle \pi(g)v - v + \beta(g), w \rangle \; d\mu(g) - 2\int_G \langle \pi(g)v - v + \beta(g), w \rangle \; d\check{\mu}(g)$$

□

A useful special case is the following:

Lemma 6.5.2.

$$E^\mu_\beta(v) - E^\mu_\beta = \int_G \|\pi(g)v - v\|^2 d\mu(g) - 2\int_G \langle \beta(g), v \rangle \; d(\mu + \check{\mu})(g)$$

Lemma 6.5.3. *The energy function is continuous: if β_n are cocycles such that $\beta_n \to \beta$ uniformly over compact sets in G (and strongly in $\mathcal{H}$) and such that the β_n and β are uniformly in $L^2(\mu \times \| \cdot \|)$ (all dominated by the same L^2 function) then for all $v \in \mathcal{H}$*

$$E^\mu_\beta(v) = \lim_n E^\mu_{\beta_n}(v)$$

and in particular

$$E^\mu_\beta = \lim_n E^\mu_{\beta_n}$$

If μ is compactly supported then the $L^2(\mu)$ requirement is satisfied automatically for all cocycles and the energy function is continuous everywhere.

If G is finitely (or compactly) generated and μ has finite second moment relative to the word length then likewise the energy function is continuous everywhere.

Proof. The uniform L^2 requirement (since v is fixed $\beta_n(g) + \pi(g)v - v$ are also uniformly in L^2) allows us to apply the Dominated Convergence Theorem (twice) to conclude that

$$E^\mu_\beta(v) = \int_G \|\beta(g) + \pi(g)v - v\|^2 \; d\mu(g) = \int_G \| \lim_n \beta_n(g) + \pi(g)v - v\|^2 \; d\mu(g)$$
$$= \int_G \lim_n \|\beta_n(g) + \pi(g)v - v\|^2 \; d\mu(g)$$
$$= \lim_n \int_G \|\beta_n(g) + \pi(g)v - v\|^2 \; d\mu(g) = \lim_n E^\mu_{\beta_n}(v)$$

When μ is compactly supported the energy of any cocycle is finite (since $\|\beta(g)\|$ must attain its maximum on the support of μ) and since $\beta_n \to \beta$ uniformly on compact sets in G the convergence is uniform on the support of μ meaning the Uniform Convergence Theorem can replace the need for the Dominated Convergence Theorem. Of course, compact support means $\|\beta(g)\|^2$ attains its maximum on the support of μ and so the energy of any cocycle is finite hence the function is continuous everywhere.

When G is finitely (compactly) generated and μ has finite second moment in terms of word length we proceed as follows. First observe that for $s, g \in G$

$$\|\beta(sg)\| = \|\pi(s)\beta(g) + \beta(s)\| \leq \|\beta(g)\| + \|\beta(s)\|$$

Let S be a finite (compact) symmetric generating set for G and write $|g|$ for the word length in S. Then

$$\|\beta(g)\| \leq |g| \max_{s\in S} \|\beta(s)\|$$

from the above (repeatedly). Therefore

$$E^\mu_\beta(v) = \int_G \|\beta(g) + \pi(g)v - v\|^2 \; d\mu(g) \leq \int_G |g|^2 \; d\mu(g) \max_{s\in S} \|\beta(s) + \pi(s)v - v\|^2$$

is finite since S is a finite (compact) set and μ has finite second moment (meaning that $\int |g|^2 \; d\mu(g) < \infty$). Hence energy is finite everywhere as in the compactly supported case. Moreover, if $\beta_n \to \beta$ then $\beta_n(s) \to \beta(s)$ uniformly over $s \in S$ and so

$$\max_{s\in S} \|\beta_n(s)\|^2 \to \max_{s\in S} \|\beta(s)\|^2$$

hence

$$\|\beta_n(g)\|^2 \leq |g|^2 \big(\max_{s\in S} \|\beta(s)\|^2 + 1 \big)$$

for sufficiently large n which is a fixed $L^2(G, \mu)$ function that then uniformly bounds the $\|\beta_n(g)\|^2$. Hence the energy function is continuous everywhere. □

6.5.3 Directional Derivatives

Consider the **directional derivative of the energy function**:

$$D_w E^\mu_\beta = \lim_{t\to 0} \frac{E^\mu_\beta(tw) - E^\mu_\beta}{t}$$

Lemma 6.5.4.

$$D_w E^\mu_\beta = -2 \int_G \langle \beta(g), w \rangle \; d(\mu + \check{\mu})(g)$$

Proof. Applying Lemma 6.5.2 to the directional derivative:

$$D_w E^\mu_\beta = \lim_{t\to 0} \frac{1}{t}\Big(\int_G t^2 \|\pi(g)w - w\|^2 \; d\mu(g) - 2\int_G t\langle \beta(g), w\rangle \; d(\mu+\check{\mu})(g)\Big)$$
$$= -2\int_G \langle \beta(g), w\rangle \; d(\mu+\check{\mu})(g)$$

□

Lemma 6.5.5.

$$E^\mu_\beta(v) - E^\mu_\beta = \int_G \|\pi(g)v - v\|^2 \; d\mu(g) + D_v E^\mu_\beta$$

Proof. Lemma 6.5.4 and Lemma 6.5.2. □

Lemma 6.5.6. *For $c \in \mathbb{R}$,*

$$D_{cv} E^\mu_\beta = c D_v E^\mu_\beta$$

Proof. By Lemma 6.5.4 twice,

$$D_{cv} E^\mu_\beta = -2\int_G \langle \beta(g), cv\rangle \; d(\mu+\check{\mu})(g) = -2c\int_G \langle \beta(g), w\rangle \; d(\mu+\check{\mu})(g) = cD_v E^\mu_\beta$$

□

Lemma 6.5.7. *The derivative of the energy function is continuous for each fixed direction: if $v \in \mathcal{H}$ and β_n are cocycles such that $\beta_n \to \beta$ uniformly over compact sets in G (and strongly in $\mathcal{H}$) and such that the β_n and β are uniformly in $L^2(\mu)$ (all dominated by the same L^2 function) then*

$$D_v E^\mu_\beta = \lim_n D_v E^\mu_{\beta_n}$$

If μ is compactly supported then the $L^2(\mu)$ requirement is satisfied automatically for all cocycles and the derivative of the energy function is continuous everywhere.

If G is finitely (or compactly) generated and μ has finite second moment relative to the word length then likewise the derivative of the energy function is continuous everywhere.

Proof. Fix $v \in \mathcal{H}$. By Lemma 6.5.5 we have that

$$\lim_n D_v E^\mu_{\beta_n} = \lim_n E^\mu_{\beta_n}(v) - E^\mu_{\beta_n} - \int_G \|\pi(g)v - v\|^2 \; d\mu(g)$$

and therefore, by continuity of energy (Lemma 6.5.3),

$$\lim_n D_v E^\mu_{\beta_n} = E^\mu_\beta(v) - E^\mu_\beta - \int_G \|\pi(g)v - v\|^2 \; d\mu(g) = D_v E^\mu_\beta$$

□

6.5.4 Minima of the Energy

Definition 6.6. A cocycle β **minimizes the energy** when

$$E_\beta^\mu \leq E_\beta^\mu(v)$$

for all $v \in \mathcal{H}$.

Lemma 6.5.8. *A cocycle β minimizes the energy if and only if all the directional derivatives of the energy function for β are zero.*

Proof. Assume that $D_v E_\beta^\mu = 0$ for all $v \in \mathcal{H}$. Then by Lemma 6.5.5,

$$E_\beta^\mu(v) - E_\beta^\mu = \int_G \|\pi(g)v - v\|^2 \, d\mu(g) + D_v E_\beta^\mu = \int_G \|\pi(g)v - v\|^2 \, d\mu(g) \geq 0$$

so β minimizes the energy.

Now suppose that for some $v \in \mathcal{H}$ we have $D_v E_\beta^\mu \neq 0$. Replacing v by $-v$ (Lemma 6.5.6) if necessary, we may assume that there is $\delta > 0$ such that

$$D_v E_\beta^\mu \leq -\delta$$

From the definition of the directional derivative this means for each $\epsilon > 0$ there is some $t > 0$ such that

$$\frac{1}{t}\left(E_\beta^\mu(tv) - E_\beta^\mu\right) \leq -\delta + \epsilon$$

and therefore, taking $\epsilon = \delta/2$,

$$E_\beta^\mu(tv) - E_\beta^\mu \leq -\frac{t\delta}{2}$$

meaning that β does not minimize the energy. ☐

6.5.5 "Centers" of Cocycles

Definition 6.7. Let $\beta : G \to \mathcal{H}$ be a cocycle and μ a probability measure on G. The **μ-center of β** is the vector

$$\mu(\beta) := \int_G \beta(g) \, d\mu(g)$$

A cocycle is then μ-harmonic if and only if its μ-center is 0 (Lemma 6.1.1).

Definition 6.8. The **μ-convolution of a vector** v is

$$\pi(\mu)v := \int_G \pi(g)v \, d\mu(g)$$

6.5.6 Energy Decreases under Convolution

Lemma 6.5.9. *Let $\beta : G \to \mathcal{H}$ be a cocycle and let $\mu \in P(G)$. Write $\tilde{\mu} \in P(G)$ to mean $\tilde{\mu} = {}^{\mu+\check{\mu}}/_2$. Then*

$$E^\mu_\beta(\tilde{\mu}(\beta)) \leq E^\mu_\beta$$

with equality if and only $\tilde{\mu}(\beta)$ is a fixed point for the linear (π) action of G on $\mathcal{H}$. If the only fixed point is zero then equality holds if and only if β is $\tilde{\mu}$-harmonic.

Proof. Note first that $\pi(g^{-1})\beta(g) = \beta(g^{-1}g) - \beta(g^{-1}) = -\beta(g^{-1})$. For $v \in \mathcal{H}$ and $g \in G$ (treating $\mathcal{H}$ as real; if it is complex the obvious modifications are required),

$$\begin{aligned}\|\beta(g)+\pi(g)v - v\|^2 - \|\beta(g)\|^2 &= \|\pi(g)v - v\|^2 + 2\langle \beta(g), \pi(g)v - v\rangle \\ &= \|\pi(g)v\|^2 + \|v\|^2 - 2\langle \pi(g)v, v\rangle + 2\langle \pi(g^{-1})\beta(g), v\rangle - 2\langle \beta(g), v\rangle \\ &= 2\|v\|^2 - 2\langle \pi(g)v, v\rangle - 2\langle \beta(g^{-1}) + \beta(g), v\rangle\end{aligned}$$

and therefore

$$E^\mu_\beta(v) - E^\mu_\beta = 2\|v\|^2 - 2\langle \pi(\mu)v, v\rangle - 2\langle \mu(\beta) + \check{\mu}(\beta), v\rangle$$

Now observe that

$$\|\pi(\mu)v\| \leq \int \|\pi(g)v\| \; d\mu(g) = \|v\|$$

and therefore

$$E^\mu_\beta(v) - E^\mu_\beta \leq 4\|v\|^2 - 4\langle \frac{\mu(\beta) + \check{\mu}(\beta)}{2}, v\rangle$$

Then taking $v = {}^1/_2(\mu(\beta) + \check{\mu}(\beta)) = \tilde{\mu}(\beta)$ we obtain that

$$E^\mu_\beta(v) - E^\mu_\beta \leq 0$$

For the inequalities above to all be equality requires that $\pi(g)\tilde{\mu}(\beta) = \pi(h)\tilde{\mu}(\beta)$ for all g, h in the support of μ. Hence in this case $\tilde{\mu}(\beta)$ is a fixed point for the linear action. □

6.6 Harmonicity and Energy

Theorem 6.9. *Let π be a unitary representation of a group G on a Hilbert space $\mathcal{H}$ with no nonzero fixed points and $\varphi : G \to \mathcal{H}$ a π-cocycle and let $\mu \in P(G)$ be symmetric. Then φ is μ-harmonic if and only if it minimizes the μ-energy.*

Proof. Since μ is symmetric, $\tilde{\mu} = \check{\mu} = \mu$. Assume φ minimizes the energy. Then $E^\mu_\varphi \leq E^\mu_\varphi(v)$ for all $v \in \mathcal{H}$. By Lemma 6.5.9 we have that $E^\mu_\varphi(\mu(\varphi)) \leq E^\mu_\varphi$ and equality occurs if and only if φ is μ-harmonic (since there are no nonzero fixed points). Hence minimizing the energy implies φ is harmonic.

Now assume that φ is harmonic. By Lemma 6.5.8, harmonicity implies that $D_v E^\mu_\varphi = 0$

for all $v \in \mathcal{H}$. Now by Lemma 6.5.4, for any v,

$$E_\varphi(v) - E_\varphi = \int_G \|\pi(g)v - v\|^2 + D_v E_\varphi = \int_G \|\pi(g)v - v\|^2 \, d\mu(g) \geq 0$$

and therefore φ minimizes the energy. □

6.7 Existence of Unique Minima

Theorem 6.10. *Let π be a unitary representation of a group G on a Hilbert space $\mathcal{H}$ and $\mu \in P(G)$ a symmetric probability measure on G with support generating G.*

Assume that either μ is compactly supported or that G is finitely or compactly generated and μ has finite second moment relative to the word length.

Let $[\beta]$ be a reduced cohomology class containing an $L^2(\mu)$ representative. There exists a unique cocycle $\varphi \in [\beta]$ such that φ minimizes the μ-energy.

Proof. First we establish uniqueness. Suppose φ and ϕ are both cocycles in the same reduced cohomology class so that both minimize the energy. Since they are in the same reduced cohomology class there exists a sequence v_n such that $\phi(g) = \lim_n \varphi(g) + \pi(g)v_n - v_n$ (strongly) uniformly over compact sets in G.

Since φ minimizes the energy we have $D_v E^\mu_\varphi = 0$ for all v by Lemma 6.5.8. By Lemma 6.5.2, for any v,

$$E^\mu_\varphi(v) - E^\mu_\varphi = \int_G \|\pi(g)v - v\|^2 \, d\mu(g) + D_v E^\mu_\varphi = \int_G \|\pi(g)v - v\|^2 \, d\mu(g)$$

By the continuity of energy (Lemma 6.5.3) we have that

$$0 = E^\mu_\phi - E^\mu_\varphi = \lim_n E^\mu_\varphi(v_n) - E^\mu_\varphi = \lim_n \int_G \|\pi(g)v_n - v_n\|^2 \, d\mu(g)$$

and therefore, since $\phi - \varphi$ is a cocycle and $\phi(g) - \varphi(g) = \lim_n \pi(g)v_n - v_n$ (uniformly over compact sets in G, strongly in $\mathcal{H}$), again by the continuity of energy (Lemma 6.5.3)

$$E^\mu_{\phi-\varphi} = \lim_n E^\mu_0(v_n) = 0$$

and so

$$\int_G \|\phi(g) - \varphi(g)\|^2 \, d\mu(g) = 0$$

meaning that $\phi(g) = \varphi(g)$ for μ-almost every g which by continuity (and that the support of μ generates G) gives for all g and so $\phi = \varphi$ meaning there is at most one cocycle in each class that minimizes the energy.

We now show that a minimizer in fact exists. Fix a cocycle φ and take a sequence of vectors $v_n \in \mathcal{H}$ such that

$$E^\mu_\varphi(v_n) \downarrow \inf_{v\in\mathcal{H}} E^\mu_\varphi(v)$$

Define

$$\beta_n(g) = \varphi(g) + \pi(g)v_n - v_n$$

so that the β_n are cocycles cohomologous to φ. By the Parallelogram Law (that $\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2$),

$$\begin{aligned}
\frac{1}{2}\int_G \|\beta_n(g) - \beta_m(g)\|^2 \, d\mu(g) &= \frac{1}{2}\int_G 2\|\beta_n(g)\|^2 + 2\|\beta_m(g)\|^2 - \|\beta_n(g) + \beta_m(g)\|^2 \, d\mu(g) \\
&= E^\mu_\varphi(v_n) + E^\mu_\varphi(v_m) - 2\int_G \|\frac{1}{2}\big(\beta_n(g) + \beta_m(g)\big)\|^2 \, d\mu(g) \\
&= E^\mu_\varphi(v_n) + E^\mu_\varphi(v_m) - 2E^\mu_\varphi(\frac{v_n + v_m}{2}) \\
&\leq E^\mu_\varphi(v_n) + E^\mu_\varphi(v_m) - 2\inf_{v\in\mathcal{H}} E^\mu_\varphi(v)
\end{aligned}$$

and therefore

$$\lim_{n,m\to\infty} \int_G \|\beta_n(g) - \beta_m(g)\|^2 \, d\mu(g) = 0$$

meaning that for almost every g we have that (again by Lemma 6.5.3 we have continuity)

$$\|\beta_n(g) - \beta_m(g)\| \to 0 \qquad \text{as } n, m \to \infty$$

Since this a Cauchy sequence of vectors (for each g) it has a strong limit point, call it $\beta(g)$:

$$\beta(g) = \lim_{n\to\infty} \beta_n(g) = \lim_{n\to\infty} \varphi(g) + \pi(g)v_n - v_n$$

So β is evidently a cocycle almost cohomologous to φ. We remark at this point that were the infimum attained by some vector v then β will simply be $\varphi(g) + \pi(g)v - v$.

The proof will be complete if we show that β minimizes the energy, so it is enough (by Lemma 6.5.8) to show that $DE^\mu_\beta = 0$. Suppose that $D_w E^\mu_\beta \neq 0$ for some w. Then, by the continuity of the derivative of the energy function (Lemma 6.5.7),

$$\lim_n D_w E^\mu_{\beta_n} \neq 0$$

so there exists $\delta > 0$ and a subsequence of n so that

$$D_w E^\mu_{\beta_n} \leq -\delta$$

(we may replace w by $-w$ to force the derivative to be negative using Lemma 6.5.6).

By Lemma 6.5.6 this means that for any $c \geq 0$,

$$D_{cw} E^\mu_{\beta_n} \leq -c\delta$$

and then by Lemma 6.5.2

$$E^\mu_{\beta_n}(cw) - E^\mu_{\beta_n} = \int_G \|\pi(g)cw - cw\|^2 \, d\mu(g) + D_{cw}E^\mu_{\beta_n} \leq 4c^2\|w\|^2 - c\delta$$

Now $E^\mu_{\beta_n} = E^\mu_\varphi(v_n) \downarrow \inf_{v\in\mathcal{H}} E^\mu_\varphi(v)$ so for any $\epsilon > 0$ and large n,

$$E^\mu_{\beta_n}(cw) - E^\mu_{\beta_n} \geq \inf_{v\in\mathcal{H}} E^\mu_\varphi(v) - \big(\inf_{v\in\mathcal{H}} E^\mu_\varphi(v) + \epsilon\big) = -\epsilon$$

and therefore for any $\epsilon > 0$, by taking $c = {}^{\delta}/_{8\|w\|^2}$, we have

$$-\epsilon \leq c(4c\|w\|^2 - \delta) = c\Big(\frac{\delta}{2} - \delta\Big) = -\frac{c\delta}{2} = -\frac{\delta^2}{16\|w\|^2}$$

and therefore $\delta^2 \leq 16\|w\|^2\epsilon$ which contradicts that $\delta > 0$ (since δ is fixed and we can take $\epsilon \to 0$). So in fact $D_v E^\mu_\beta = 0$ which completes the proof. □

Theorem 6.3 now follows from Theorem 6.9 and Theorem 6.10.

Chapter 7

Injectivity of Reduced Cohomology

Let G be a locally compact group and Γ a countable subgroup of G. Let π be a representation of G. Then $\pi|_\Gamma$ is a representation of Γ (restrict the domain to Γ). Likewise, if β is a (G,π)-cocycle then $\beta|_\Gamma$ is a $(\Gamma,\pi|_\Gamma)$-cocycle (all that has to be verified is the cocycle identity which is immediate). Moreover, if β is a (G,π) (almost) coboundary then $\beta|_\Gamma$ is a $(\Gamma,\pi|_\Gamma)$ (almost) coboundary using the same (sequence of) vectors. There is therefore a natural map, called the **restriction map**, from the (G,π) (reduced) cohomology to that of $(\Gamma,\pi|_\Gamma)$.

The purpose of this chapter is to use the result from the previous chapter on existence and uniqueness of harmonic representatives to show that in certain cases this restriction map for reduced cohomology is injective. Consequences of this appear in Chapter 12: The Margulis-Zimmer Conjecture.

7.1 Simultaneously Harmonic Measures

The main consequence of the existence and uniqueness of harmonic representatives of reduced cohomology is:

Theorem 7.1. *Let Γ be a* dense *finitely generated subgroup of a locally compact group G and let π be a unitary representation of G on a Hilbert space $\mathcal{H}$. Assume that there exist probability measures μ on G and ν on Γ such that the restriction of a μ-harmonic map on G is a ν-harmonic map on Γ and such that μ is symmetric and compactly supported on a generating set for G and that ν is symmetric and finitely supported on a generating set for Γ. Then the natural map $\overline{H^1}(G,\pi) \to \overline{H^1}(\Gamma,\pi|_\Gamma)$ is injective.*

Proof. Take $[\beta] \in \overline{H^1}(G,\pi)$ and $\varphi \in [\beta]$ the unique μ-harmonic representative (Theorem 6.2). Then $\varphi|_\Gamma$ is a cocycle for $(\Gamma,\pi|_\Gamma)$ and by hypothesis $\varphi|_\Gamma$ is ν-harmonic. It is therefore the unique ν-harmonic representative of the image of $[\beta|_\Gamma]$ in $\overline{H^1}(\Gamma,\pi|_\Gamma)$. Now take $[\beta'] \in \overline{H^1}(G,\pi)$ to be another reduced cohomology class and let φ' be the unique μ-harmonic representative for it. Suppose that $[\beta]$ and $[\beta']$ map to the same reduced cohomology class for $(\Gamma,\pi|_\Gamma)$. Then $\varphi|_\Gamma$ and $\varphi'|_\Gamma$ are both ν-harmonic and in the same reduced cohomology class so by uniqueness, $\varphi(\gamma) = \varphi'(\gamma)$ for all $\gamma \in \Gamma$. Therefore $\varphi = \varphi'$ on the closure of Γ which by density is all of G. But then, by uniqueness of φ as a μ-harmonic cocycle, $[\beta] = [\beta']$. □

7.2 Totally Disconnected Groups

The aim of this section is to prove that when the ambient group is totally disconnected the restriction map for reduced cohomology to a dense subgroup is injective (that is, there exist probability measures as required above).

7.2.1 Measures on Dense Subgroups

Let G be a totally disconnected second countable locally compact compactly generated group and Γ a dense finitely generated subgroup of G. Let K be a compact open subgroup of G (which exists since G is totally disconnected). Let S_0 be a symmetric finite generating set for Γ. Take an arbitrary symmetric, compactly supported probability measure ρ on G such that S_0 is contained in the support of ρ. Let m_K be the Haar measure of G restricted to K and normalized to be a probability measure (possible since K is compact). Let $\mu = m_K * \rho * m_K$ be the convolution of ρ by m_K on both sides. Then μ is a bi-K-invariant probability measure on G.

Let C be the support of μ which is compact (since the supports of ρ and m_K are compact) and contains S_0. Consider the compact open subgroup

$$K_0 = \bigcap_{g \in C} gKg^{-1}$$

which is open and nontrivial since each conjugate of K contains an open subgroup (as G is totally disconnected) and since C is compact.

Let S_1 be a symmetric system of representatives for G/K_0. There are only finitely many elements of S_1 in C since C is compact. Let $S = S_0 \cup (S_1 \cap C)$. Then S_0 is a finite symmetric set which generates Γ and is contained in C and contains at least one representative of every coset of K_0 in C.

Observe that for $\gamma \in S$, it holds that $\gamma^{-1} K_0 \gamma \subseteq K$ since $\gamma \in C$. Define the measure ν on Γ as follows: for $\gamma \in S$ set

$$\nu(\gamma) = \frac{1}{|S \cap \gamma K_0|} \mu(\gamma K_0)$$

which is well-defined since S is finite (by $|S \cap \gamma K_0|$ we mean cardinality). Then ν is finitely supported on a symmetric generating set for Γ (though ν may not itself be symmetric).

7.2.2 Harmonicity

Let φ be a μ-harmonic map. Then for any $k \in K$ and $g \in G$ we have

$$\varphi(gk) = \int_G \varphi(gkh)\ d\mu(h) = \int_G \varphi(gh)\ dk\mu(h) = \int_G \varphi(gh)\ d\mu(h) = \varphi(g)$$

since $k\mu = \mu$ by the left-K-invariance of μ. So φ is right-K-invariant.

Also, for any $k \in K_0$ and any $g \in C$, we have that $kg \in K_0 g \subseteq gK$ so $kg = gk'$ for some $k' \in K$ and so $\varphi(kg) = \varphi(gk') = \varphi(g)$ by the right-K-invariance of φ.

Then for $g \in \gamma K_0$ we have $\varphi(g) = \varphi(\gamma)$ and so

$$\begin{aligned}\int_\Gamma \varphi(\gamma)\, d\nu(\gamma) &= \sum_{\gamma \in S} \frac{1}{|S \cap \gamma K_0|} \varphi(\gamma) \int_G \mathbb{1}_{\gamma K_0}(g)\, d\mu(g) = \sum_{\gamma \in S} \frac{1}{|S \cap \gamma K_0|} \int_G \varphi(\gamma) \mathbb{1}_{\gamma K_0}(g)\, d\mu(g) \\ &= \sum_{\gamma \in S} \frac{1}{|S \cap \gamma K_0|} \int_G \varphi(g) \mathbb{1}_{\gamma K_0}(g)\, d\mu(g) = \int_G \varphi(g) \sum_{\gamma \in S} \frac{1}{|S \cap \gamma K_0|} \mathbb{1}_{\gamma K_0}(g)\, d\mu(g) \\ &= \int_G \mathbb{1}_{\operatorname{supp} \mu}(g) \varphi(g)\, d\mu(g) = \int_G \varphi(g)\, d\mu(g) = \varphi(e) = 0\end{aligned}$$

so $\varphi|_\Gamma$ is ν-harmonic at the identity. Recall that a cocycle is harmonic if and only if it is harmonic at the identity (Lemma 6.1.1) so $\varphi|_\Gamma$ is ν-harmonic.

Let $\check{\nu}$ be the symmetric opposite of ν: $\check{\nu}(\gamma) = \nu(\gamma^{-1})$ for γ in the support of ν. For $g \in \gamma K_0$ with $\gamma \in S$, we have that, writing $g = \gamma k$ for some $k \in K_0$,

$$\varphi(g^{-1}) = \varphi(k^{-1}\gamma^{-1}) = \varphi(\gamma^{-1})$$

by the left-K_0-invariance of φ (on C). Then

$$\begin{aligned}\int_\Gamma \varphi(\gamma)\, d\check{\nu}(\gamma) &= \int_\Gamma \varphi(\gamma^{-1})\, d\nu(\gamma) \\ &= \sum_{\gamma \in S} \frac{1}{|S \cap K_0|} \varphi(\gamma^{-1}) \int_G \mathbb{1}_{\gamma K_0}(g)\, d\mu(g) \\ &= \sum_{\gamma \in S} \frac{1}{|S \cap \gamma K_0|} \varphi(\gamma^{-1}) \int_G \mathbb{1}_{\gamma K_0}(g)\, d\mu(g) \\ &= \sum_{\gamma \in S} \frac{1}{|S \cap \gamma K_0|} \int_G \varphi(\gamma^{-1}) \mathbb{1}_{\gamma K_0}(g)\, d\mu(g) \\ &= \sum_{\gamma \in S} \frac{1}{|S \cap \gamma K_0|} \int_G \varphi(g^{-1}) \mathbb{1}_{\gamma K_0}(g)\, d\mu(g) \\ &= \int_G \varphi(g^{-1}) \sum_{\gamma \in S} \frac{1}{|S \cap \gamma K_0|} \mathbb{1}_{\gamma K_0}(g)\, d\mu(g) \\ &= \int_G \varphi(g^{-1})\, d\mu(g) = \int_G \varphi(g)\, d\mu(g) = \varphi(e) = 0\end{aligned}$$

where the last line follows by the symmetry of μ.

So φ is both ν- and $\check{\nu}$-harmonic. Let $\nu^* = {}^1\!/\!_2(\nu + \check{\nu})$ be the symmetrization of ν. Then φ is ν^*-harmonic and obviously ν^* is a symmetric finitely supported measure on Γ.

7.2.3 Restriction is Injective

Theorem 7.1 (applied to μ and ν^*) then gives injectivity of the map $\overline{H^1}(G,\pi) \to \overline{H^1}(\Gamma, \pi|_\Gamma)$ for the case when G is totally disconnected:

Theorem 7.2. *Let G be a totally disconnected locally compact compactly generated group and Γ a dense finitely generated subgroup of G. Let π be a unitary representation of G on a Hilbert space. Then the natural restriction map $\overline{H^1}(G,\pi) \to \overline{H^1}(\Gamma, \pi|_\Gamma)$ is injective.*

III

Normal Subgroups of Commensurators

Chapter 8

Commensuration

A natural idea when studying infinite groups is to take into account the fact that finite groups have essentially been completely classified. Many structural results about countable groups can be obtained by understanding them "up to finite index" in the sense that if one can identify a finite index subgroup that has certain properties then one has in some sense understood the group. This is referred to as a "geometric" approach to studying groups and is one of many ideas in the field of geometric group theory.

This chapter discusses commensuration, the analogue of normalization in this geometric context. Normal subgroups are characterized by all conjugates of the subgroup being the subgroup–a normal subgroup is invariant under conjugation; commensurated subgroups are those whose conjugates agree with the original subgroup up to finite index–commensurated subgroups are invariant under conjugation up to finite index.

The classification of finite groups proceeds by reducing the general case to the simple case by observing that any finite group is either simple already or has a nontrivial normal subgroup. The quotient by the normal subgroup is then a smaller finite group and the normal subgroup is also a finite group. This process must eventually terminate with a collection of finite simple groups. The classification of finite simple groups then forms the bulk of the work in classifying the finite groups.

We would like a similar method for infinite groups that also takes into account the geometric idea of identifying groups up to finite index. This geometric approach to understanding group structure leads naturally to the study of the commensurator of a subgroup: the "geometric normalizer".

8.1 Geometric Approach

Recall that a subgroup is of finite index when there are only finitely many cosets in the quotient space. If G is a countable group and H is a finite index subgroup then all of the "infinite" structure of G can be seen in H (since G is a union of finitely many copies of H).

We will in principle identify groups that agree on a finite index subgroup (sometimes up to an isomorphism or conjugation). Formalizing this idea will occupy the rest of the chapter.

8.1.1 Just Infinite Groups

Rather than attempt to classify countable groups by always quotienting out by normal subgroups, we take into account that finite groups are classified. The terminating case of the algorithm is then not just simple groups but in fact just infinite groups:

Definition 8.1. A countable group G is **just infinite** when every normal subgroup $N \triangleleft G$ is either finite $|N| < \infty$ or has finite index $[G : N] < \infty$.

A just infinite group is the smallest object we need to classify since the finite groups N and G/N are already classified.

8.2 Commensurable Subgroups

We now formalize the notion of agreement up to finite index in the form of commensuration.

8.2.1 Commensurate Subgroups

Two groups which are commensurate will be thought of as "the same":

Definition 8.2. Let G be a group and $A < G$ and $B < G$ be subgroups. Then A and B are **commensurate** or **commensurable** when $A \cap B$ has finite index in both A and B.

Often the group G is not specified and this is intended to mean that there exists some group G and isomorphic copies of A and B in G such that the isomorphic copies are commensurate in the above sense. Commensurable is sometimes reserved to refer to subgroups which are conjugate to commensurate subgroups. A useful relationship between lattices and commensuration is that:

Proposition 8.2.1. *Let Γ be a lattice in a locally compact group G and Γ' be a subgroup of G commensurate to Γ. Then Γ' is a lattice in G.*

8.2.2 Commensurated Subgroups

Accounting for the idea that two commensurate subgroups are the same, we can broaden the definition of normal subgroup to commensurated subgroup by requiring only that the conjugates be commensurate.

Definition 8.3. Let A be a group and $B < A$ be a subgroup. Then A **commensurates** B when for every $a \in A$ the subgroup $B \cap aBa^{-1}$ has finite index in both B and aBa^{-1}:

$$B <_{comm} A \quad \text{when} \quad \forall a \in A \ \ [B : B \cap aBa^{-1}] < \infty, [aBa^{-1} : B \cap aBa^{-1}] < \infty$$

and B is then a **commensurated subgroup** of A.

8.3 The Commensurator

The normalizer of a subgroup is the largest group that normalizes the subgroup. The commensurator is defined analogously in the context of commensurate groups being treated as the same.

8.3.1 Formal Definition

Mimicking the definition of the normalizer we define:

Definition 8.4. The **commensurator of** B **in** A is the group

$$\mathrm{Comm}_A(B) = \{a \in A : [B : B \cap aBa^{-1}] < \infty \text{ and } [aBa^{-1} : B \cap aBa^{-1}] < \infty\}$$

that is, the largest subgroup of A which commensurates B. If the ambient group is clear from context we will write simply $\mathrm{Comm}(B)$.

8.3.2 An Example

Let $G = \mathrm{SL}_n(\mathbb{R})$ be the ambient group and $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ the lattice. The commensurator is

$$\mathrm{Comm}_{\mathrm{SL}_n(\mathbb{R})}(\mathrm{SL}_n(\mathbb{Z})) = \mathrm{SL}_n(\mathbb{Q})$$

the matrices with rational entries.

This can be seen as follows. Let $q \in \mathrm{SL}_n(\mathbb{Q})$. Then for any $g \in \mathrm{SL}_n(\mathbb{Z})$ the matrix qgq^{-1} will have rational entries with denominators bounded (in terms of the largest denominator in q). The integer matrices will then have finite index in this group (the index in fact will depend on the denominators in q). So for each $q \in \mathrm{SL}_n(\mathbb{Q})$ the group $\mathrm{SL}_n(\mathbb{Z}) \cap q\mathrm{SL}_n(\mathbb{Z})q^{-1}$ will have finite index in both $\mathrm{SL}_n(\mathbb{Z})$ and $q\mathrm{SL}_n(\mathbb{Z})q^{-1}$. Clearly taking a matrix with an irrational entry and conjugating will lead to infinite index and so the largest subgroup of $\mathrm{SL}_n(\mathbb{R})$ which can commensurate $\mathrm{SL}_n(\mathbb{Z})$ is $\mathrm{SL}_n(\mathbb{Q})$.

8.3.3 Finite Orbits

There is a well-known characterization of commensuration in terms of the orbits of the action on the coset space.

Proposition 8.3.1. *Let $A < B$ be a subgroup of a group. Then A is commensurated by B if and only if the orbits of the A action on B/A are finite.*

This is an easy consequence of the definition but nevertheless can be quite useful.

8.3.4 Bounded Index

The example of $\mathrm{SL}_n(\mathbb{Z})$ shows that it can happen that while each conjugate has finite index in the subgroup there need be no uniform bound on the indices.

Proposition 8.3.2. *Let $A < B$ be a commensurated subgroup such that $[A : A \cap bAb^{-1}] \leq K$ and $[bAb^{-1} : A \cap bAb^{-1}] \leq K$ for some fixed K uniformly over $b \in B$. Then $A \triangleleft B$.*

The proof of this well-known fact is omitted since we will not need it here.

8.4 Relative Profinite Completions

The main issue arising when passing from normal subgroups to commensurated subgroups is that the quotient space is no longer a group. For example, if one wishes to show a group G is just infinite this can be accomplished by letting N be an infinite normal subgroup and showing that G/N is a finite group (there are a wealth of techniques for proving groups are finite). This is in fact the approach taken by Margulis in the Normal Subgroup Theorem [Mar91]. However, if one wishes to show that all commensurated subgroups are trivial (up to finite index), as will be the case in the Margulis-Zimmer Conjecture (see Chapter 12: The Margulis-Zimmer Conjecture), then this approach fails since the quotient is not a group.

8.4.1 Motivation

The relative profinite completion will be the replacement for the quotient group. It is a locally compact group constructed from a group and a commensurated subgroup that reflects the structure of the pair that has been studied in the context of group actions and representations ([Sch80], [Tza00], [Tza03]). In particular, a normal subgroup will lead to a discrete relative profinite completion that agrees with the quotient group.

We will be most interested in proving that certain commensurated subgroups have finite index in the group commensurating them. The relative profinite completion will be compact precisely when the commensurated subgroup is finite index.

In general, the relative profinite completion can be thought of as the "totally disconnected" version of the quotient space obtained by trying to "impose" a group structure onto it that behaves like that of a quotient group when the subgroup is normal. The reader is referred to Shalom and Willis [SW09] for further details and proofs.

8.4.2 Formal Definition

Definition 8.5. Let A be a countable group and $B < A$ such that A commensurates B. Consider the group of symmetries of A/B which we denote by $\mathrm{Symm}(A/B)$ and observe that the left action of A on A/B gives a homomorphism $\tau : A \to \mathrm{Symm}(A/B)$. Endow $\mathrm{Symm}(A/B)$ with the topology of pointwise convergence and define the **relative profinite completion of** B **w.r.t** A, denoted $A//B$, to be the closure of $\tau(A)$ in $\mathrm{Symm}(A/B)$ with this topology.

The phrase completion is slightly misleading since the kernel of τ vanishes but when the kernel of τ is trivial this is a true completion. In the special case that B is normal then the kernel is all of B and the relative profinite completion $A//B$ is discrete and isomorphic to A/B.

8.4.3 The Compact Open Subgroup

The closure of $\tau(B)$ in the topology of pointwise convergence of symmetries will be compact since the orbits of the B action on A/B are finite (this is precisely where we need that B is commensurated by A). It will be open since $aB \cap B = \emptyset$ for each $a \notin B$. Hence $\overline{\tau(B)}$ is a compact totally disconnected group. In fact $\{\tau(aBa^{-1}) : a \in A\}$ will be subbase of neighborhoods of the identity.

Now $\overline{\tau(B)}$ is a subgroup of countable index in $A//B$ since B is of countable index in A (as A is countable). Hence $A//B$ is locally compact and totally disconnected:

Proposition 8.4.1. *Let A be a countable group and $B < A$ be commensurated by A. Then the relative profinite completion $A//B$ is a totally disconnected locally compact group and the image of A is dense in $A//B$ and the image of B is precompact in $A//B$.*

8.4.4 The Universal Property

The relative profinite completion has a certain universal property among totally disconnected groups related to A and B. Specifically, for any H a totally disconnected locally compact group and K a compact open subgroup of H, define $\tau_{H,K} : H \to \mathrm{Symm}(H/K)$ as before (K is necessarily commensurated by H, see e.g. [SW09]).

Lemma 8.4.2 ([SW09])**.** *Let H be a totally disconnected locally compact group and K a compact open subgroup of H, define $\tau_{H,K} : H \to \mathrm{Symm}(H/K)$ as before (K is necessarily commensurated by H). Then $\tau_{H,K}$ is a continuous open map with closed range.*

Moreover, $H//K$ is isomorphic to $H/\ker(\tau_{H,K})$ and in fact $\ker(\tau_{H,K})$ is the largest normal subgroup of H that is contained in K.

A consequence of this is:

Lemma 8.4.3 ([SW09])**.** *Let $B < A$ be any subgroup of a countable group, H a totally disconnected locally compact group and K a compact open subgroup of H. Let $\varphi : A \to H$ be a homomorphism such that (i) $\varphi(A)$ is dense in H; and (ii) $\varphi^{-1}(K) = B$.*

Then B is commensurated by A and $A//B$ is isomorphic to $H//K$. In particular, if H is simple then $A//B$ is isomorphic to H.

From this we can deduce the following universal property:

Theorem 8.6 (Shalom-Willis [SW09])**.** *Let B be a commensurated subgroup of a group A and let H be a totally disconnected group with a compact open subgroup $K < H$. Let $\varphi : A \to H$ a homomorphism such that (i) $\varphi(A)$ is dense in H; and (ii) $\varphi^{-1}(K) = B$.*

Then there exists a continuous surjective homomorphism $\psi : H \to A//B$ such that $\psi \circ \varphi : A \to H \to A//B$ is the natural homomorphism.

We also remark that:

Lemma 8.4.4. *Let B be a commensurated subgroup of a group A and let $\tau : A \to A//B$ be the canonical map. Then $\tau(A) \cap \overline{\tau(B)} = \tau(B)$.*

Proof. Let $K = \overline{\tau(B)}$ be a compact open subgroup of $H = A//B$. Observe that $\tau^{-1}(K) = B$ by the definition of $A//B$ (see the previous facts). Let $L = \tau(A) \cap K$. Then

$$\tau^{-1}(L) = \tau^{-1}(\tau(A) \cap K) = \tau^{-1}(\tau(A)) \cap \tau^{-1}(K) = A \cap B = B$$

hence $\tau(B) = \tau(\tau^{-1}(L)) = L$ as claimed. □

8.4.5 An Example

We present an example of the relative profinite completion to aid the reader's intuition. Consider the groups

$$\mathrm{SL}_n(\mathbb{Z}) < \mathrm{SL}_n(\mathbb{Z}[1/p])$$

where p is some prime.

Clearly $\mathrm{SL}_n(\mathbb{Z})$ is commensurated by $\mathrm{SL}_n(\mathbb{Z}[1/p])$ since for any fixed $\gamma \in \mathrm{SL}_n(\mathbb{Z}[1/p])$ there is some $m \in \mathbb{N}$ such that for every entry $\gamma_{i,j}$ of γ we have that $p^m\gamma_{i,j} \in \mathbb{Z}$. Therefore $\gamma\mathrm{SL}_n(\mathbb{Z})\gamma^{-1} \cap \mathrm{SL}_n(\mathbb{Z})$ has finite index in $\mathrm{SL}_n(\mathbb{Z})$ and in $\gamma\mathrm{SL}_n(\mathbb{Z})\gamma^{-1}$ since p and m are fixed. Of course $m \to \infty$ as γ ranges over $\mathrm{SL}_n(\mathbb{Z}[1/p])$ and this is not a normal subgroup.

Observe that the natural homomorphism

$$\varphi : \mathrm{SL}_n(\mathbb{Z}[1/p]) \to \mathrm{SL}_n(\mathbb{Q}_p)$$

has the properties that $\varphi(\mathrm{SL}_n(\mathbb{Z}[1/p]))$ is dense and that $\varphi^{-1}(\mathrm{SL}_n(\mathbb{Z}_p)) = \mathrm{SL}_n(\mathbb{Z})$ where $\mathbb{Z}_p$ is the p-adic integers. Hence the above Lemmas apply and we observe that

$$\mathrm{SL}_n(\mathbb{Z}[1/p])//\mathrm{SL}_n(\mathbb{Z}) \simeq \mathrm{SL}_n(\mathbb{Q}_p)//\mathrm{SL}_n(\mathbb{Z}_p)$$

by the second Lemma above. Now

$$\mathrm{SL}_n(\mathbb{Q}_p)//\mathrm{SL}_n(\mathbb{Z}_p) \simeq \mathrm{PSL}_n(\mathbb{Q}_p)$$

by the first Lemma since the largest normal subgroup of $\mathrm{SL}_n(\mathbb{Q}_p)$ is its center. Therefore, we have derived that

$$\mathrm{SL}_n(\mathbb{Z}[1/p])//\mathrm{SL}_n(\mathbb{Z}) \simeq \mathrm{PSL}_n(\mathbb{Q}_p)$$

and so the relative profinite completion is what one would expect (in that the completion of $\mathbb{Z}[1/p]$ over $\mathbb{Z}$ in any reasonable sense is the p-adic numbers).

8.4.6 Correspondence of Properties

Lemma 8.4.5. *Let A be a countable group and $B < A$ a commensurated subgroup. Then $[B : A] < \infty$ if and only if $A//B$ is compact.*

Proof. Assume that $[A : B] < \infty$. Then the image of B in $A//B$ has finite index in $A//B$. But the image of B is precompact hence $A//B$ is the finite union of compact sets hence is compact.

Now assume that $A//B$ is compact. Let K be the closure of the image of B which is compact. Now the images of B and aB are disjoint for $a \notin B$ so $K \cap aK = \emptyset$ for $a \notin B$ (since K is open). Since $A//B$ is compact there can be only finitely many disjoint cosets of K meaning there are only finitely many disjoint cosets in A/B. □

Lemma 8.4.6. *Let A be a countable group and $B < A$ a commensurated subgroup. If A is finitely generated then $A//B$ is compactly generated.*

Proof. Since $\overline{B}$ is compact in $A//B$ if we take a finite generating set S for A then the set $\bigcup_{s\in S} s\overline{B}$ is a compact set which generates $A//B$. ☐

Chapter 9

Normal Subgroups of Commensurators of Lattices

Let G be a topological group that is locally compact, second countable and compactly generated. We shall refer to such groups as **lcsc c.g.** from here on. Let Γ be a lattice in G and let Λ be a subgroup of the commensurator of Γ such that Λ is dense in G. That is:

$$\Gamma < \Lambda < \mathrm{Comm}_G(\Gamma) < G$$

where G is lcsc c.g., Γ is discrete and G/Γ has finite volume (Haar measure), and Λ is dense in G and Λ commensurates Γ (see below).

Our aim is to study the normal subgroups of Λ in this context. The celebrated Normal Subgroup Theorem of Margulis [Mar79],[Mar91] establishes the virtual triviality of normal subgroups of the lattice Γ when G is a higher-rank Lie group:

Theorem 9.1 (Margulis)**.** *Let Γ be an irreducible lattice in a higher-rank semisimple Lie group. Then Γ is just infinite: every normal subgroup of Γ is finite or of infinite index.*

This has been extended [BS05] to (most) lattices in products of locally compact groups.

9.1 Normal Subgroups Contain the Lattice

The first result we present is that infinite normal subgroups of the commensurator of a lattice necessarily contain (up to finite index) the lattice:

Theorem 9.2. *Let G be a lcsc c.g. group and let $\Gamma < G$ be a finitely generated integrable lattice in G. Let $\Lambda < \mathrm{Comm}(\Gamma)$ be a subgroup of the commensurator of Γ in G such that Λ is dense in G and $\Gamma < \Lambda$. Assume that*

(i) *G is not a compact extension of an abelian group*

(ii) *for every closed normal subgroup M of G which is not cocompact the intersection $M \cap \Lambda$ is finite.*

Then for any infinite normal subgroup N of Λ the intersection $N \cap \Gamma$ has finite index in Γ.

Corollary 9.3. *Let G be a lcsc c.g. group that is topologically simple and not a compact extension of an abelian group. Let $\Gamma < G$ be a finitely generated integrable lattice in G. Let $\Lambda < \mathrm{Comm}(\Gamma)$ be a subgroup of the commensurator such that Λ is dense in G and contains Γ. Then any infinite normal subgroup of Λ contains a finite index subgroup of Γ.*

9.1.1 The Reduction Step

The theorem is a consequence of the following:

Theorem 9.4. *Let G be a lcsc c.g. group and let $\Gamma < G$ be a finitely generated integrable lattice in G. Let $\Lambda < \mathrm{Comm}(\Gamma)$ be a subgroup of the commensurator of Γ in G such that Λ is dense in G and $\Gamma < \Lambda$.*

Let N be a normal subgroup of Λ such that

(i) Γ *maps* onto Λ/N *via the coset map* $\Lambda \to \Lambda/N$

(ii) $\overline{N}$ *is cocompact in* G

(iii) $\overline{[N,N]}$ *is cocompact in* G*.*

Then Λ/N is finite.

Proof. (of Theorem 9.2 from Theorem 9.4). To see how the second implies the first, realize that since $N \triangleleft \Lambda$ this means that $\overline{N} \triangleleft \overline{\Lambda} = G$. From the hypothesis of the first theorem this means that either $\overline{N}$ is cocompact in G or else $\overline{N} \cap \Lambda$ is finite. If $\overline{N} \cap \Lambda$ is finite then obviously N is as well (since $N < \Lambda$ so $N = N \cap \Lambda \subseteq \overline{N} \cap \Lambda$ is finite) but the Theorem requires that N be infinite. So we have that $\overline{N}$ is cocompact in G.

Since $[N,N]$ is a characteristic normal subgroup of N (the commutator subgroup is always characteristic) $[N,N] \triangleleft \Lambda$ and so $\overline{[N,N]} \triangleleft \overline{\Lambda} = G$. Then by hypothesis either $\overline{[N,N]} \cap \Lambda$ is finite or $\overline{[N,N]}$ is cocompact in G.

Suppose that $\overline{[N,N]} \cap \Lambda$ is finite. Then $[N,N] \subseteq \Lambda$ so $[N,N] \subseteq \overline{[N,N]} \cap \Lambda$ is finite. Let $h_1, h_2 \in [\overline{N}, \overline{N}]$. Then $h_1 = \lim n_{1,m}$ and $h_2 = \lim n_{2,m}$ for $n_{1,m}, n_{2,m} \in N$ and so

$$h_1 h_2 h_1^{-1} h_2^{-1} = \lim n_{1,m} n_{2,m} n_{1,m}^{-1} n_{2,m}^{-1}$$

Since $n_{1,m} n_{2,m} n_{1,m}^{-1} n_{2,m}^{-1} \in [N,N]$ which is finite there is a subsequence along which

$$n_{1,m} n_{2,m} n_{1,m}^{-1} n_{2,m}^{-1} = n \in [N,N]$$

is constant. Then $h_1 h_2 h_1^{-1} h_2^{-1} = n \in [N,N]$ and we conclude that $[\overline{N}, \overline{N}] = [N,N]$ is finite.

Now $\overline{N}$ is cocompact and $\overline{N}$ is compactly generated since G is compactly generated (and it is cocompact) so by Lemma 9.1.1 the center $Z(\overline{N})$ is finite index in $\overline{N}$. Since $Z(\overline{N})$ is a characteristic normal subgroup of $\overline{N}$ we have $Z(\overline{N}) \triangleleft G$. But $Z(\overline{N})$ has finite index in a cocompact group so is itself cocompact. Hence G is a compact extension of an abelian group (as the center is abelian) contradicting our hypothesis. So we instead have that $\overline{[N,N]}$ is cocompact in G.

Let $\Lambda' = \Gamma \cdot N$. Then Λ' is a subgroup of (and possibly equal to) Λ and Λ' commensurates Γ. By definition of Λ' we have that Γ maps onto Λ'/N via the coset map $\gamma \mapsto \gamma N$. Of course Λ' is dense in $\overline{\Lambda'}$.

We apply the second theorem to the groups $\Gamma < \Lambda' < \overline{\Lambda'}$ and $N = N \cap \Lambda'$ is normal in Λ'. This is valid since $\overline{N}$ and $\overline{[N,N]}$ are cocompact in G and so $\overline{N}$ and $\overline{[N,N]}$ are cocompact in $\overline{\Lambda'}$.

The second theorem implies that Λ'/N is finite. Then $\Gamma \cdot N/N$ is finite and so $\Gamma/\Gamma \cap N$ is finite (group isomorphism theorem). $\square$

We made use of the following general fact about topological groups in the above proof:

Lemma 9.1.1. *Let H be a locally compact compactly generated group such that $[H,H]$ is finite. Then the center $Z(H)$ has finite index in H.*

Proof. Let $K \subseteq H$ be a compact generating set. For $x \in K$ consider the orbit of x under conjugation by H: $h \mapsto hxh^{-1}$. Since $[H,H]$ is finite we know that $hxh^{-1}x^{-1}$ only takes on finitely many values and so for each x the orbit $\{hxh^{-1} : h \in H\}$ is finite. Therefore there exists a finite index subgroup H_x of H that fixes x: $H_x = \{h \in H : hxh^{-1} = x\}$ has finite index. Note that H_x commutes with x.

Now H_x is compactly generated since H is (and H_x has finite index) so let $Q_x \subseteq H_x$ be a compact generating set for H_x. For $q \in Q_x$ observe that $qxq^{-1}x^{-1} = e$ by construction. By the continuity of the action of H on itself there is then an open neighborhood U_x of x such that $qyq^{-1}y^{-1} = e$ for all $q \in Q_x$ and all $y \in U_x$. This can be seen as follows: if no such neighborhood exists then there exists $x_n \to x$ and $q_n \in Q$ such that $q_n x_n q_n^{-1} x_n^{-1} \neq e$. Since $q_n x_n q_n^{-1} x_n^{-1} \in [H,H]$ is a finite set there is a subsequence on which $q_n x_n q_n^{-1} x_n^{-1} = z \neq e$ is constantly equal to $z \in [H,H]$. Take a further subsequence along which $q_n \to q \in Q_x$ since Q_x is compact. Then $q_n x_n q_n^{-1} x_n^{-1} \to qxq^{-1}x^{-1}$ and $q_n x_n q_n^{-1} x_n^{-1} = z$ hence $qxq^{-1}x^{-1} = z \neq e$ contradicting that $q \in H_x$. Hence there is such a neighborhood U_x.

Therefore for all $x \in K$ there is an open neighborhood U_x of x such that for all $q \in Q_x$ and all $y \in U_x$ we have $qyq^{-1}y^{-1} = e$. Since Q_x generates H_x this means that U_x commutes with H_x. Now $K \subseteq \bigcup_{x \in K} U_x$ is an open cover of a compact set hence there is a finite subcover: $K \subseteq \bigcup_{j=1}^{\ell} U_{x_j}$ for some $x_1, \ldots, x_\ell \in K$. Let

$$H_0 = \bigcap_{j=1}^{\ell} H_{x_j}$$

Then H_0 commutes with $U_{x_1}, \ldots, U_{x_\ell}$ hence it commutes with K and therefore H_0 commutes with all of H. Now H_0 has finite index in H since it is a finite intersection of finite index subgroups of H and since it commutes with H we know that $H_0 \subseteq Z(H)$ which therefore has finite index. $\square$

We now turn to proving the second theorem.

Proof. (of Theorem 9.4) By Theorem 9.5 (see below) Λ/N has property (T). By Theorem 9.10 (below) Λ/N is amenable. Together these imply that Λ/N is finite (Proposition B.3.1). $\square$

9.1.2 Property (T)

Theorem 9.5. *Let G be a lcsc c.g. group and let $\Gamma < G$ be a finitely generated integrable lattice in G. Let $\Lambda < \mathrm{Comm}(\Gamma)$ be a subgroup of the commensurator of Γ in G such that Λ is dense in G and $\Gamma < \Lambda$.*

Let N be a normal subgroup of Λ such that

(i) *Γ maps onto Λ/N via the coset map*

(ii) *$\overline{N}$ is cocompact in G*

(iii) *$\overline{[N, N]}$ is cocompact in G.*

Then Λ/N has property (T).

As with many other proofs of property (T), we make essential use of (first) cohomology and cocycles. Specifically, the proof follows the pattern of supposing the existence of a noncoboundary cocycle and deriving a contradiction from there.

Proof. To show that Λ/N has property (T) we will show that every irreducible representation of Λ/N on a Hilbert space has no nontrivial reduced cohomology, that is: $\overline{H^1}(\Lambda/N, \pi) = 0$. Note that while Λ may not be finitely generated, since Γ maps onto Λ/N and Γ is finitely generated we know that Λ/N is finitely generated. By Theorem 4.2 of [Sha06] (reproduced as Theorem 5.17) the vanishing of all reduced cohomology for irreducible representations implies property (T) (for lcsc c.g. and finitely generated groups hence for Λ/N). Let $\pi : \Lambda/N \to \mathcal{H}$ be an irreducible representation and suppose that $\overline{H^1}(\Lambda/N, \pi) \neq 0$.

The series of Lemmas below complete the proof in stages: first Lemma 9.1.2 shows that such a representation must be finite-dimensional; second Lemma 9.1.3 shows that such a finite-dimensional representation must have finite image; and third Lemma 9.1.4 shows that such a finite-dimensional representation with finite image must in fact be trivial.

We then obtain a contradiction and therefore will have shown that Λ/N in fact has property (T) since π was an arbitrary irreducible unitary representation. □

Lemma 9.1.2. *Let $\pi : \Lambda/N \to \mathcal{H}$ be an irreducible representation and suppose that there is nontrivial reduced cohomology: $\overline{H^1}(\Lambda/N, \pi) \neq 0$. Then π is finite-dimensional.*

Proof. Treat π as a representation of Λ with $\pi(N)$ being trivial. By Theorem 10.3 of [Sha00a] (reproduced as Theorem 9.9) π extends continuously to a unitary representation of G since $\overline{H^1}(\Lambda, \pi) \neq 0$ and π is irreducible (so there can be no nontrivial invariant subspaces). That Theorem applies since Γ maps onto Λ/N and so for a Λ-cocycle b if $b|_\Gamma$ is almost a coboundary then so is b.

Note that π is irreducible as a Λ representation (since it is as a Λ/N representation so as a Λ representation where N acts trivially it also must be) and therefore by the density of Λ, the extension to G is also irreducible. Since $\pi(N)$ is trivial, $\pi(\overline{N})$ will be trivial (the closure being taken in G). Then since $\overline{N}$ is cocompact in G the representation π on G is in fact an irreducible representation on $G/\overline{N}$ which is compact, meaning π is finite-dimensional. □

Lemma 9.1.3. *Let $\pi : \Lambda/N \to \mathcal{H}$ be a finite-dimensional irreducible representation and suppose that $\overline{H^1}(\Lambda/N, \pi) \neq 0$. Then $\pi(\Gamma)$ is finite.*

Proof. Suppose that $\pi(\Gamma)$ is infinite. Treat π as an irreducible Λ-representation. Since Γ maps onto Λ/N then π is an irreducible Γ-representation. Let Γ_{00} be a finite index subgroup of Γ and suppose that Γ_{00} has almost invariant vectors for π. Take $\Gamma_0 < \Gamma_{00}$ to be a finite index subgroup that is normal in Γ (which always exists since Γ_{00} has finite index in Γ). Then Γ_0 is a finite index normal subgroup of Γ with almost invariant vectors for π.

As π is finite-dimensional there is then an invariant vector for Γ_0 since there is a sequence of unit vectors u_n such that $\|\pi(\gamma_0)u_n - u_n\| \to 0$ for all γ_0 in a finite generating set of Γ_0 and the unit ball is (strongly) compact by finite-dimensionality hence there is a limit point u of some sequence of the u_n which must be an invariant unit vector. Now for $\gamma \in \Gamma$ we can write $\gamma = a\gamma_0$ for some a in a finite set A of representatives and some $\gamma_0 \in \Gamma_0$. Then $\pi(\gamma)u = \pi(a)\pi(\gamma_0)u = \pi(a)u$. Now since Γ_0 is normal, for all $\gamma_0 \in \Gamma_0$ and $a \in A$ we have that $\gamma_0 a = a\gamma_0'$ for some $\gamma_0' \in \Gamma_0$ and therefore $\pi(\gamma_0)\pi(a)u = \pi(a)\pi(\gamma_0')u = \pi(a)u$ for all $a \in A$ and $\gamma_0 \in \Gamma_0$ so $\pi(a)u$ is also Γ_0-invariant.

Let V be the space of Γ_0-invariant vectors (which is nontrivial). For $v \in V$ and $\gamma \in \Gamma$ we see that $\pi(\gamma)v = \pi(a)\pi(\gamma_0)v = \pi(a)v$ for some $a \in A$ and we have already shown that $\pi(a)V \subseteq V$. Hence V is a Γ-invariant subspace. Since π is irreducible (for Γ) then V is the entire space. But then $\pi(\Gamma) = \pi(A)$ is finite as claimed.

So instead assume that every Γ_{00} of finite index does not have almost invariant vectors for π. Let β be a nontrivial π-cocycle that is not a coboundary and consider the associated affine action of Λ. By Corollary 9.7, an application of the work of Gelander, Karlsson and Margulis, the affine action of Λ extends to a continuous isometric action of G. Now N is in the kernel of the representation hence $\overline{N}$ is in the kernel of the G-action. But $\overline{N}$ is cocompact so the G-action is the action of a compact group and hence has a fixed point. Then the Λ-action has a fixed point v. So $\pi(\lambda)v + \beta(\lambda) = v$ for all $\lambda \in \Lambda$. But this means that β is a coboundary, a contradiction. □

Corollary 9.7 is the key to the above proof and is a direct consequence of the work of Gelander, Karlsson and Margulis. We stress that we need to already know that π is finite-dimensional to apply their result and that in general (when π is infinite-dimensional) one cannot transition from almost invariant vectors to invariant vectors as we did above. The following appears as Theorem 8.1 in [GKM08]:

Theorem 9.6 (Gelander, Karlsson, Margulis)**.** *Let G be a locally compact, second countable, compactly generated group and Γ a cocompact lattice in G. Let $\Lambda < \mathrm{Comm}_G(\Gamma)$ containing Γ and which is dense in G. Let X be a completely uniform convex BNPC metric space. Assume that Λ acts by isometries on X such that any subgroup $\Gamma_0 < \Lambda$ commensurable to Γ satisfies that the displacement (see below) $d_{\Sigma_0} \to \infty$ where Σ_0 is a finite generating set of Γ_0, and has no parallel orbits. Assume further that the action is C-minimal. Then the Λ-action extends uniquely to a continuous isometric G-action.*

In Remark 8.9 of the paper, following the proof of the above Theorem, it is discussed how to remove the requirement that Γ be cocompact and in this case it is only necessary that Γ be p-integrable for some $1 < p < \infty$ (and our theorem will require that Γ be 2-integrable).

The displacement mentioned above is defined as $d_\Sigma(x) = \max_{\sigma\in\Sigma} d(\sigma x, x)$ and the condition required is that $d_\Sigma(x) \to \infty$ as $x \to \infty$ (that is, leaves compact sets).

Corollary 9.7. *Let G be a locally compact, second countable, compactly generated group and Γ an integrable lattice in G. Let $\Lambda < \mathrm{Comm}_G(\Gamma)$ such that $\Gamma < \Lambda$ and Λ is dense in G. Let $\pi : \Lambda \to \mathcal{H}$ be an irreducible unitary representation of Λ on a Hilbert space. Assume that every finite index subgroup of Γ does not admit almost invariant vectors for π. Then any affine action of Λ coming from a π-cocycle extends to a continuous isometric G-action.*

Proof. If Λ acts on a Hilbert space via an affine action associated to a unitary representation then the Hilbert space is a completely uniform convex BNPC metric space where Λ acts by isometries. The representation being irreducible will guarantee that the action is C-minimal. Hilbert space is geodesically complete hence there can be no parallel orbits (as remarked in [GKM08]).

Let $\lambda \cdot x$ denote an affine action on $\mathcal{H}$ coming from a π-cocycle $\beta : \Lambda \to \mathcal{H}$. Then $\lambda \cdot x = \pi(\lambda)x + \beta(\lambda)$. Observe that $\|\pi(\lambda)x - x\| \le \|\lambda\cdot x - x\| + \|\beta(\lambda)\|$.

Let Γ_0 be commensurable with Γ. Let Σ_0 be a finite generating set for Γ_0. Suppose that d_{Σ_0} is uniformly bounded on some unbounded set ($d(a, b) = \|a - b\|$ here). Then there exists $v_n \in \mathcal{H}$ with $\|v_n\| \to \infty$ such that $\|\sigma \cdot v_n - v_n\| \le C$ for all $\sigma \in \Sigma_0$. Write $u_n = \|v_n\|^{-1}v_n$. Then u_n are unit vectors and

$$\|\pi(\sigma)u_n - u_n\| = \|v_n\|^{-1}\|\pi(\sigma)v_n - v_n\| \le \|v_n\|^{-1}C + \|v_n\|^{-1}\|\beta(\sigma)\|$$

Since Σ_0 is finite there is a finite maximum for $\|\beta(\sigma)\|$. Hence $\|\pi(\sigma)u_n - u_n\| \to 0$ for all σ in a generating set for Γ_0. Hence u_n would be a sequence of almost invariant vectors for the finite index subgroup $\Gamma \cap \Gamma_0$ of Γ contradicting our assumption.

Then by the Theorem the Λ-action extends to a continuous isometric G-action as claimed. □

Lemma 9.1.4. *Let $\pi : \Lambda/N \to \mathcal{H}$ be a finite-dimensional irreducible representation and suppose that $\overline{H^1}(\Lambda/N, \pi) \neq 0$. Supposing $\pi(\Gamma)$ is finite leads to a contradiction.*

Proof. Since $\pi(\Gamma)$ is finite, there is some $\Gamma_0 < \Gamma$ of finite index such that $\pi(\Gamma_0)$ is trivial ($\pi(\Gamma)$ being finite means some operator in the range is the image of a finite index subgroup and therefore the trivial operator is also). Set $\Lambda_0 < \Lambda$ by $\Lambda_0 = \Gamma_0 N$. Still we must have that $H^1(\Lambda_0, \pi) \neq 0$ since otherwise every cocycle from Λ would have Λ_0 in its kernel and therefore be a cocycle from the finite group $\Lambda/\ker$ and as such would necessarily be a coboundary (Lemma 5.2.1).

There is therefore a nontrivial cocycle $\varphi : \Lambda_0 \to \mathbb{R}$ which is of course a homomorphism (i.e. a character). Since Γ_0 is a finitely generated integrable lattice in G, by Theorem 9.8 (which is Theorem 0.8 in [Sha00a] combined with the discussion in section 10 of that paper on nonuniform lattices which are integrable), $\varphi|_{\Gamma_0}$ extends to a nontrivial character on $G_0 = \overline{\Lambda_0}$ (note that it need not be the case that this extension agrees with φ on Λ_0, just on Γ_0). Call this extension $\psi : G_0 \to \mathbb{R}$.

Now ψ vanishes on $[N, N]$ hence it vanishes on $\overline{[N, N]}$ (the closure being taken in G_0). But $\overline{[N, N]}$ is cocompact in G_0 (since its closure in G is cocompact in G) hence ψ is a character on the compact group $G_0/\overline{[G_0, G_0]}$. Now any character on a compact group is trivial (Lemma 5.2.2) but this contradicts that ψ is a nontrivial character. □

Extending Affine Actions and Characters

The following result from Shalom's work [Sha00a] (discussed in more detail in [Sha00b]) appears as Theorem 0.8 in [Sha00a] with a discussion in section 10.4 of that paper indicating how integrability is in fact enough (and cocompactness is not strictly necessary):

Theorem 9.8 (Shalom)**.** *Let $\Gamma < \Lambda < \mathrm{Comm}_G(\Gamma) < G$ where G is locally compact, second countable, compactly generated and Γ is a finitely generated integrable lattice and Λ is dense in G. Let $\varphi : \Lambda \to \mathbb{C}$ be an additive homomorphism. Then $\varphi|_\Gamma$ extends continuously to G.*

The following appears as Theorem 10.3 in [Sha00a] (combined with the remarks in section 10.4 on nonuniform lattices):

Theorem 9.9 (Shalom)**.** *Let $\Gamma < \Lambda < \mathrm{Comm}_G(\Lambda) < G$ such that Λ is dense in G (and G is locally compact, second countable, compactly generated) and Γ is a finitely generated integrable lattice in G. Let $(\pi, \mathcal{H})$ be a unitary Λ-representation and $b \in Z^1(\Lambda, \pi)$ a cocycle. If $b|_\Gamma$ is not almost a coboundary then there exists a nonzero Λ-invariant subspace on which the linear Λ-action extends continuously to a unitary representation of G.*

9.1.3 Amenability

Theorem 9.10. *Let G be a locally compact, second countable group and let $\Gamma < G$ be a lattice in G. Let $\Lambda < \mathrm{Comm}(\Gamma)$ be a subgroup of the commensurator of Γ in G such that Λ is dense in G and $\Gamma < \Lambda$.*

Let N be a normal subgroup of Λ such that $\overline{N}$ is cocompact in G and such that Γ maps onto Λ/N via the coset map. Then Λ/N is amenable.

The main ingredient in the amenability proof is the SAT Factor Theorem (Theorem 3.9) along the lines of Margulis' original Factor Theorem used in the amenability half of the Normal Subgroup Theorem and along the lines of the Factor Theorem in [BS05] used for the Normal Subgroup Theorem for lattices in products.

Proof. (of Theorem 9.10 from Theorem 3.9) To show that Λ/N is amenable it is enough to show that for any compact metric space where Λ/N acts continuously that there is a Λ/N invariant probability measure. Let Z be a compact metric space that Λ/N acts on continuously. Then Λ acts on Z with the action of N being trivial.

Let α be an admissible probability measure on G (admissible meaning the support is a generating set and some convolution power is nonsingular w.r.t. Haar measure) and let (X, ν) be the Poisson Boundary of (G, α). By Lemma 3.5.1 the action of Γ on (X, ν) is SAT.

The action of G on (X, ν) is amenable (see Zimmer [Zim84] for the definition of amenable action and a proof of this fact) and therefore the action of Γ on (X, ν) is also amenable (Γ being closed in G). There is then a Γ-equivariant map $\varphi : X \to P(Z)$. Let $Y = P(Z)$ and $\eta = \varphi_* \nu \in P(Y)$ so that $\varphi : (X, \nu) \to (Y, \eta)$ is a Γ-map.

By hypothesis Γ maps onto Λ/N via the coset map $\gamma \mapsto \gamma N$ so for any $\lambda \in \Lambda$ there is some $\gamma \in \Gamma$ such that $\gamma N = \lambda N$. This means that for any $\lambda \in \Lambda$ there exists $\gamma \in \Gamma$ and $n_1, n_2 \in N$ such that $\gamma n_1 = \lambda n_2$. Now

$$\lambda\eta = \lambda n_2 \eta = \gamma n_1 \eta = \gamma\eta$$

since η is N-invariant (as N acts trivially on Z). Therefore the Λ-action on (Y, η) is quasi-invariant so (Y, η) is in fact a Λ-space.

Then by the Factor Theorem φ extends to being a G-map to a measurably Λ-isomorphic G-space (Y', η'). Since N acts trivially on Z the same is true on $Y = P(Z)$ and therefore $\overline{N}$ acts trivially on Y'. Moreover η is invariant under N and therefore η' is $\overline{N}$-invariant.

Let $Q = G/\overline{N}$. Then Q is a compact group by hypothesis. Since η' is quasi-invariant under G it also is under Q. Let $\eta'' = \int_Q q\eta' dq$ be the average of the translates of η' under the normalized Haar measure on Q. Then η'' is Q-invariant hence G-invariant and in the same measure class as η'. Let η''' be the isomorphic image of η'' on Y. So η''' is a Λ-invariant probability measure on $Y = P(Z)$. Take ρ to be the barycenter of η''': for $f \in C(Z)$

$$\rho(f) = \int_{P(Z)} \zeta(f) \; d\eta'''(\zeta)$$

Then $\rho \in P(Z)$ is Λ-invariant since

$$\lambda\rho = \int_{P(Z)} \lambda\zeta \; d\eta'''(\zeta) = \int_{P(Z)} \zeta \; d\lambda\eta'''(\zeta) = \rho$$

since η''' is Λ-invariant. Hence ρ is a Λ/N-invariant probability measure on Z which completes the proof.

□

9.1.4 The Conditions Are Necessary

Our main theorem imposes two conditions on the group G. We present examples of groups where the conclusion does not hold showing that these conditions are in fact necessary for such a statement.

Obviously if Λ is not assumed dense in G then our requirements on G must instead be applied to the closure of Λ so that condition is necessary for the statement of the Theorem to make sense.

9.1.5 Compact Extensions of Abelian Groups

First let us look at the case when $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$. Let $\Lambda = \mathbb{Z}[\sqrt{2}]$. Since $1, \sqrt{2}$ generate the reals, Λ is dense in $\mathbb{R}$. For $a + b\sqrt{2} \in \Lambda$ we have that

$$(a + b\sqrt{2}) + \mathbb{Z} + (-a - b\sqrt{2}) = \mathbb{Z}$$

so Λ commensurates Γ (already we can see how abelian groups pose a problem). Let $N = \sqrt{2}\mathbb{Z}$. Then N is an infinite normal subgroup of Λ (since it is a subgroup and the ambient group is abelian) but $N \cap \Gamma = \sqrt{2}\mathbb{Z} \cap \mathbb{Z} = 0$ does not have finite index in Γ.

One can see that the above generalizes to any abelian group G and lattice Γ (since the commensurator $\mathrm{Comm}_G(\Gamma) = G$). Specifically, to see that it is not enough to merely assume G is nonabelian, consider the groups

$$\Gamma = \mathbb{Z}^n < \Lambda = \mathrm{SO}(n, \mathbb{Q}) \ltimes (\mathbb{Q}[\sqrt{2}])^n < G = \mathrm{SO}(n, \mathbb{R}) \ltimes \mathbb{R}^n$$

Since $\mathrm{SO}(n)$ is an algebraic group, Λ is dense in G (the $\mathbb{Q}$-points of an algebraic group are always dense in the $\mathbb{R}$-points). Moreover, $\mathbb{Q}\sqrt{2}$ commensurates $\mathbb{Z}^n$ as does $\mathrm{SO}(n, \mathbb{Q})$ (both since $\mathbb{Q}$ commensurates $\mathbb{Z}$) and G/Γ is compact (since $\mathrm{SO}(n, \mathbb{R})$ is compact and $\mathbb{R}^n/\mathbb{Z}^n$ is compact) and Γ is obviously discrete in G hence Γ is a lattice. Therefore the groups satisfy the basic setup of our Theorem.

However, the group $N = \sqrt{2}\ \mathbb{Q}^n$ is an infinite normal subgroup of Λ (since it sits in the abelian part and for $g \in \mathrm{SO}(n, \mathbb{Q})$ we see that $g\sqrt{2}(q_1, \cdots, q_n) = \sqrt{2}\ g(q_1, \cdots, q_n) \in \sqrt{2}\ \mathbb{Q}^n$ since $\mathrm{SO}(n, \mathbb{Q})$ acts on $\mathbb{Q}^n$) but $N \cap \Gamma$ is the trivial group so the conclusion of our Theorem does not hold. Since $\mathrm{SO}(n, \mathbb{R})$ is compact, G is a compact extension of an abelian group. It is easy to see that the only nontrivial normal subgroup of G is $\mathbb{R}^n$ (since $\mathrm{SO}(n, \mathbb{R})$ acts transitively on $\mathbb{R}^n$ and is simple) and therefore the only closed normal subgroup of G which is not cocompact is the trivial group which obviously has finite intersection with Λ so the second condition of our Theorem is satisfied. This shows that the requirement that G not be a compact extension of an abelian group is in fact necessary.

9.1.6 Noncocompact Normal Subgroups

Let G be a lcsc c.g. group, Γ a lattice in G and $\Lambda < \mathrm{Comm}(\Gamma)$ a subgroup of the commensurator of Γ which is dense in G. Suppose there exists $M \triangleleft G$ not cocompact such that $M \cap \Lambda$ is infinite. Then $N = M \cap \Lambda$ is an infinite normal subgroup of Λ but $M \cap \Gamma$ cannot have finite index in Γ since M is not cocompact.

Concretely, let $G = \mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ and $\Gamma = \mathrm{SL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$. Then the commensurator $\Lambda = \mathrm{SL}_n(\mathbb{Q}) \ltimes \mathbb{Q}^n$ is obviously dense and there is a normal subgroup $M = \mathbb{R}^n$ of G which is not cocompact. Then $N = \mathbb{Q}^n$ is a normal subgroup of Λ and $N \cap \Gamma = \mathbb{Z}^n$ is a normal subgroup of Γ of infinite index.

Clearly G is not a compact extension of an abelian group since $\mathrm{SL}_n(\mathbb{R})$ has no cocompact abelian subgroup. Therefore the requirement on non-cocompact closed normal subgroups is also necessary.

9.1.7 On the Integrability Condition

As one can see, the requirement that the lattice be integrable is only necessary to make use of auxiliary results as part of the property (T) half of the proof. In the papers those results appear in, it is shown that in fact one can replace the need for integrability by the so-called weak cocompactmess of Γ in G. We say that a lattice Γ is weakly cocompact in G when the trivial representation is not weakly contained in $L^2_0(G/\Gamma)$ (i.e. does not almost have invariant vectors). The interested reader should consult [GKM08] and [Sha00b] for more information about this notion that was originally due to Margulis [Mar91].

9.2 The Normal Subgroup Theorem

We now develop a true normal subgroup theorem for (dense) commensurating groups.

Corollary 9.11. *Let G be a lcsc c.g. group and let $\Gamma < G$ be a finitely generated integrable lattice in G. Let $\Lambda < \mathrm{Comm}(\Gamma)$ be a subgroup of the commensurator of Γ in G such that Λ is dense in G and $\Gamma < \Lambda$.*

Assume that G is not a compact extension of an abelian group. Assume that for every closed normal subgroup M of G which is not cocompact the intersection $M \cap \Lambda$ is finite.

Then there is a one-to-one, onto correspondence between commensurability classes of infinite normal subgroups of Λ and commensurability classes of open normal subgroups of the relative profinite completion $\Lambda//\Gamma$.

By commensurability class we mean that two normal subgroups are in the same class when their intersection has finite index in both (they are commensurate directly, no need for conjugation). Note that in particular the finite index normal subgroups of Λ correspond to the finite index open normal subgroups of $\Lambda//\Gamma$.

Proof. Let $H = \Lambda//\Gamma$. For this proof, the topology involved in closures is the topology of H and not G. Let $\tau : \Lambda \to H$ be the canonical map ($\tau = \tau_{\Lambda,\Gamma}$) and let $\Lambda' = \tau(\Lambda)$ and $\Gamma' = \tau(\Gamma)$ be the images of Λ and Γ in H, respectively.

Let $N \triangleleft \Lambda$ be an infinite normal subgroup. By Theorem 9.2, $\Gamma_0 = N \cap \Gamma$ has finite index in Γ. Therefore $\overline{\tau(\Gamma_0)}$ has finite index in $\overline{\Gamma'}$ (the closures are taken in H).

Since $\overline{\Gamma'}$ is a compact open subgroup so is $\overline{\tau(\Gamma_0)}$ (since it has finite index). Hence $\overline{\tau(N)}$ contains an open subgroup and so is itself open (H being totally disconnected).

Since $N \triangleleft \Lambda$ we have $\overline{\tau(N)} \triangleleft \overline{\Lambda'} = H$. Therefore $\overline{\tau(N)}$ is an open normal subgroup of H. This shows that for any infinite normal subgroup N of Λ there is a corresponding open normal subgroup $\overline{\tau(N)}$ of H.

We now show that this mapping is injective up to finite index. Let $N \triangleleft \Lambda$ and write $\Lambda' = \tau(\Lambda)$ and $N' = \tau(N)$ and $\Gamma' = \tau(\Gamma)$. Set $\Lambda_0 = \Lambda' \cap \overline{N'}$.

Since N' is dense in $\overline{N'}$ and $\overline{\Gamma'}$ is open, $\overline{N'} \subseteq \overline{\Gamma'}N'$.

Let $\lambda \in \Lambda_0$. Then $\lambda = xn$ for some $x \in \overline{\Gamma'}$ and some $n \in N'$. So $\lambda n^{-1} \in \Lambda'$ (since $N' \subseteq \Lambda'$) but also $\lambda n^{-1} = x \in \overline{\Gamma'}$.

Now $\Lambda' \cap \overline{\Gamma'} = \Gamma'$ (from the definition of relative profinite completion, Lemma 8.4.4). Hence $\lambda n^{-1} \in \Gamma'$ and therefore $\lambda \in \Gamma' N'$. So $\Lambda_0 \subseteq \Gamma' N'$. Therefore

$$[\Lambda_0 : N'] \leq [\Gamma' N' : N'] = [\Gamma' : \Gamma' \cap N'] \leq [\Gamma : \Gamma \cap N] < \infty$$

Now let N_1 and N_2 both be infinite normal subgroups of Λ such that $\overline{\tau(N_1)} = \overline{\tau(N_2)}$ (i.e. that map to the same open normal subgroup of H). Then $\tau(N_1)$ has finite index in $\Lambda_0 = \tau(\Lambda) \cap \overline{\tau(N_1)}$ and $\tau(N_2)$ also has finite index in Λ_0. Therefore

$$[\Lambda_0 : \tau(N_1 \cap N_2)] = [\Lambda_0 : \tau(N_1) \cap \tau(N_2)] \leq [\Lambda_0 : \tau(N_1)][\Lambda_0 : \tau(N_2)] < \infty$$

and also

$$[\Lambda_0 : \tau(N_1 \cap N_2)] = [\Lambda_0 : \tau(N_1)][\tau(N_1) : \tau(N_1 \cap N_2)]$$

and so $\tau(N_1 \cap N_2)$ has finite index in $\tau(N_1)$.

Now write $N = N_1 \cap N_2$ and $L = \ker(\tau) \cap N$ and $L_1 = \ker(\tau) \cap N_1$. Then $\tau(N)$ is isomorphic to N/L and $[\tau(N_1) : \tau(N)] < \infty$. We also see that

$$\tau(N_1) \cong N_1/L_1 \cong (N_1/L)/(L_1/L)$$

by the Isomorphism Theorems. Now

$$[L_1 : L] = [\ker(\tau) \cap N_1 : \ker(\tau) \cap N_1 \cap N_2] \leq [\Gamma : \Gamma \cap N_2]$$

since $\ker(\tau) \subseteq \Gamma$. By the Third Isomorphism Theorem

$$[N_1 : N] = [N_1/L : N/L] \leq [\tau(N_1) : \tau(N)][\Gamma : \Gamma \cap N_2] < \infty$$

since $[\Gamma : \Gamma \cap N_2] < \infty$ by the Theorem. Hence the correspondence is injective up to finite index in the sense that two normal subgroups which map to the same open normal subgroup are the same up to finite index.

Finally, for any open normal subgroup M of H set $N = \tau^{-1}(M \cap \tau(\Lambda))$. Then $N = \tau^{-1}(M)$. Now for $\lambda \in \Lambda$

$$\tau(\lambda N \lambda^{-1}) = \tau(\lambda)\tau(N)\tau(\lambda)^{-1} = \tau(\lambda)(M \cap \tau(\Lambda))\tau(\lambda)^{-1} = M \cap \tau(\Lambda)$$

since M is normal. Hence $\lambda N \lambda^{-1}) \subseteq \tau^{-1}(M) = N$. Thus $N \triangleleft \Lambda$. Moreover

$$\overline{\tau(N)} = \overline{\tau(\tau^{-1}(M))} = \overline{M \cap \tau(\Lambda)} = M$$

since Λ is dense and M is closed, hence the mapping $M \mapsto \tau^{-1}(M)$ inverts the previous mapping. Thus the correspondence is surjective. □

Normal Subgroup Theorem Examples

We present several corollaries of our Normal Subgroup Theorem by considering certain classes of locally compact groups. In effect we develop Normal Subgroup Theorems for commensurators of lattices in (topologically) simple groups; a strengthening of the the work of Bader and Shalom [BS05]; and some additional examples of classes of groups where our result applies.

10.1 Just Infinite Groups

Our results in this chapter will be phrased in terms of just infinite and just noncompact groups. Recall the following definitions:

Definition 10.1. A discrete group Λ is **just infinite** when every normal subgroup $N \triangleleft \Lambda$ is either finite or has finite index in Λ (in symbols: $|N| < \infty$ or $[\Lambda : N] < \infty$).

Definition 10.2. A locally compact group G is **just noncompact** when every nontrivial closed normal subgroup $N \triangleleft G$ is cocompact or compact.

10.2 Commensurators in Simple Groups

The first class of locally compact groups we specialize to are the topologically simple groups (those with no nontrivial closed normal subgroups).

Corollary 10.3. *Let G be a lcsc c.g. group that is not a compact extension of an abelian group and that is simple as a topological group. Let $\Gamma < G$ be a finitely generated integrable lattice and $\Lambda < \mathrm{Comm}(\Gamma)$ be a subgroup of the commensurator of Γ such that Λ is dense in G and $\Gamma < \Lambda$. Suppose further that $\Lambda // \Gamma$ has no infinite index noncompact open normal subgroups. Then Λ is just infinite.*

A key example is the group $\Lambda = \mathrm{PSL}_2(\mathbb{Z}[1/p])$ for some prime p. Take $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and $G = \mathrm{PSL}_2(\mathbb{R})$ so that Γ is a lattice in G and Λ commensurates Γ and is dense in G. Since G is a simple topological group (and clearly not a compact extension of an abelian group) the hypotheses of the Theorem are satisfied.

We saw previously that $\Lambda // \Gamma = \mathrm{PSL}_2(\mathbb{Q}_p)$ which is also simple and therefore the hypotheses of the Corollary are satisfied. We then conclude that $\mathrm{PSL}_2(\mathbb{Z}[1/p])$ is just infinite.

10.3 Irreducible Lattices in Products

We now prove a version of the Normal Subgroup Theorem for lattices in products due to Bader and Shalom [BS05]. This is the most straightforward example of the fact that often lattices are themselves commensurators of other lattices. The first statement we prove is a restricted form of Bader-Shalom's result (that applies only in the case when one of the groups is totally disconnected):

Corollary 10.4 (Bader-Shalom). *Let $\Lambda < G \times H$ be a finitely generated irreducible integrable lattice in a product of lcsc c.g. groups such that H is totally disconnected. If G is nondiscrete and not a compact extension of an abelian group and G is just noncompact and H has no infinite index noncompact open normal subgroups then Λ is just infinite.*

Proof. Set $\Gamma = \Lambda \cap G \times K$ where K is a compact open subgroup of H. Then $\mathrm{proj}_K \Gamma$ is dense in K (the projection of Λ to H is dense since the lattice is irreducible).

In general, if L is a lattice in a locally compact second countable group M and U is an open subgroup of M then $L \cap U$ is a lattice in U. Applying this to Λ we find that Γ is a lattice in $G \times K$ since K is open.

Since Λ is integrable we see that Γ is also (take the fundamental domain for Λ and generating set and intersect them accordingly). Likewise the finite generation of Λ implies that of Γ.

Since K is compact the projection of Γ to G is a lattice in G: that Γ has finite covolume in $G \times K$ implies that it does projected to G; if the projection were not discrete then there would exist $\gamma_n, \gamma \in \Gamma$ such that $\mathrm{proj}\ \gamma_n \to \mathrm{proj}\ \gamma$ and if we let W be a compact neighborhood of $\mathrm{proj}\ \gamma$ then $\{\mathrm{proj}\ \gamma_n : n \in \mathbb{N}\} \subseteq W$ so $\{\gamma_n : n \in \mathbb{N}\} \subseteq W \times K$ which is compact hence γ_n has a convergent subsequence contradicting the discreteness in $G \times K$.

Since K is commensurated by H, being a compact open subgroup, we have that Γ is commensurated by Λ. Of course the projection of Λ to G is dense and therefore $\mathrm{proj}_G \Gamma < \mathrm{proj}_G \Lambda < G$ satisfy the setup of our main theorem.

Since G is just noncompact there are no noncocompact closed normal subgroups. Also H has no infinite index noncompact open normal subgroups.

Now $\Lambda//\Gamma = H/\ker(\tau_{H,K})$ since $\mathrm{proj} : \Lambda \to H$ is a homomorphism such that $\overline{\mathrm{proj}(\Lambda)} = H$ and $\mathrm{proj}^{-1}(K) = \Lambda \cap G \times K = \Gamma$ hence Theorem 8.6 implies $\Lambda//\Gamma$ is isomorphic to $H//K$ which in turn is isomorphic to $H/\ker(\tau_{H,K})$. But $\ker(\tau_{H,K}) \lhd H$ is compact (since $\ker_{H,K}$ is contained in K which is compact) hence $\Lambda//\Gamma$ is isomorphic to H modulo a compact normal subgroup. Since H has no infinite index noncompact open normal subgroups neither does this quotient. Then $\mathrm{proj}_G \Lambda$ is just infinite since there are no infinite index open normal subgroups of this quotient.

Let $N \lhd \Lambda$ be infinite. Consider the map $\rho : \Lambda \to \mathrm{proj}_G \Lambda$ given by $\rho(\lambda) = \mathrm{proj}_G \lambda$. Then $\ker(\rho) = \Lambda \cap \{e\} \times H$ and $\ker(\rho) \lhd \Lambda$. Since Λ is discrete in $G \times H$ we know that $\ker(\rho)$ is discrete in $G \times H$. Hence $\ker(\rho)$ is discrete in H. As H is just noncompact and $\ker(\rho)$ is closed, either $\ker(\rho)$ is finite or cocompact.

If $\ker(\rho)$ is cocompact then $\Lambda/\ker(\rho)$ is an irreducible lattice in $G \times (H/\ker(\rho))$ and projects discretely to G since $H/\ker(\rho)$ is compact but also projects densely since the lattice is irreducible meaning that G itself is discrete contradicting our hypothesis.

On the other hand, if $\ker(\rho)$ is finite then $[\Lambda : N] \leq [\mathrm{proj}_G \Lambda : \mathrm{proj}_G N]|\ker(\rho)|$. Now $[\mathrm{proj}_G \Lambda : \mathrm{proj}_G N] = \infty$ would imply that $\mathrm{proj}_G N$ is finite since $\mathrm{proj}_G N \lhd \mathrm{proj}_G \Lambda$ and

$\text{proj}_G\ \Lambda$ is just infinite. But $\text{proj}_G\ N$ being finite and $\ker(\rho)$ being finite together imply that N is finite. So we conclude that $[\Lambda : N] < \infty$.

Therefore we conclude that Λ is just infinite since every infinite normal subgroup is of finite index. □

While the above statement does not apply to all products of groups, our methods allow us to strengthen the restricted form of their Theorem to include the converse statement:

Corollary 10.5. *Let $\Lambda < G \times H$ be a finitely generated irreducible integrable lattice where G and H are lcsc c.g. groups, G is nondiscrete and not a compact extension of an abelian group and H is totally disconnected. Then Λ is just infinite if and only if G is just noncompact and H has no infinite index noncompact open normal subgroups.*

Proof. By Corollary 10.4 we know that Λ is just infinite when G is just noncompact and H has no nontrivial noncompact open normal subgroups.

Suppose G has a noncompact closed normal subgroup M. Then $M \times H$ is a closed normal subgroup of $G \times H$ which is not cocompact. Set $N = \Lambda \cap M \times H$. Then N is an infinite normal subgroup of Λ (if it were finite then M would be compact since Λ projects densely to G) but cannot have finite index since $M \times H$ is not cocompact. Hence if G is not just noncompact then Λ cannot be just infinite.

Now suppose that H has an infinite index noncompact open normal subgroup M. Let Γ be the lattice in constructed in the previous Corollary (10.4) which has the property that $\Lambda // \Gamma = H/\ker(\tau_{H,K})$. Now $\ker(\tau_{H,K}) \subseteq K$ (Lemma 8.4.2) so $\ker(\tau_{H,K})$ is compact. Let M' be the image of M in $H/\ker(\tau_{H,K})$, a closed normal subgroup. If M' were finite index then $(H/\ker(\tau_{H,K}))/(M/M \cap \ker(\tau_{H,K}))$ is finite and so H/M would be finite and then M is finite index. If M' were compact then M would be compact since $\ker(\tau_{H,K})$ is compact. Then by the one-to-one correspondence of open normal subgroups of $H/\ker(\tau_{H,K})$ with infinite normal subgroups of Λ (Corollary 9.11) M' must correspond to an infinite normal subgroup N of Λ which is of infinite index. □

10.4 Further Examples

We now state various results on lattices in specific lcsc c.g. groups showing that they are integrable and have dense commensurators. Our purpose here is to demonstrate that our Theorem applies to a wide variety of known examples and implies new facts about (most of) them.

The class of groups where all lattices are known to be integrable includes all simple and semisimple Lie groups [Lub95], [Sha00b] and Kac-Moody groups [Sha00b]. Of course, cocompact lattices (and in fact weakly cocompact lattices, see below) are integrable regardless of ambient group.

Groups where lattices are known to have dense commensurators include simple and semisimple (higher-rank) Lie (and p-adic Lie) groups [Mar91], automorphism groups of trees [Liu94], [Lub95], [BL01], and Kac-Moody groups and buildings lattices [CR09].

These mean our Theorem applies to cocompact (and sometimes nonuniform) lattices in the above groups. When a group of one of the types listed above is known to be simple and not a compact extension of an abelian group (easy conditions to verify) then we immediately obtain the one-to-one correspondence describing normal subgroups of the commensurator (and every dense subgroup of it).

Chapter 11

Commensurators of Tree Lattices

As we have indicated, the theory of lattices in Lie groups is fairly complete. More recently, the study of lattices in automorphism groups of trees, being the natural totally disconnected counterpart of Lie groups, has focused on extending the results for Lie groups to tree automorphisms. The reader is referred to the works of Bass, Burger, Farb, Lubotzky, Mozes, Zimmer and many others; we refer in particular [BL01], [BM00] and [Moz98] for more information.

We outline the basic theory of tree lattices and use our Normal Subgroup Theorem to make some progress on a question about commensurators of tree lattices: namely for a uniform lattice in the automorphism group of regular tree the commensurator is just infinite if and only if the corresponding relative profinite completion is has no infinite index noncompact open normal subgroups.

11.1 Tree Automorphisms

Let T be a locally finite tree: that is, T is a graph containing no cycles and such that each vertex has only finitely many edges connected to it. The automorphism group of the tree is denoted by $\mathrm{Aut}(T)$ and is the collection of all maps $T \to T$ taking vertices to vertices preserving the edges: if v_1 and v_2 share an edge then so do their images. The requirement that T be locally finite is necessary to ensure that $\mathrm{Aut}(T)$ is a locally compact group.

11.2 Tree Lattices

The term **tree lattice** is used to refer to a lattice in the automorphism group of a tree. The reader is referred to Bass and Lubotzky [BL01] for a thorough study and survey of tree lattices.

Definition 11.1. For a closed subgroup $H < \mathrm{Aut}(T)$ define the **vertex stabilizer subgroup** in H of a vertex $x \in T$ by

$$H_x = \{h \in H : hx = x\}$$

Proposition 11.2.1. *A subgroup $H < \mathrm{Aut}(T)$ is discrete precisely when H_x is finite for some vertex $x \in T$ (equivalently for every vertex in T).*

Proposition 11.2.2. *A discrete subgroup $\Gamma < \mathrm{Aut}(T)$ will have a finite volume fundamental domain, that is be a lattice, if and only if*

$$\sum_{x \in \Gamma \backslash T} |\Gamma_x|^{-1} < \infty$$

The reader is referred to [FMT07] for details and a proof of this fact.

Proposition 11.2.3. *A lattice will be uniform (cocompact) when* $\Gamma\backslash T$ *is a finite graph.*

11.3 A Question About Commensurators

Our theorem makes progress in particular on a question of Lubotzky, Mozes and Zimmer regarding simplicity of commensurators of lattices in tree automorphism groups [LMZ94] (see also [FMT07] and [FHT08]).

Question 11.2 (Lubotzky-Mozes-Zimmer)**.** *When is the commensurator of a uniform lattice in a tree automorphism group simple?*

11.4 Commensurators of Tree Lattices

Let $\mathrm{Comm}(\Gamma)$ be the commensurator of Γ in $G = \mathrm{Aut}(T)$. Provided G is simple, our main theorem implies that the infinite normal subgroups of $\mathrm{Comm}(\Gamma)$ are in one-one correspondence with the noncompact closed normal subgroups of $\mathrm{Comm}(\Gamma)//\Gamma$:

Theorem 11.3. *Let* Γ *be a lattice in* $\mathrm{Aut}(T)$ *where* T *is a locally finite tree. Assume that* $\mathrm{Aut}(T)$ *is simple and* $\mathrm{Comm}(\Gamma)$ *is dense. Then* $\mathrm{Comm}(\Gamma)$ *is just infinite if and only if* $\mathrm{Comm}(\Gamma)//\Gamma$ *has no infinite index noncompact open normal subgroups.*

11.5 Uniform Regular Tree Lattices

The tree T is called regular (or k-regular) when every vertex has the same number of edges k. There is of course only one such tree for each k. Tits [Tit70] has shown that the automorphism group of a regular tree is virtually simple (there is a finite index simple closed subgroup). Liu has shown that commensurators of uniform lattices in tree automorphism groups are dense [Liu94] and therefore we have the following contribution to the question of Lubotzky, Mozes and Zimmer:

Theorem 11.4. *Let* Γ *be a uniform lattice in the automorphism group of a regular tree. Then* $\mathrm{Comm}(\Gamma)$ *is just infinite if and only if* $\mathrm{Comm}(\Gamma)//\Gamma$ *has no infinite index noncompact open normal subgroups. Moreover, any infinite normal subgroup of* $\mathrm{Comm}(\Gamma)$ *contains (up to finite index)* Γ.

IV

Commensurated Subgroups of Lattices

Chapter 12

The Margulis-Zimmer Conjecture

We conclude the dissertation with the statement of a conjecture of Margulis and Zimmer dating to the late 1970s and our progress on this conjecture (half of the conjecture is proven here–to be precise: the property (T) half). This conjecture is the natural extension of the Margulis Normal Subgroup Theorem for lattices to the true "up to finite index" point of view the theory is based on.

12.1 The Framework for the Conjecture

Before we can state the conjecture we need to establish the exact nature of the objects being studied. Specifically, the arithmeticity and normal subgroup theorems for lattices in Lie groups are necessary to even formulate the statement. The reader unfamiliar with algebraic groups, arithmetic lattices or Margulis' Arithmeticity Theorem should consult Appendix C: Algebraic Groups for definitions and statements.

12.1.1 Objects of Study

Let $\mathbf{G}_1, \ldots, \mathbf{G}_\ell$ be connected simple algebraic groups which are defined over local fields $k_1, \ldots, k_\ell$ all of characteristic zero. Write $G = \mathbf{G}_1(k_1) \times \cdots \times \mathbf{G}_\ell(k_\ell)$. We shall require that the rank of G be at least two, meaning that either the k_j-rank of $\mathbf{G}_j$ is at least two for some j or that $\ell \geq 2$ (the definition of the rank can be found in Appendix C: Algebraic Groups, Definition C.14) and that G be compactly generated.

Let $\Gamma < G$ be a finitely generated strongly irreducible lattice in G (that is, the projection of Γ to any proper subproduct of the $G_j(k_j)$ is dense). Let $\Lambda < \Gamma$ be a commensurated subgroup of Γ.

12.1.2 Arithmeticity

By Margulis' Arithmeticity Theorem (Theorem C.21), since G is higher-rank Γ is necessarily S-arithmetic and so we may work entirely in the framework of S-arithmetic groups. To this end, let K be a global field and let V be the set of inequivalent valuations on K. Let $V_\infty \subset V$ denote the archimedean valuations. Let $V_\infty \subseteq S \subseteq V$ and denote by $\mathcal{O}_S$ the ring of S-integers in K (and let $\mathcal{O}$ be the ring of integers in K). According to the Arithmeticity Theorem Γ is commensurable with the image of $\mathbf{G}(\mathcal{O}_S)$ under an isogeny from $\prod_{v \in S} \mathbf{G}(K_v)$ to G (here K_v denotes the completion of K under the valuation v) where $\mathbf{G}$ is an absolutely simple, simply connected algebraic group defined over K.

12.2 Standard Commensurated Subgroups

The Normal Subgroup Theorem for lattices tells us that lattices in higher-rank Lie groups have no nontrivial normal subgroups. When interested in studying the commensurated subgroups of a lattice in a Lie group, however, there are obvious examples of nontrivial commensurated subgroups:

Definition 12.1. Let Γ be an S-arithmetic group for some set of valuations $V_\infty \subseteq S \subseteq V$ as above (Γ is commensurable with $\mathbf{G}(\mathcal{O}_S)$). Let $V_\infty \subseteq S' \subseteq S$. Any subgroup $\Lambda < \Gamma$ which is commensurable with $\mathbf{G}(\mathcal{O}_{S'})$ is called a **standard commensurated subgroup of** Γ.

The fact that standard commensurated subgroups are in fact commensurated is obvious. For example, $\mathrm{SL}_2(\mathbb{Z}[1/p])$ commensurates $\mathrm{SL}_2(\mathbb{Z})$ and sits as a lattice in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$ and it is clear that $\mathrm{SL}_2(\mathbb{Z})$ corresponds to $S' = \{\infty\} \subsetneq S = \{p, \infty\}$. More generally, $\mathbf{G}(\mathcal{O}_{S'}) < \mathbf{G}(\mathcal{O}_S)$ is commensurated for the same reason: $\mathcal{O}_S$ adds only a finite number of additional valuations to $\mathcal{O}_{S'}$.

12.3 The Conjecture

Margulis and Zimmer formulated the following conjecture, the extension of the arithmeticity theorem to the commensurated setting which is the natural setting (up to finite index) for this type of result:

Conjecture. *Let Γ be a strongly irreducible lattice in a higher-rank simple algebraic group. Then any infinite commensurated subgroup of Γ is a standard commensurated subgroup.*

By the Margulis Arithmeticity Theorem this statement is equivalent to:

Conjecture. *Let K be a global field, $\mathcal{O}$ the ring of integers, and $\mathbf{G}$ an absolutely simple, simply connected algebraic group defined over K. Let V be the set of inequivalent valuations on K and $V_\infty \subseteq V$ the archimedean valuations. For $V_\infty \subseteq S \subseteq V$ let $\mathcal{O}_S$ be the ring of S-integers and let Γ be a finite index subgroup of $\mathbf{G}(\mathcal{O}_S)$. Assume that the S-rank of $\mathbf{G}$ is at least two. Let $\Lambda < \Gamma$ be an infinite commensurated subgroup of Γ. Then Λ is a standard commensurated subgroup of Γ.*

Chapter 13

The Property (T) "Half"

The Margulis-Zimmer conjecture extends the Normal Subgroup Theorem to commensurated subgroups. We make some progress on this conjecture here, specifically we prove "half" of the conjecture in the sense that we formulate the correct target group that "ought" to be finite and show that it does indeed have property (T). The amenability "half", still unsolved, would then complete the conjecture.

13.1 Statement of the Theorem

Our main goal for the rest of this chapter is to prove:

Theorem 13.1. *Let K be a global field, $\mathcal{O}$ the ring of integers, and $\mathbf{G}$ an absolutely simple, simply connected algebraic group defined over K. Let V be the set of inequivalent valuations on K and $V_\infty \subseteq V$ the archimedean valuations. Let $V_\infty \subseteq S \subseteq V$ and $\mathcal{O}_S$ be the ring of S-integers.*

Let Γ be a finite index subgroup of $\mathbf{G}(\mathcal{O}_S)$ and assume that the S-rank of $\mathbf{G}$ is at least two. Let $\Lambda < \Gamma$ be a commensurated subgroup of Γ.

Define $V_\infty \subseteq S' \subseteq S$ to be the smallest subset of S such that a finite index subgroup of Λ sits inside $\mathbf{G}(\mathcal{O}_{S'})$. Let Λ_0 be this finite index subgroup and let $\Gamma_0 = \Gamma \cap \mathbf{G}(\mathcal{O}_{S'})$ (a standard commensurated subgroup).

Then exactly one of the following holds:

- *$\Gamma_0 // \Lambda_0$ has property (T)*
- *the S'-rank of $\mathbf{G}$ is one*

A concrete example may be useful at this point:

Corollary 13.2. *Let $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$ (for some prime p) and consider $\mathrm{SL}_2(\mathbb{Z}[1/p])$ (which is a lattice in G when embedded diagonally). Let $\Lambda < \mathrm{SL}_2(\mathbb{Z}[1/p])$ be an infinite commensurated subgroup of $\mathrm{SL}_2(\mathbb{Z}[1/p])$. Then one of the following happens:*

- *$\mathrm{SL}_2(\mathbb{Z}[1/p]) // \Lambda$ has property (T)*
- *Λ is (up to finite index) a subgroup of $\mathrm{SL}_2(\mathbb{Z})$.*

13.2 Proof of the Theorem

Let $\mathbf{G}$ be an algebraic group defined over a global field K and let V be the set of inequivalent valuations on K with V_∞ the archimedean valuations. Let $\mathcal{O}$ be the ring of integers in K. Let $V_\infty \subseteq S \subseteq V$ and write $\mathcal{O}_S$ for the ring of S-integers in K.

Let $\Gamma = \mathbf{G}(\mathcal{O}_S)$ which, as remarked above, is, up to finite index, the most general form of an irreducible lattice in $\prod_{v\in S}\mathbf{G}(K_v)$.

Let $\Lambda < \Gamma$ be an infinite commensurated subgroup of Γ. Note that since Γ is finitely generated, the relative profinite completions formed below are all compactly generated.

13.2.1 The Reduction Step

We begin with a reduction step. The reason for this is that it can happen that the projection of Λ to some G_j is bounded. In such a situation, Λ will have a finite index subgroup which is still commensurated by Γ but projects trivially to that G_j, which is to say that this finite index subgroup lies entirely in the product of the other G's. Specifically, we choose the minimal subset $S' \subseteq S$ such that Λ is commensurate with a subgroup of $\prod_{v\in S'}\mathbf{G}(K_v)$.

Now consider $G' = \prod_{v\in S'}\mathbf{G}(K_v)$ and $\Gamma' = \mathbf{G}(\mathcal{O}_{S'})$. Write Λ' for the finite index subgroup of Λ which is contained in Γ'. Clearly Λ' is commensurated by Γ' (a consequence of how S' was chosen).

If S' is empty then Λ is finite but our hypothesis is that Λ is infinite. In the case that S' contains only one element v there are two possibilities. Either the sole factor $\mathbf{G}(K_v)$ has property (T) or it does not. In case it has property (T) we have that $\Gamma = \mathbf{G}(\mathcal{O}_{\{v\}})$ also has property (T) (Kazhdan's Theorem). For the moment, assume the following is true:

Proposition 13.2.1. *Let Γ be a finitely generated group and $\Lambda < \Gamma$ a commensurated subgroup. If Γ has property (T) then $\Gamma//\Lambda$ has property (T).*

So we are left with the case that $\mathbf{G}(\mathcal{O}_{\{v\}})$ does not have (T). This in turn means that Λ' is a commensurated subgroup of the rank one group $\mathbf{G}(\mathcal{O}_{\{v\}})$ which was referred to as the second possibility in the statement of the Theorem.

The remaining case is when S' contains at least two valuations. In this case $\Lambda' < \Gamma' < G'$ satisfy the conditions of our theorem in their own right (the rank of G' is at least two since S' has at least two elements). In addition, Λ' is not commensurate with a subgroup of $\mathbf{G}(\mathcal{O}_{S''})$ for any $S'' \subsetneq S'$.

Assume for the moment that the following holds:

Proposition 13.2.2. *Let $\Lambda < \Gamma$ be a commensurated subgroup of a strongly irreducible lattice Γ in a product of simple algebraic groups. Assume that there are at least two factors in the product and Λ is not commensurate with the ring of S'-integers for any $S' \subsetneq S$. Then $\Gamma//\Lambda$ has property (T).*

Applying this to $\Lambda' < \Gamma'$ we obtain that $\Gamma'//\Lambda'$ has property (T). Now Γ' is a standard commensurated subgroup of Γ and Λ' has finite index in Λ so we have shown that we are in the second case of the Theorem.

13.2.2 The Reduced Cohomology Argument

Proof. (of Propositions 13.2.1 and 13.2.2) Write $H = \Gamma//\Lambda$ for the relative profinite completion of Λ with respect to Γ. Suppose that H does not have property (T) (if it does then we are done since $\Gamma//\Lambda$ then has (T)). Then, by Shalom's Theorem (Theorem 5.17), there is an irreducible unitary representation π of H on a Hilbert space such that the reduced cohomology of π is nontrivial: $\overline{H^1}(H,\pi) \neq 0$.

Since H is a totally disconnected group and Γ is dense in H, by Theorem 7.2, the restriction map $\overline{H^1}(H,\pi) \to \overline{H^1}(\Gamma,\pi|_\Gamma)$ is injective. Therefore $\overline{H^1}(\Gamma,\pi|_\Gamma) \neq 0$.

Proposition 13.2.1 follows immediately as this would contradict that Γ has (T).

Continuing now in the context of Proposition 13.2.2, Γ sits as an irreducible lattice in a product of at least two groups and $\pi|_\Gamma$ is an irreducible representation of Γ (since Γ is dense in H, if $\pi|_\Gamma$ had a proper invariant subspace then so would π).

By Corollary 5.19 there is then a unitary representation σ of one of the factors G_j such that $\overline{H^1}(\Gamma,\pi|_\Gamma) = \overline{H^1}(G_j,\sigma)$ and such that $\sigma|_\Gamma = \pi|_\Gamma$.

Since $\pi|_\Gamma$ is irreducible so is σ (since the projection of Γ is dense in G_j). Suppose there is a fixed point for G_j under σ. Then σ is trivial since it is irreducible and so any σ-cocycle is a homomorphism from G_j to $\mathcal{H}$ and therefore its inner product with a fixed vector is a character (homomorphism into $\mathbb{R}$ or $\mathbb{C}$). But G_j is a simple algebraic group and any character must vanish on $[G_j,G_j] \triangleleft G_j$ hence either G_j is abelian or there are no characters. Of course simple abelian groups are finite (every element generates the group so for an element x we can write $x = n(2x)$ for some n and then x has finite order so the whole group does) so that possibility is ruled out and we conclude that in fact every σ-cocycle is zero. But $\overline{H^1}(G_j,\sigma) \neq 0$ contradicts this. Hence there are no fixed points for G_j.

Since Λ is precompact in H there is a fixed point for Λ under π. Hence there is a fixed point for the projection of Λ to G_j. Then by the Howe-Moore Theorem (Theorem 5.4) it must be the case that the projection of Λ is bounded (the matrix coefficients tend to zero as g leaves compact sets since there are no G_j fixed points but there is a fixed point for Λ which obviously has constant matrix coefficient).

By Proposition 13.2.3 (below) the projection of Λ to any archimedean valuation component cannot be bounded. Therefore the factor it projects boundedly to is totally disconnected (corresponds to a non-archimedean valuation $v \in S$).

Now a bounded subgroup of a totally disconnected group is necessarily contained in a compact open subgroup of the totally disconnected group. Since the compact open subgroups all have finite index in one another this means that Λ has a finite index subgroup contained in the closure of $\mathbf{G}(\mathcal{O}_{S\setminus\{v\}})$. But this means that Λ is commensurate with a subgroup of $\mathbf{G}(\mathcal{O}_{S\setminus\{v\}})$ which contradicts the hypothesis of the theorem (which amounts to the fact that we have already performed the reduction step).

This contradiction allows us to conclude that $\Gamma//\Lambda$ has (T). □

Proposition 13.2.3. *Let $\Lambda < \Gamma = \mathbf{G}(\mathcal{O}_S)$ be a commensurated subgroup of a lattice in $\prod_{v\in S}\mathbf{G}(K_v)$. The projection of Λ to $\mathbf{G}(K_v)$ is unbounded for every archimedean v (that is, for $v \in V_\infty$).*

Proof. Suppose the projection of Λ to $\mathbf{G}(K_v)$ is bounded. Then $K = \overline{\text{proj}\ \Lambda}$ is a compact subgroup of $\mathbf{G}(K_v)$. By Proposition 13.2.4, the projection of Λ to $\mathbf{G}(K_v)$ is Zariski dense hence K is Zariski dense in $\mathbf{G}(K_v)$. But since v is archimedean this means that $K = \mathbf{G}(K_v)$

(see e.g. Milne [Mil11] Proposition IV.3.4) and so $\mathbf{G}(K_v)$ is compact contradicting that we have already performed the reduction step. □

Proposition 13.2.4. *Let $\Lambda < \Gamma = \mathbf{G}(\mathcal{O}_S)$ be a commensurated subgroup of a lattice in $\prod_{v \in S} \mathbf{G}(K_v)$. The projection of Λ to $\mathbf{G}(K_v)$ is Zariski dense.*

Proof. Write $\Lambda_0 = \mathrm{proj}_{\mathbf{G}(K_v)} \Lambda$ and $\Gamma_0 = \mathrm{proj}_{\mathbf{G}(K_v)} \Gamma$. Since Λ is commensurated by Γ we have that Λ_0 is commensurated by Γ_0. Since K_v is archimedean, $\mathbf{G}(K_v)$ is Zariski connected. Now $\overline{\Lambda_0}$ (the closure being taken in the Zariski topology of $\mathbf{G}(K_v)$) is commensurated by Γ_0 (Lemma 13.2.5). All closures below are in the Zariski topology.

Since $\overline{\Lambda_0}$ is Zariski closed, it is Zariski connected (since $\mathbf{G}(K_v)$ is Zariski connected because it is algebraically simple and the connected component is normal in it) and for each $a \in \Gamma_0$ we have that $\overline{\Lambda_0} \cap a\overline{\Lambda_0}a^{-1}$ has finite index in $\overline{\Lambda_0}$, by Lemma 13.2.6 it must be that $a\overline{\Lambda_0}a^{-1} = \overline{\Lambda_0}$.

Hence $\overline{\Lambda_0}$ is normalized by Γ_0 and hence by the Zariski closure of Γ_0. But Γ is an irreducible lattice so $\overline{\Gamma_0} = \mathbf{G}(K_v)$ (in fact the projection is dense even in the usual topology). So the Zariski closed subgroup $\overline{\Lambda_0}$ is normal in the algebraically simple group $\mathbf{G}(K_v)$. Being nontrivial this means that $\overline{\Lambda_0} = \mathbf{G}(K_v)$ and therefore Λ_0 is Zariski dense. □

Lemma 13.2.5. *Let $A < B < G$ where G is an algebraic group and A is commensurated by B. Then the Zariski closure of A is commensurated by B.*

Proof. All closures in this proof are Zariski. Let $b \in B$. Then $[A : A \cap bAb^{-1}] < \infty$ and $[bAb^{-1} : A \cap bAb^{-1}] < \infty$. Therefore, $[\overline{A} : \overline{A \cap bAb^{-1}}] < \infty$ since finite index passes to closures (Lemma A.4.2). Then $[\overline{A} : \overline{A} \cap b\overline{A}b^{-1}] < \infty$ (since $\overline{A \cap bAb^{-1}} \subseteq \overline{A} \cap b\overline{A}b^{-1}$). Likewise, $[b\overline{A}b^{-1} : \overline{A} \cap b\overline{A}b^{-1}] < \infty$. Hence $\overline{A}$ is commensurated by B. □

Lemma 13.2.6. *Let H be an algebraic (Zariski closed) subgroup of an algebraic group. Then H is connected if and only if the only Zariski closed subgroup L of H with $[H : L] < \infty$ is $L = H$.*

Proof. First assume H is connected and let L be a Zariski closed subgroup of finite index. Write $n = [H : L]$ and let $h_1, \ldots, h_n$ be a system of representatives for H/L. Then $H = \bigcup_{j=1}^{n} h_j L$. Now the sets $h_j L$ are separated from one another since $\overline{h_j L} \cap \overline{h_{j'} L} = h_j L \cap h_{j'} L = \emptyset$ for $j \neq j'$. But this means that if $n > 1$ then H is not connected.

Now assume that H is not connected. Then the connected component of H is a finite index closed algebraic subgroup which is not equal to H. □

APPENDICES

Appendix A

Group Theory

Groups play a central role in modern mathematics, being an abstract generalization of the ideas of addition and multiplication, and arise in virtually every field.

Fundamental in the study of groups is the understanding of group actions. Our work centers on group actions on analytic spaces, namely metric and measure spaces, and on group actions on Hilbert spaces. The main results in this dissertation on the structure of certain classes of groups follow directly from results about their actions.

A.1 Groups

Definition A.1. A **group** is a set G together with a binary operation $\cdot : G \times G \to G$ and a distinguished element $e \in G$ such that

- $g \cdot e = g$ for all $g \in G$;
- for all $g \in G$ there exists $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$; and
- $g \cdot (h \cdot k) = (g \cdot h) \cdot k$ for all $g, h, k \in G$

The binary operation is usually written as multiplication, omitting the $\cdot$. The group is **abelian** when $g \cdot h = h \cdot g$ for all $g, h \in G$ in which case we often write $+$ for the operation. The map $g \mapsto g^{-1}$ maps G onto itself and is called the **inverse map**.

Much of our work involves studying the structure and classification of certain classes of infinite groups. Examples of groups include the integers $\mathbb{Z}$, the real numbers $\mathbb{R}$ and the rational numbers $\mathbb{Q}$, and also such objects as the two-by-two matrices with determinant one $\mathrm{SL}_2(\mathbb{R})$.

A.1.1 Homomorphisms

Definition A.2. Let G and H be groups. A map $\varphi : G \to H$ is called a **homomorphism** when φ preserves the group operations:

$$\varphi(e_G) = e_H \qquad \varphi(gh) = \varphi(g)\varphi(h) \qquad \varphi(g^{-1}) = \varphi(g)^{-1}$$

Definition A.3. A homomorphism that is one-one and onto is called an **isomorphism**.

Definition A.4. Let G be a group and $g \in G$. The map $\varphi_g : G \to G$ by $\varphi_g(h) = ghg^{-1}$ is a homomorphism called the **conjugation** by g.

A.1.2 Subgroups

Subsets of groups inherit the operations of the group; those that are closed under these operations are groups in their own right, called subgroups. Much of our focus will be on understanding specific types of subgroups in larger groups.

Definition A.5. Let G be a group. A subset $H \subseteq G$ is called a **subgroup** when $e \in H$ and the group operations of G restricted to H are closed, that is if $h, k \in H$ then $hk \in H$ and $h^{-1} \in H$. This will be written as $H < G$.

A.1.3 Normal Subgroups

The most important class of subgroups are the normal subgroups: those which remain fixed under conjugation. Briefly this is because they are the only subgroups on which morphisms can vanish and therefore groups can be decomposed over their normal subgroups.

Many of the major results in this dissertation and in the background leading up to our work focus on showing the existence or nonexistence of normal subgroups.

Definition A.6. Let G be a group. A subgroup $N < G$ is called **normal** when the conjugation of N by any element of G is N, that is for all $g \in G$ and $n \in N$ we have that $gng^{-1} \in N$, also written $gNg^{-1} = N$. Normality will be written as $N \triangleleft G$.

Definition A.7. A group G is **simple** when it has no nontrivial normal subgroups (the trivial ones being G and $\{e\}$).

Definition A.8. The **kernel** of a homomorphism φ is the set $\{g \in G : \varphi(g) = e\}$. The kernel is written $\ker(\varphi)$.

The kernel of any homomorphism is a normal subgroup: $\ker(\varphi) \triangleleft G$.

A.1.4 Quotients and Cosets

Given a group and a subgroup it is natural to ask to what extent we can write each group element in terms of the subgroup. This process is referred to as quotienting, i.e. dividing out by a subgroup.

Definition A.9. Let G be a group and H a subgroup. The **quotient** of G **modulo** H, written G/H, is the set of equivalence classes of elements of g under right multiplication by H. That is

$$G/H = \{gH : g \in G\} \qquad \text{where } gH = \{gh : h \in H\}$$

Definition A.10. Let G be a group and $H < G$ a subgroup. The elements of G/H are called **cosets** of H in G, that is gH is a coset for each $g \in G$.

Definition A.11. Let G be a group and $H < G$ a subgroup. The **normalizer** of H in G is

$$N_G(H) = \{g \in G : gHg^{-1} = H\}$$

The normalizer of a subgroup is the largest subgroup in which the group is normal.

A.1.5 Abelian Groups

Definition A.12. A group G is **abelian** when $gh = hg$ for all $g, h \in G$, that is the group operation is commutative.

Obviously any subgroup of an abelian group is normal.

Definition A.13. Let G be a group. The **center** of G is the subgroup

$$Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\ \}$$

That is, the center is the elements which commute with the rest of the group. A group is abelian if and only if $Z(G) = G$.

A.1.6 Finite Index Subgroups

Subgroups of finite index have a special role in the geometric study of groups. Geometrically speaking, groups that share a finite index subgroup are "the same" in that they exhibit the same geometric and dynamical properties. Our work, following in a long tradition, will adopt the point of view that groups that are equal up to finite index are in fact equivalent.

Definition A.14. Let G be a group and H a subgroup. The **index** of H in G is

$$[G : H] = \big|G/H\big|$$

where $|\cdot|$ is the cardinality.

Definition A.15. A subgroup H of a group G has **finite index** when $[G : H] < \infty$.

Note the simple fact that if $[G : H] = 2$ then $G = H \cup kH$ for some $k \in G$ and therefore for $h \in H$ we have $hk \in kH$ (since $hk \in H$ would imply $k \in H$) which means that $k^{-1}hk \in H$ so $k^{-1}Hk = H$ meaning $H \triangleleft G$.

A.1.7 Quotient Groups

For a group G and a subgroup $H < G$ the coset space G/H is not a group in general. However, in the special case when $N \triangleleft G$ the quotient becomes a group itself:

Definition A.16. Let $N \triangleleft G$ be a normal subgroup in a group. Then G/N is called the **quotient group** of G by N.

For $gN, hN \in G/N$ observe that

$$gN \cdot hN = \{gn_1hn_2 : n_1, n_2 \in N\}$$

and since $n_1h = hn'$ for some $n' \in N$ (because $h^{-1}Nh = N$ so $Nh = hN$) we have that

$$gn_1hn_2 = ghn'n_2 \in ghN$$

and therefore

$$gN \cdot hN = ghN$$

defines a group operation (likewise $g^{-1}N = g^{-1}gNg^{-1} = Ng^{-1} = (gN)^{-1}$).

A.1.8 Systems of Representatives

When writing a group element in terms of cosets, it is common to fix a system of representatives for the quotient operation.

Definition A.17. Let G be a group and H a subgroup. A **system of representatives** for G/H is a (finite or infinite) collection $g_1, g_2, \ldots$ in G such that no two g_j are in the same coset ($g_jH \cap g_\ell H = \emptyset$ for $j \neq \ell$) and such that the union of the cosets is all of G: $\bigcup_j g_jH = G$.

Definition A.18. A system S is said to be **symmetric** when $g \in S$ implies $g^{-1} \in S$. This is written $S = S^{-1}$.

Observe that if g, g^{-1} are in the same coset then $g^{-1} \in gH$ so $g^{-1} = gh$ for some $h \in H$ meaning that $g^2 = h^{-1} \in H$. Then $g \in H$ or $\sqrt{g} \in H$ so either $g \in H$ or $gHg^{-1} = H$. Therefore one can always take a system of representatives that is symmetric except for representatives in the normalizer which must be taken one or the other only.

A.2 Group Actions

A major theme in the study of groups is to understand the structure of a group in terms of its actions. An action of a group on a set is when each element of the group moves around the elements of the set in a manner compatible with the group operations.

Definition A.19. Let G be a group and S a set. A map $\cdot : G \times S \to S$ such that

- $e \cdot s = s$ for all $s \in S$
- $g \cdot (h \cdot s) = (gh) \cdot s$ for all $g, h \in G$ and $s \in S$

is called an **action** of the group G on the set S and will be written $G \curvearrowright S$.

Definition A.20. Let $G \curvearrowright S$. The **kernel of the group action** is

$$\ker(G \curvearrowright S) = \{g \in G : g \cdot s = s \text{ for all } s \in S\ \}$$

and is always a normal subgroup: $\ker(G \curvearrowright S) \triangleleft G$.

Definition A.21. Let $G \curvearrowright S$. The **stabilizer** of an element in the set is

$$\mathrm{stab}_G(s) = \{g \in G : g \cdot s = s\}$$

and the **orbit** is

$$\mathrm{orb}_G(s) = Gs = \{s' \in S : \exists g \in G \quad g \cdot s = s'\} = \{g \cdot s : g \in G\}$$

The main result about orbits and stabilizers is the orbit-stabilizer theorem: $|Gs| = [G : \mathrm{stab}(s)]$. That is, the size of the orbit of s equals the index of the stabilizer (when everything is finite).

A.3 Countable Groups

The theory of infinite groups is simplest in the case of countable groups without a topology (more correctly, with the discrete topology).

Definition A.22. A group G is called **countable** when the underlying set has countable (finite or infinite) cardinality.

A.3.1 Finitely Generated Groups

Finitely generated groups are the easiest countably infinite groups to work with. Being finitely generated means that while the group is infinite, there is a finite set of elements in terms of which every element can be written as a product.

Definition A.23. Let G be a group and $A \subseteq G$. The group generated by A, written $\langle A \rangle$, is the smallest subset of G containing A that is closed under the group operations multiplication and inversion. This is a subgroup of G.

Definition A.24. A group G is **finitely generated** when there is a finite set $S \subseteq G$ such that the group generated by S is all of G: $\langle S \rangle = G$. Such S is called a **generating set**.

Definition A.25. Let G be a finitely generated group and S a generating set. The **word length** on G relative to S is defined by

$$|g|_S = \min\{n \in \mathbb{N} : \exists s_1, \ldots, s_n \in S \quad g = s_1 s_2 \cdots s_n\}$$

When S is clear from context we will write $|\cdot|$ with no subscript.

Definition A.26. Let G be a finitely generated group. A generating set S is called **symmetric**, written $S = S^{-1}$ when $s \in S$ implies $s^{-1} \in S$.

A.4 Topological Groups

When the underlying set G is not countable it is often the case that there is a natural topology on it. For example, the set of real numbers $\mathbb{R}$ with addition is a group and the real numbers have a nondiscrete topology on them. It is easy to see that the group operations are continuous with respect to this topology. Likewise, the n-by-n matrices with real entries (and determinant one) $\mathrm{SL}_n(\mathbb{R})$ have a topology under which the group operations are continuous.

Definition A.27. A **topological group** is a set G together with a topology on G and a group structure such that the group operations of multiplication and inversion are continuous with respect to the topology.

Countable discrete groups are special cases of topological groups (in fact any group can be made a topological group by imposing the discrete topology but when the underlying set is uncountable this is not generally helpful).

Definition A.28. A **homomorphism** of topological groups is an ordinary homomorphism of the underlying groups that is continuous with respect to their topologies.

Definition A.29. A **subgroup** of a topological group is an ordinary subgroup that is also closed in the topology of the group. Likewise, normal subgroup means closed normal subgroup.

Definition A.30. A topological group is **simple** or **topologically simple** when there are no nontrivial closed normal subgroups.

This extends generally to all aspects of group structure. When the group is topological, all group operation related mappings are required to be continuous.

A.4.1 Locally Compact Groups

A subclass of topological groups that has been particularly well-studied are the groups which are locally compact topologically. Recall that a topological space is locally compact when every point has a compact neighborhood.

Definition A.31. A topological group is called **locally compact** when the underlying topology is locally compact. We will refer to such groups as **locally compact groups** with the implicit indication that they are topological.

This applies in general to topological properties: a topological group is said to have a topological property (such as compactness, second countability, etc.) when the topology the group is endowed with has that property.

A.4.2 Polish Groups

A generalization of locally compact groups still well enough behaved to study analytically are the Polish groups:

Definition A.32. A topological group is called **Polish** when the underlying topology is Polish: it is separable and completely metrizable.

A.4.3 Compact Generation

Compactly generated groups are the topological analogue of finitely generated groups: when the group is discrete–meaning we can treat it as a topological group with the discrete topology–compact generation is the same as finite generation.

Definition A.33. A locally compact group G is **compactly generated** when there is a compact set $K \subseteq G$ such that $\langle K \rangle = G$.

Compactly generated groups are the analogue of finitely generated groups, and in fact compactly generated countable groups are simply finitely generated groups.

A.4.4 Discrete Subgroups

Any countable group can be endowed with the discrete topology in which case the group operations multiplication and inversion are automatically continuous.

Definition A.34. A group G is called **discrete** when there is no additional topological structure placed on G.

Definition A.35. A subgroup Γ of a topological group G is called a **discrete subgroup** when Γ is discrete in the topology of G.

Examples of countable discrete groups include the integers $\mathbb{Z}$ and the two by two matrices with integer entries and determinant one: $\mathrm{SL}_2(\mathbb{Z})$. Both are discrete groups and also are discrete subgroups of $\mathbb{R}$ and $\mathrm{SL}_2(\mathbb{R})$ respectively. However $\mathbb{Q}$ is a discrete group but as a subgroup of $\mathbb{R}$ it is not discrete (in fact it is dense).

A.4.5 Finite Index, Normality and Closure

We also mention that basic properties of subgroups carry to closures, in particular normality and finite index:

Lemma A.4.1. *Let $A < B < G$ where G is a Polish topological group and A and B are arbitrary subgroups. If $A \triangleleft B$ then $\overline{A} \triangleleft \overline{B}$.*

Proof. Let $b \in \overline{B}$ and $a \in \overline{A}$. Then $b = \lim b_n$ for some $b_n \in B$ and $a = \lim a_n$ for some $a_n \in A$. Since $A \triangleleft B$ for each n we know that $b_n a_n b_n^{-1} = a_n'$ for some $a_n' \in A$. Now by (joint) continuity of group multiplication

$$bab^{-1} = \lim_n b_n a_n b_n^{-1} = \lim_n a_n' \in \overline{A}$$

hence $b\overline{A}b^{-1} \subseteq \overline{A}$ for all $b \in \overline{B}$. □

Lemma A.4.2. *Let $A < B < G$ where G is a Polish topological group and A and B are arbitrary subgroups. If $[B : A] < \infty$ then $[\overline{B} : \overline{A}] \leq [B : A] < \infty$ (the topology being any of those of G).*

Proof. Write $B = \bigcup_{j=1}^N b_j A$ where $b_1, \ldots, b_N$ is a system of representatives for B/A (we are writing $N = [B : A] < \infty$). Let $x \in \overline{B}$. Then there exists $x_n \in \bigcup_{j=1}^N b_j A$ such that $x_n \to x$. Since the union is finite there is some j such that an infinite subsequence of the x_n are in $b_j A$. Therefore $x \in \overline{b_j A} = b_j \overline{A}$. Hence

$$\overline{B} = \overline{\bigcup_{j=1}^N b_j A} \subseteq \bigcup_{j=1}^N b_j \overline{A}$$

and therefore $[\overline{B} : \overline{A}] \leq N < \infty$. □

A.5 Measures on Groups

The next natural step after introducing topology to groups is to introduce measures and therefore be able to reason analytically. Most readers will have been familiar with all material presented thus far, but the introduction of measures onto groups is our first departure from the usual theory of groups.

A.5.1 Borel Measures

Let G be a locally compact topological group. The σ-algebra generated by the compact sets of G is called the Borel sets of G.

Definition A.36. A **regular Borel measure** on a locally compact group is a countably additive measure on the Borel sets of the group such that the measure of any compact set is finite and the Borel sets are all (inner and outer) regular.

A.5.2 Haar Measure

A key fact about locally compact groups is the existence of a translation invariant σ-finite measure on them.

Theorem A.37 (Haar)**.** *Let G be a locally compact group. There exists a regular Borel measure* Haar *on G that is unique up to a multiplicative constant which is translation invariant:* $\mathrm{Haar}(gB) = \mathrm{Haar}(B)$ *for all Borel sets B in G and all $g \in G$. This measure is called a* ***Haar measure*** *on G.*

We remark that integration against Haar measure can be defined exactly as integration against the Lebesgue measure is defined.

A.5.3 Probability Measures on Groups

Definition A.38. The **space of probability measures on** G, written $P(G)$, consists of all regular Borel measures on G that assign G a total measure of one.

When G is countable and discrete the probability measures on G are just the $\ell^1(G)$ functions $\mu : G \to [0, 1]$ such that $\sum_g \mu(g) = 1$.

Definition A.39. For a measure μ on a locally compact group G the **integral** is defined as

$$\int_G f(g)\ d\mu(g)$$

for any Borel function f on G analogously to integration for Lebesgue measure.

Definition A.40. Let G be a locally compact group and $\mu_1, \mu_2 \in P(G)$ be probability measures on G. The **convolution** of μ_1 and μ_2, written $\mu_1 * \mu_2$, is defined as

$$\int_G f\ d\mu_1 * \mu_2 = \int_G \int_G f(gh)\ d\mu_2(h)\ d\mu_1(g)$$

and $\mu_1 * \mu_2 \in P(G)$ when $\mu_1, \mu_2 | in P(G)$ so $* : P(G) \times P(G) \to P(G)$ is a binary operation.

Proposition A.5.1. *Let G be a countable discrete group and $\mu_1, \mu_2 \in P(G)$. Then*

$$\mu_1 * \mu_2(g) = \sum_{h \in G} \mu_2(hg)\mu_1(h)$$

Proposition A.5.2. *The space of probability measures $P(G)$ on a locally compact group is a convex space with a binary operation (convolution).*

Definition A.41. A probability measure $\mu \in P(G)$ on a locally compact group G is **admissible** when the support of μ generates G algebraically and some convolution power of μ is nonsingular with respect to Haar measure.

A.5.4 Symmetric Measures

Probability measures on groups generally cannot be invariant under translation. In fact, a translation invariant measure on a group is a Haar measure and as such a translation invariant probability measure on a group can only exist when the group is finite (or compact).

That said, even though we cannot ask for a measure that is invariant under group multiplication (i.e. translation), we can still ask for measures which are invariant under the inverse map of the group.

Definition A.42. Let G be a group and $\mu \in P(G)$ a probability measure on it. The **symmetric opposite** of μ is written $\check{\mu}$ and defined by

$$\check{\mu}(B) = \mu\{g^{-1} : g \in B\}$$

for all measurable $B \subseteq G$. When G is discrete this simply means that

$$\mu(g^{-1}) = \mu(g)$$

for all $g \in G$.

Definition A.43. Let G be a group. A (probability) measure $\mu \in P(G)$ on G is **symmetric** when $\check{\mu} = \mu$.

Note that if μ is a symmetric measure on G then for any function $f : G \to \mathbb{R}$ we have that

$$\int_G f(g)\ d\mu(g) = \int_G f(g^{-1})\ d\mu(g)$$

which is most commonly how we will make use of the symmetry of a measure.

A.5.5 Moments

The moments of a measure are a way of quantifying how much of the measure is concentrated on the elements with small word length. Moments of probability measures on Euclidean space play a key role in probability theory and we will make use of them in a similar fashion.

Definition A.44. Let Γ be a countable discrete finitely generated group and S a generating set. Let $\mu \in P(\Gamma)$ be a probability measure on Γ. The **first moment** of μ relative to word length from S is

$$M_1(S, \mu) = \sum_{\gamma \in \Gamma} |\gamma|_S \mu(\gamma)$$

The **second moment** (higher order moments being defined similarly) is

$$M_2(S, \mu) = \sum_{\gamma \in \Gamma} |\gamma|_S^2 \mu(\gamma)$$

Let S_1 and S_2 be finite generating sets for the same finitely generated group Γ. Write

$$C_{1,2} = \max_{s_1 \in S_1} |s_1|_{S_2}$$

to be the maximum word length of the elements of S_1 relative to S_2. Observe that

$$|\gamma|_{S_1} \leq C_{1,2} |\gamma|_{S_2}$$

and by symmetry

$$|\gamma|_{S_2} \leq C_{2,1} |\gamma|_{S_1}$$

and therefore

$$M_k(S_1, \mu) \leq C_{1,2}^k M_k(S_2, \mu) \leq (C_{1,2} C_{2,1})^k M_k(S_1, \mu)$$

which means that either they are both finite or both infinite.

Definition A.45. Let Γ be a finitely generated group and $\mu \in P(\Gamma)$. Then μ has **finite first moment relative to word length** when for some (equivalently for any) finite generating set S we have $M_1(S, \mu) < \infty$. Likewise μ has **finite second moment relative to word length** when the same holds for M_2.

A.6 Lattices

Of particular interest are the countable subgroups of locally compact groups that "contain" information about the topological group. Rigidity theory focuses on when information about locally compact groups and their actions can be transferred to countable subgroups and when information about countable subgroups can be transferred back.

Definition A.46. Let G be a locally compact group. A subgroup Γ is a **lattice** in G when it is discrete in the topology of G and has finite covolume: there exists a fundamental domain F for G/Γ (a Borel set, which one may taken to be open or closed, such that $F\Gamma = G$) that has finite Haar measure, $\mathrm{Haar}(F) < \infty$.

Examples of lattices include $\mathbb{Z} < \mathbb{R}$ and $\mathrm{SL}_n(\mathbb{Z}) < \mathrm{SL}_n(\mathbb{R})$.

A.6.1 Irreducibility

Definition A.47. A lattice Γ in a locally compact group G is **(strongly) irreducible** when the projection of Γ to any G/H is dense for any closed normal subgroup H.

A lattice Γ in a product of simple locally compact groups is **irreducible** when its projection to each factor is dense (and strongly so when its projection to any subproduct is dense).

A.6.2 Cocompactness

The first example of a lattice that we mentioned was $\mathbb{Z} < \mathbb{R}$ and it is easy to see that $\mathbb{R}/\mathbb{Z}$ is of finite Haar measure (on the reals of course Haar measure is simply the usual Lebesgue measure) and in fact that it has a compact fundamental domain: $\mathbb{Z} \cdot [0,1] = \mathbb{R}$.

Definition A.48. Let Γ be a lattice in a locally compact group G. Then Γ is **cocompact** or **uniform** when there is a compact fundamental domain for G/Γ.

On the other hand, $\mathrm{SL}_n(\mathbb{Z}) < \mathrm{SL}_n(\mathbb{R})$ is also a lattice (a theorem of Borel and Harish-Chandra) but it is not cocompact. This is easy to see in the case of SL_2 using the standard picture on the plane (there are "cusps" in the fundamental domain that cannot be compactified away).

A.6.3 Integrability

A more relaxed condition than cocompactness that is often enough to perform analysis in a similar fashion is integrability:

Definition A.49. Let G be a locally compact, second countable topological group and Γ a finitely generated lattice in G. Then Γ is **integrable** (this is more precisely 2-integrable but we will refer to it simply as integrable) when there exists a fundamental domain X for G/Γ such that

$$\int_X |\alpha(g,x)|^2 \; dm(x) < \infty$$

where $\alpha : G \times X \to \Gamma$ is given by $\alpha(g,x) = \gamma$ if and only if $gx\gamma \in X$ and where $|\cdot|$ denotes the word length in Γ (the choice of generating set will not affect the finiteness of the integral) and m is the Haar measure on G.

Our result on normal subgroups of commensurators of lattices will require this property on the lattices in order to utilize results of [GKM08] and [Sha00b]. We remark that this condition is only necessary for the property (T) half of the proof, the amenability half carries through even in its absence.

Clearly if Γ is cocompact (i.e. uniform) then it is integrable. As mentioned in [Sha00b] lattices in simple Lie groups and Kac-Moody groups are known to always have this property. See Section 10.4 for further discussion.

A.7 Lie Groups

The original motivation for the study of topological groups was the work of Lie on manifolds admitting group structure. Lie groups are topological groups with the topology coming from a differentiable manifold, that is:

Definition A.50. A **Lie group** is a differentiable manifold equipped with group operations compatible with the smooth structure on the manifold.

Definition A.51. Let G be a Lie group. A subgroup $H < G$ is called a **Lie subgroup** when H is topologically a subgroup of G (a closed subgroup) and H inherits the differentiable structure from G in such a way that H is a submanifold.

The reader is referred to [Mil11] and [Var74] and [Che46] for more information on Lie groups. A key example of a Lie group is $\mathrm{SL}_n(\mathbb{R})$, the special liner group consisting of n by n matrices with real entries and determinant one. Other examples are $\mathbb{R}$, the real numbers, and $\mathbb{T}^n$, the n-torus.

A.7.1 Semisimple Lie Groups

The representation theory of Lie groups, and most of rigidity theory for lattices in Lie groups, is most complete in the context of semisimple Lie groups. Though we will not make use of the semisimple property directly in our work, we should mention what it means.

Definition A.52. Let G be a Lie group and $G_1, \ldots, G_k$ be Lie subgroups of G. If the map $(g_1, \ldots, g_k) \mapsto g_1 \cdots g_k$ from $G_1 \times \cdots \times G_k \to G$ is surjective and has a finite kernel then we say that G is the **almost direct product** of $G_1, \ldots, G_k$.

Definition A.53. Let G be a connected Lie group. We say G is **almost simple** when the center of G, written $Z(G)$, is finite and $G/Z(G)$ is simple (no nontrivial normal Lie subgroups).

Definition A.54. A connected Lie group G is **semisimple** when it is the almost direct product of almost simple connected Lie groups.

A broader description of semisimplicity and related notions in a more general setting is presented in Appendix C but given the nature of the results we will be discussing later it seems appropriate to mention it and define it here along with Lie groups.

A.7.2 Lattices in Lie Groups

Lattices in Lie groups have been well-studied and generally behave "similarly" to $\mathrm{SL}_n(\mathbb{Z}) < \mathrm{SL}_n(\mathbb{R})$. In particular, rigidity theory is best-developed in the context of lattices in semisimple Lie groups, which are known to have a variety of properties including integrability.

Furstenberg's boundary theory, discussed in Chapter 2: Stationary Dynamical Systems, was originally developed for semisimple Lie groups and their lattices and Margulis' work on arithmeticity and normal subgroups, discussed in Appendix C: Algebraic Groups and Chapter 9: Normal Subgroups of Commensurators of Lattices, likewise takes place in the setting of semisimple Lie groups.

Lattices in the more general setting of algebraic groups is discussed in Appendix C in further detail. Our normal subgroup theorem in Chapter 9 applies to lattices in the most general setting possible, that of locally compact groups, but our commensurated subgroups theorem applies only to Lie groups and algebraic groups.

A.7.3 The Rank of a Lie Group

The rank of a Lie group, and more generally for algebraic groups, which are discussed in Appendix C, is effectively the maximal dimension of a diagonal subgroup. Precisely:

Definition A.55. Let G be a connected Lie group. The **Cartan subgroup** of G is the centralizer of a maximal torus in G.

A Cartan subgroup of a connected Lie group will be connected and nilpotent, and all possible maximal tori lead to conjugate subgroups so we often speak of “the” Cartan subgroup.

Definition A.56. The **rank** or **real rank** of a connected Lie group G is the dimension of a maximal torus in G.

For example, $\mathrm{SL}_2(\mathbb{R})$ is a rank-one Lie group and more generally $\mathrm{SL}_n(\mathbb{R})$ has rank $n-1$. Also note that the rank of $G_1 \times G_2$ equals the rank of G_1 plus that of G_2.

A.8 Further Examples

We present now some other examples of locally compact groups. These examples are the most commonly arising topological groups and our results apply to them in particular.

A.8.1 p-adic Lie Groups

Lie groups are manifolds with smooth structure, meaning they are groups over the real or complex numbers. This can be generalized to the p-adic numbers giving rise to the p-adic Lie groups. A key example of a p-adic Lie group is $\mathrm{SL}_n(\mathbb{Q}_p)$ where $\mathbb{Q}_p$ are the p-adic numbers (the completion of the rationals under the p-adic valuation).

p-adic Lie groups will be totally disconnected (see below) since the underlying field is totally disconnected. Lattices in p-adic Lie groups can be obtained using the p-adic integers $\mathbb{Z}_p$ in place of the integers as in the Lie group case.

One can also form products of groups, for example $G = \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Q}_p)$, and form lattices by diagonal embedding: $\mathrm{SL}_2(\mathbb{Z}[1/p])$ is a lattice in G.

A.8.2 Automorphism Groups of Trees

Another area where topological groups arise is in the study of automorphism groups of structures. An automorphism is a map from a structure X to itself preserving the algebraic (and possibly analytic) structure on X.

Let X be a graph (a collection of vertices and edges). The automorphisms of X are the maps sending vertices to vertices such that two vertices are connected by an edge if and only if their images under the map are connected by an edge. The set of automorphisms of a structure forms a group under composition and inversion and is written $Aut(X)$.

Let T be a tree (graph with no cycles). Then $Aut(T)$ will be a locally compact group which is totally disconnected. Lattices in $Aut(T)$ can be thought of as follows: cocompact lattices correspond to quotient actions on finite graphs and noncocompact lattices to "profinite" graphs. This idea can be extended to automorphism groups of simplicial complexes in the obvious way.

A.9 Totally Disconnected Groups

Definition A.57. A **totally disconnected group** is a locally compact group that is totally disconnected in its topology.

As remarked above, p-adic Lie groups and automorphism groups of trees are totally disconnected groups. The structure of totally disconnected groups is not as well-understood as that of Lie groups though recently progress has been made by Willis, see [SW09].

Proposition A.9.1. *A totally disconnected group admits a compact open subgroup and in fact there is a neighborhood base of compact open sets.*

Amenability and Property (T)

Two properties of infinite groups have achieved a special level of importance. Amenability is when a group has a type of averaging property in the sense that it is possible to take averages over the group in connection with its actions on objects such as functions or measures in such a way that the average is invariant under translation. Property (T) is quite opposite to this: it means that whenever the group acts unitarily on a Hilbert space with almost invariant vectors (a sequence of vectors such that the group moves each vector less and less far apart) then in fact there is an invariant vector.

B.1 Amenability

Amenability was originally defined by von Neumann as a resolution to the Banach-Tarski paradox (specifically, a sufficient condition for a group to not lead to the "paradox" is that it be amenable). The concept is easier to define and work with in the case of countable discrete groups, so we will start there, and then discuss the locally compact definition. The reader is referred to [Run02] for more information.

B.1.1 Discrete Group Definition

In the case of discrete groups, amenability is the defined as having an invariant mean:

Definition B.1. A discrete group Γ is **amenable** when there exists a finitely additive invariant mean (finitely additive measure) on Γ. That is, there exists a function m defined on all subsets of Γ such that

- $m(\Gamma) = 1$
- $m(\bigcup_{j=1}^{k} B_k) = \sum_{j=1}^{k} m(B_k)$ for any finite collection of disjoint subsets $B_1, \ldots, B_k \subseteq \Gamma$
- $m(gB) = m(B)$ for all $g \in G$ and $B \subseteq \Gamma$

The mean essentially gives a reasonable answer to the question, what is the probability that a random element of Γ belongs to a given subset.

For example, let m be defined on subsets of $\mathbb{Z}$ by, for $B \subseteq \mathbb{Z}$,

$$m(B) = \limsup_{N\to\infty} \frac{1}{2N+1}\big|B \cap \{-N, \ldots, N\}\big|$$

the upper density of the subset. Then $m(\mathbb{Z}) = 1$ and for any fixed t, writing $B + t = \{b + t :$

$b \in B\}$,

$$\begin{aligned}
&\frac{1}{2N+1}\big|B+t\cap\{-N,\ldots,N\}\big| \\
&\qquad\qquad = \frac{1}{2N+1}\big|B\cap\{-N-t,\ldots,N-t\}\big| \\
&\qquad\qquad = \frac{2(N+t)+1}{2N+1}\,\frac{1}{2(N+t)+1}\big|B\cap\{-N-t,\ldots,N+t\}\big| \pm \frac{2t}{2N+1}
\end{aligned}$$

and so m is invariant:

$$m(B+t) = \limsup_{N\to\infty}\frac{1}{2N+1}\big|B+t\cap\{-N,\ldots,N\}\big| = m(B)$$

and finally we see that if $B_1,\ldots,B_k$ are disjoint that

$$\big|\bigcup_{j=1}^{k} B_j\cap\{-N,\ldots,N\}\big| = \sum_{j=1}^{k}\big|B_j\cap\{-N,\ldots,N\}\big|$$

and so $m(\bigcup B_j) = \sum m(B_j)$ for finite disjoint unions. However we remark that this fails for infinite unions ($m(\{n\}) = 0$ for each n but $m(\bigcup_n\{n\}) = m(\mathbb{Z}) = 1$). So m is in fact an invariant mean on $\mathbb{Z}$ proving $\mathbb{Z}$ is amenable.

B.1.2 Locally Compact Definition

When the group is locally compact the definition becomes more intricate:

Definition B.2. Let G be a locally compact group and Haar some Haar measure on G. A linear functional $m : L^\infty(G,\mathrm{Haar}) \to \mathbb{R}$ is a **mean** when $m(\mathbb{1}) = 1$ (here $\mathbb{1}$ is the identity function which is constantly one) and when $m(f) \geq 0$ for all $f \geq 0$ (meaning $f(x) \geq 0$ almost everywhere).

Definition B.3. A mean m on a group G is **left-invariant** when the left action of G on $L^\infty(G,\mathrm{Haar})$ preserves m: for $g \in G$ let $L_g : L^\infty(G,\mathrm{Haar}) \to L^\infty(G,\mathrm{Haar})$ by $L_g f(x) = f(gx)$ and then m is left-invariant when $m(L_g f) = m(f)$ for all f. Right-invariance is defined similarly.

Definition B.4. A locally compact group G is **amenable** when there is a left-invariant (equivalently, right-invariant) mean on G.

The discrete definition is actually a special case of the locally compact definition so there is no problem in defining amenability this way.

B.1.3 Equivalences for Discrete Groups

There are large number of well-known equivalent conditions in the discrete case, an indication of why amenability plays such a key role in infinite group theory. For completeness we will list many of them here, though we will not need most of them in what follows.

Theorem B.5. *Let Γ be a discrete group. The following are equivalent:*

- Γ *is amenable*
- *there exists finite subsets, called Følner sets, $F_n \subseteq \Gamma$ such that $|\gamma F_n \triangle F_n|/|F_n| \to 0$ for each $\gamma \in \Gamma$ and such that $\cup_n F_n = \Gamma$ (Følner)*
- *there exists a sequence of probability measures $\mu_n \in P(\Gamma)$ such that $\|\gamma\mu_n - \mu_n\| \to 0$ for each $\gamma \in \Gamma$ (Day)*
- *there exists a sequence of unit vectors $x_n \in \ell^2(\Gamma)$ such that $\|\gamma x_n - x_n\| \to 0$ for each $\gamma \in \Gamma$ (Dixmier)*
- *if $\mu \in P(\Gamma)$ is symmetric then convolution by μ is a norm one operator (Kesten)*
- *if Γ acts isometrically on a separable Banach space with a weakly closed convex invariant subset of the unit ball then Γ has a fixed point in that set*

B.1.4 Equivalences for Locally Compact Groups

Many of the above conditions carry over to the locally compact case when modified appropriately. We mention the ones we will make use of:

Theorem B.6. *Let G be a locally compact group. The following are equivalent:*

- *G is amenable*
- *there exists compact subsets, Følner sets, $K_n \subseteq G$ such that $\mathrm{Haar}(gK_n \triangle K_n)/\mathrm{Haar}(K_n) \to 0$ and such that $\cup_n K_n = G$*
- *if G acts isometrically and continuously on a separable Banach space with a weakly closed convex invariant subset of the unit ball then G has a fixed point in that set*

In the following chapters we will see additional equivalent conditions.

B.1.5 Examples

Some examples of amenable groups are

- the integers $\mathbb{Z}$ (use Følner sets $F_n = \{-n, \ldots, n\}$)
- finite groups (use the counting measure normalized to total mass one)
- compact groups
- solvable groups

- direct products of amenable groups
- subgroups of amenable groups
- finitely generated groups of subexponential growth

Some examples of nonamenable groups are

- nonabelian free groups with two or more generators
- any group containing a free subgroup on two or more generators
- $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 2$
- $\mathrm{SL}_n(\mathbb{R})$ for $n \geq 2$
- more generally, any finitely generated linear group is either solvable or nonamenable (Tits alternative)

B.2 PROPERTY (T)

Property (T) is a strong anti-amenability property that was introduced by Kazhdan in connection with studying lattices in Lie groups. The intuition for property (T) is that if a group acts on a Hilbert space unitarily with nontrivial almost invariant vectors then there is actually a nontrivial invariant vector. The name arises from an equivalent condition that the trivial representation is an isolated point in the unitary dual.

Property (T) plays a key role in the study of rigidity for Lie groups and lattices and in the study of lattices in general locally compact groups. It is also a crucial aspect of group representation theory and appears in many forms in the areas of operator algebras and geometric group theory.

A very complete introduction to Property (T) along with an overview of many of its applications and some results it plays a key role in can be found in the book of Bekka, de la Harpe and Valette [BDV08].

B.2.1 DEFINITION

The most commonly used definition of property (T), and the one we will adopt, is in terms of almost invariant vectors for unitary representations. The reader is referred to Chapter 5: Unitary Representations for a description of (the aspects that we will use of) the theory of unitary representations.

Definition B.7. Let G be a locally compact or countable discrete group and π a unitary representation of G on a Hilbert space $\mathcal{H}$. A vector $x \in \mathcal{H}$ is (K, ϵ)**-invariant** for a compact set $K \subseteq G$ and an $\epsilon > 0$ when $\|gx_n - x_n\| < \epsilon$ for all $g \in K$.

Definition B.8. Let G be a locally compact or countable discrete group and π a unitary representation of G on a Hilbert space $\mathcal{H}$. A sequence of vectors $x_n \in \mathcal{H}$ is **K-almost-invariant** for a compact set $K \subseteq G$ when $\|gx - x\| \to 0$ for all $g \in K$.

Definition B.9. Let G be a group and π a unitary representation of G on a Hilbert space. A vector x is **invariant** when $\pi(g)x = x$ for all $g \in G$. When $x \neq 0$ is invariant we say that (G, π) **admits a nontrivial invariant vector**.

Definition B.10. Let G be a group and π a unitary representation on a Hilbert space. Let $K \subseteq G$ be a compact set and x_n a sequence of *unit* vectors such that $\|\pi(k)x_n - x_n\| \to 0$ for all $k \in K$. Then $\{x_n\}$ is a **sequence of K-almost-invariant unit vectors**. When for every compact K there is such a sequence we say that (G, π) **admits almost invariant (unit) vectors**.

Definition B.11. Let G be a locally compact or countable discrete group. Then G has **property** (T) when any unitary representation π of G on a Hilbert space that admits almost invariant (unit) vectors also admits a nonzero invariant vector.

This can be sharpened quantitatively in that it is equivalent to say that there exists a fixed $\epsilon > 0$ and compact $K \subseteq G$, both depending only on G, such that any unitary representation that has a (K, ϵ)-invariant vector in fact has an invariant vector. The ϵ is referred to as the **Kazhdan constant** for G.

B.2.2 Equivalent Conditions

As with amenability, part of the power of property (T) is that it has a variety of equivalent conditions:

Theorem B.12. *Let G be a locally compact or countable discrete group. The following are equivalent:*

- *G has property (T) (almost invariant vectors implies invariant vectors)*
- *the trivial unitary representation is an isolated point in the space of representations of G (the unitary dual) under the Fell topology*
- *if F_n are positive definite functions on G converging on compact sets to 1 then F_n converge to 1 uniformly*
- *every continuous affine isometric action of G on a real Hilbert space admits a fixed point*

We remark that the trivial representation being isolated in the unitary dual (the second condition above) is the motivation for the name property (T) which is meant to indicate the trivial representation is isolated.

B.2.3 Consequences

We list now some general facts about property (T), all well-known and generally easy:

- quotients of groups with property (T) also have property (T)
- if G has property (T) then the abelianization of G is compact (the abelianization of G is $G/[G,G]$ where $[\cdot,\cdot]$ is the commutator: $[A,B] = \{aba^{-1}b^{-1} : a \in A, b \in B\}$)
- a countable group Γ that has property (T) is finitely generated (and likewise a locally compact group with property (T) is compactly generated)
- a lattice in a Lie group has property (T) if and only if the Lie group does (Kazhdan)

We remark also that while not every property (T) group is finitely presented (as was conjectured by Kazhdan), every property (T) group is the quotient of a finitely presented group (as shown by Shalom).

B.2.4 Examples

The main examples of property (T) groups are simple real Lie groups of rank two or higher, lattices in those groups, compact groups, finite groups, and a variety of hyperbolic groups.

Some groups that do not have property (T) are the integers, nonabelian free groups, noncompact solvable groups and $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{Z})$.

B.3 Mutual Exclusion

Amenability and Property (T) are "mutually exclusive" in that the intersection of these two classes of groups is trivial in the geometric sense–having both properties characterizes a group being finite (compact):

Proposition B.3.1. *Let G be a countable discrete group that both has property (T) and is amenable. Then G is finite.*

Proof. Consider the action of G on $L^2(G)$ (with the counting measure). Since G is amenable there are nontrivial almost invariant vectors $f_n \in L^2(G)$, $\|f_n\| = 1$ (Dixmier) such that $\|g \cdot f_n - f_n\| \to 0$ for all $g \in G$. Since G has property T and this is an action of G on a Hilbert space, the presence of almost invariant vectors implies the existence of a nontrivial invariant vector $f \in L^2(G)$, $f \neq 0$, such that $g \cdot f = f$ for all $g \in G$. Since f is G-invariant it is constant so we have $\infty > \|f\|^2 = \sum_{g \in G} |f(g)|^2 = |f(e)|^2 |G|$ and therefore $|G| < \infty$ since $f \neq 0$. □

Proposition B.3.2. *Let G be a locally compact group that both has property (T) and is amenable. Then G is compact.*

Proof. Exactly as above replacing the counting measure with Haar measure. □

Appendix C

Algebraic Groups

An algebraic group over a field is essentially defined as the zeroes of a set of polynomials in some number of variables. The easiest example is the special linear group $\mathrm{SL}_n(\mathbb{R})$, the set of n by n matrices with determinant one and real entries equipped with matrix multiplication. The elements of this set can be thought of as the zeroes of the polynomial equation $\det(M)-1=0$ in n^2 variables (the entries of the matrix).

The theory of algebraic groups is quite deep and old and the reader should consult [Mil11] for detailed information on algebraic groups, we only touch on the aspects of the theory necessary to formulate the Margulis-Zimmer conjecture and to prove our partial result about it in a later chapter.

C.1 Definition

Before we formally define algebraic groups we will attempt to briefly motivate how this definition was arrived at. Throughout the definition process we will use SL_n as an example to illustrate the meaning of the abstract categorical statements.

C.1.1 Motivation

As mentioned above, $\mathrm{SL}_n(\mathbb{R})$ is the algebraic group consisting of n by n matrices with determinant one and real entries. Notationally it is clear that $\mathrm{SL}_n(\mathbb{C})$ refers to the algebraic group of n by n matrices with complex entries and that more generally we can write $\mathrm{SL}_n(k)$ for any field k. In fact, the polynomial that defines the determinant in terms of the entries of the matrix is the same for all these cases. The SL_n is to be defined as a functor from rings (or fields) to groups and algebraic groups in general as functors "determined by polynomials".

C.1.2 Zero-Sets

Before formally defining algebraic groups we will need some preliminary definitions:

Definition C.1. Let k be a field and $k[X_1,\ldots,X_n]$ denote the ring adjoining n abstract variables to k. For $S\subseteq k[X_1,\ldots,X_n]$ and R a k-algebra define the **zero-set** of S in R^n to be

$$S(R)=\{(r_1,\ldots,r_n)\in R^n : f(r_1,\ldots,r_n)=0 \text{ for all } f\in S\}$$

If $R\to R'$ is a homomorphism of k-algebras then $S(R)\to S(R')$ induced in the natural way defines a morphism of zero-sets. Thus S is a functor from k-algebras to sets. Clearly the zero-set of S coincides with the zero-set of the ideal generated by S in $k[X_1,\ldots,X_n]$ and by the Hilbert basis theorem this ideal is generated by a finite set of polynomials and the quotient of $k[X_1,\ldots,X_n]$ by the ideal is a finitely generated k-algebra, sometimes called the **zero-ideal**.

C.1.3 Formal Definition

Definition C.2. Given a finitely generated k-algebra A we can define the functor F_A from k-algebras to sets by $F_A(R) = \mathrm{Hom}(A, R)$ and $F_A(f)(g) = f \circ g$ for f a homomorphism of k-algebras and $g \in \mathrm{Hom}(A, R)$. A functor from k-algebras to sets is **representable** when is it is isomorphic as a functor to a functor F_A for some k-algebra A.

Definition C.3. Let k be a field and $\mathbf{G}$ be a functor from k-algebras to groups such that the composition of $\mathbf{G}$ with the functor from groups to sets that simply forgets the group structure is representable by a finitely generated k-algebra. Then $\mathbf{G}$ is an **(affine) algebraic group** defined over k.

We will not define carefully the more general non-affine algebraic group since we make no use of those. We have already seen that SL_n is representable as the k-algebra obtained by quotienting the polynomials in k out by the zero-ideal of the determinant minus one so it is clear this definition captures the motivation correctly.

C.1.4 The Zariski Topology

The natural topology on algebraic varieties is called the Zariski topology and is usually defined by specifying the closed sets. The closed sets in k^n are defined to be the zero-sets of polynomials in at most n variables and the closed sets of a general variety, including an algebraic group, are the intersections of these with the variety. That is, an algebraic group has the Zariski topology inherited as a subspace topology from k^n.

We will make only slight use of the Zariski topology in what follows but it is always in the background when studying algebraic groups.

Definition C.4. An **algebraic subgroup** of an algebraic group is a Zariski closed subgroup.

That is, an algebraic subgroup is a closed subgroup (i.e. topological subgroup) when the group is endowed with the Zariski topology. Likewise, algebraic homomorphism means topological homomorphism with respect to the Zariski topology. We mention this as usually groups have multiple topologies, only one of which is the Zariski topology.

Generally speaking, when one refers to a property of a group as an algebraic property one means that it is a topological group property using the Zariski topology. For example, when G is a topological group we say it is topologically simple when there are no closed nontrivial normal subgroups. Likewise, a group is algebraically simple when there are no nontrivial algebraic normal subgroups (meaning there are no nontrivial closed normal subgroups in the Zariski topology).

C.1.5 Groups Over Fields

Note that when $\mathbf{G}$ is defined over a field that is not algebraically closed, for example $\mathbb{Q}$, then it is also defined over every completion of that field to an algebraic closure (or in between). Clearly SL_n is an algebraic group over $\mathbb{Q}$ hence it is over $\mathbb{R}$ and $\mathbb{C}$ but also over $\mathbb{Q}_p$, the p-adic numbers. In general any algebraic group over $\mathbb{Q}$ is algebraic over the p-adics and the reals.

We will generally be concerned with algebraic groups defined over $\mathbb{Q}$ but may localize them to the reals or the p-adics.

C.1.6 Lie Groups

Algebraic groups defined over $\mathbb{R}$, evaluated at $\mathbb{R}$, such as $\mathrm{SL}_n(\mathbb{R})$ will always be Lie groups. There are however Lie groups which are not algebraic groups, such as the simply-connected covering of $\mathrm{SL}_2(\mathbb{R})$.

C.1.7 Connected Algebraic Groups

Definition C.5. An algebraic group is **connected** when it has no finite group as a quotient, even over the algebraic closure of the underlying field.

C.1.8 Notational Language

Though algebraic groups are defined as functors, it is generally easier to speak of them as groups and use the usual group theory terminology. This is always understood to mean that at the level of the group (that is applying the functor to a specific k-algebra) the properties are group theoretic and at the level of the functor the properties are the corresponding functorial interpretation.

For example, we will refer to subgroups of algebraic groups and quotients of algebraic groups in phrases such as "the center of SL_2 is a normal subgroup" and "the quotient by the center is PSL_2". The reader can verify that the functorial definitions implied here coincide with the group theoretic properties.

C.2 Structure Theory

We now (briefly) state the well-known structure theory of algebraic groups both for completeness and to explain a bit why the focus on semisimple groups is not so restrictive.

C.2.1 Basic Classes of Algebraic Groups

There are five basic class of algebraic groups that together with extensions define all algebraic groups: finite groups, abelian varieties, semisimple groups, tori and unipotent groups.

Finite Groups

The first basic class of algebraic groups is finite groups. These can be seen to be algebraic since all are subgroups of permutation groups which are in turn realizable as subgroups of GL_n hence defined by polynomial conditions making them algebraic groups.

Abelian Varieties

Another basic class of algebraic groups is abelian varieties, those algebraic varieties defined by elliptic curves hence definable by a homogenous polynomial equation. Abelian varieties are not affine algebraic groups but they are algebraic.

Semisimple Groups

The discussion of semisimple groups is postponed to the next section since it is the class we will be most focused on.

Tori

An algebraic subgroup of $\mathrm{GL}(V)$ over a finite-dimensional vector space V is of multiplicative type when there is a basis for V over the algebraic closure of k that diagonalizes the subgroup. A group that is realizable as such subgroups is called a torus.

Unipotent Groups

The final class of algebraic groups are those arising as algebraic subgroups of $\mathrm{GL}(V)$ where there exists a basis of V over k such that the subgroup is contained in the subgroup of upper triangular matrices with ones along the diagonal.

C.2.2 Extensions

The structure theory of algebraic groups can be stated as saying that every algebraic group has a composition series with specific types of algebraic groups at each step. Specifically:

- a general algebraic group $\mathbf{G}$ contains a maximal connected component $\mathbf{G}^0$ which is normal in $\mathbf{G}$ and that is a connected algebraic group such that $\mathbf{G}/\mathbf{G}^0$ is finite
- a connected algebraic group contains a maximal affine algebraic subgroup which is normal in the group such that the quotient is an abelian variety
- a connected affine algebraic group contains a maximal connected solvable subgroup, sometimes called the radical, which is normal in the group and where the quotient is a semisimple algebraic group
- a connected affine solvable algebraic group contains a maximal normal unipotent subgroup where the quotient is a torus

C.2.3 The Structure of Algebraic Groups

Therefore any algebraic group can be decomposed into a composition series of normal subgroups such that each quotient group (and the “last” group in the series) are in the basic classes outlined above.

In particular, affine algebraic groups are precisely those that avoid abelian varieties. More importantly, an affine algebraic group has a decomposition into a finite quotient, a semisimple quotient and a solvable subgroup. Subsuming the finite quotient into the semisimple group (that is, allowing for nonconnected semisimple groups) this means that affine algebraic groups can always be written as a normal solvable subgroup with quotient semisimple. As solvable groups are generally easy to understand, the focus put on semisimple groups in the theory is not misplaced.

C.3 Semisimple Groups

Paralleling and expanding upon the material in the previous chapter on semisimple Lie groups, we define semisimplicity for algebraic groups.

C.3.1 Almost Simple Groups

Recall that a connected algebraic group is **simple** when it is noncommutative and has no nontrivial normal algebraic subgroups (as in the case of topological groups, the notion of normal for algebraic groups should mean normal algebraic subgroups).

Definition C.6. An algebraic group $\mathbf{G}$ is **almost simple** when the center of $\mathbf{G}$ is finite and the quotient by the center is simple.

SL_n is almost simple since the center (consisting of matrices of the form n^{th} root of unity times the identity) is finite and the quotient by the center is PSL_n which is simple.

C.3.2 Isogenies

Definition C.7. Let $\mathbf{G}$ and $\mathbf{H}$ be algebraic groups. A homomorphism $\mathbf{G} \to \mathbf{H}$ that is surjective and has finite kernel is an **isogeny**.

Definition C.8. Two algebraic groups $\mathbf{H}_1$ and $\mathbf{H}_2$ are **isogenous** when there exists an algebraic group $\mathbf{G}$ such that $G \to H_1$ and $G \to H_2$ are both isogenies.

Isogeneity is an equivalence relation, as is easily checked, and when the underlying field is algebraically closed there is a classification scheme for almost simple groups up to isogeny.

C.3.3 Almost Direct Products

Definition C.9. Let $\mathbf{G}$ be an algebraic group and $\mathbf{G}_1, \ldots, \mathbf{G}_n$ be algebraic subgroups such that the map $\mathbf{G}_1 \times \cdots \times \mathbf{G}_n \to \mathbf{G}$ by $(g_1, \ldots, g_n) \mapsto g_1 \cdots g_n$ is an isogeny. Then $\mathbf{G}$ is the **almost direct product** of the $\mathbf{G}_j$.

In this case, each $\mathbf{G}_j$ will be a normal subgroup of $\mathbf{G}$ and they will necessarily all commute with one another.

C.3.4 Semisimple Groups

The actual definition of semisimple is then given as:

Definition C.10. An algebraic group is **semisimple** when it is an almost direct product of almost simple groups.

The following result on lattices in semisimple groups is the starting off point for Margulis' classification theory:

Theorem C.11 (Borel-Harish-Chandra)**.** *Let* $\mathbf{G}$ *be a semisimple algebraic group. Then* $\mathbf{G}(\mathbb{Z})$ *is a lattice in* $\mathbf{G}(\mathbb{R})$*.*

C.4 $\mathbb{Q}$-Groups and Rank

In terms of rigidity theory there is a fundamental difference between the groups PSL_2 and PSL_n for $n \geq 3$. Since $\mathrm{PSL}_2(\mathbb{Z})$ is essentially a free product of two finite groups, among other things, it has a huge collection of normal subgroups since any finitely generated group is a quotient of a free group and each kernel is normal. On the other hand, Margulis' Normal Subgroup Theorem implies that $\mathrm{PSL}_n(\mathbb{Z})$ for $n \geq 3$ has no nontrivial normal subgroups (up to finite index).

C.4.1 k-rank

This phenomena can be described more generally in terms of the rank of the ambient group. Intuitively the rank of an algebraic group such as SL_n is the dimension of the maximal diagonal subgroup so $\mathrm{rank}(\mathrm{SL}_n) = n - 1$. We formalize this by:

Definition C.12. Let $\mathbf{G}$ be an algebraic group over k. The **k-rank** of $\mathbf{G}$ is the dimension of any Cartan subgroup. A **Cartan subgroup** is a maximal nilpotent subgroup such that each normal subgroup of it of finite index has finite index in the normalizer. When $\mathbf{G}$ is a linear algebraic group this becomes simply the dimension of the maximal k-split torus. This will be written $\mathrm{rank}_k(\mathbf{G})$.

C.4.2 Real and p-adic Rank

Definition C.13. Let $\mathbf{G}$ be an algebraic group over $\mathbb{Q}$. The **real rank** of $\mathbf{G}$ is the dimension of a maximal $\mathbb{R}$-split torus in $\mathbf{G}(\mathbb{R})$. The real rank will be written $\mathrm{rank}_{\mathbb{R}}(\mathbf{G})$ or $\mathrm{rank}_\infty(\mathbf{G})$.

The **p-rank**, for a prime p, is the dimension of a maximal $\mathbb{Q}_p$-split torus in $\mathbf{G}(\mathbb{Q}_p)$. This will be written as $\mathrm{rank}_p(\mathbf{G})$.

C.4.3 General Rank

Definition C.14. Let $\mathbf{G}_1, \ldots, \mathbf{G}_n$ be algebraic groups defined over local fields $k_1, \ldots, k_n$, respectively, and let G be the almost direct product of the groups $\mathbf{G}_1(k_1), \ldots, \mathbf{G}_n(k_n)$. The **rank** of G is defined as

$$\mathrm{rank}(G) = \sum_{j=1}^{n} \mathrm{rank}_{k_j}(\mathbf{G}_j)$$

For example, $\mathrm{rank}(\mathrm{SL}_n) = n - 1$ when treated as an algebraic group over $\mathbb{R}$ or $\mathbb{Q}_p$. Note that the rank is defined in terms of the local fields (completed versions of $\mathbb{Q}$ for example).

Another example is that $\mathrm{rank}(\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)) = 2$ since the real rank is one and the p-rank is one and the group is an almost direct product.

Definition C.15. An algebraic group G is said to be of **higher rank** when the rank is at least two.

C.5 Rings of Integers and S-Integers

Let $\mathbf{G}$ be an algebraic group over $\mathbb{Q}$, or more generally over any number field K. Write k for the algebraic closure of K. The ring of integers will be denoted $\mathcal{O}$ (when $K = \mathbb{Q}$ we have $\mathcal{O} = \mathbb{Z}$). We can then form the algebraic group $\mathbf{G}(\mathcal{O})$, an example of this is $\mathrm{SL}_n(\mathbb{Z})$.

C.5.1 The Ring of Integers

Definition C.16. The **ring of integers** of an algebraic number field is the set of algebraic integers in K equipped with the operations of addition, subtraction and multiplication.

Algebraic integers are just roots of monic polynomial equations with coefficients in $\mathbb{Z}$ so it is easy to see that the algebraic integers contained in the rationals are just the usual integers.

C.5.2 S-Integers

Let V be the set of all (inequivalent) valuations of K and V_∞ the archimedean valuations. When $K = \mathbb{Q}$ the valuations correspond to primes, including ∞ to represent zero (which would be the only archimedean valuation).

Definition C.17. Let S be a subset of V of valuations on K containing V_∞ the archimedean valuations. The **ring of S-integers** in K is

$$\mathcal{O}_S = \{x \in K : |x|_v \leq 1 \text{ for all } v \in S\}$$

When $K = \mathbb{Q}$ and S is a set of primes the ring of S-integers means the set of $x \in \mathbb{Q}$ such that $|x| \leq 1$ and $|x|_p \leq 1$ for each $p \in S$ where $|\cdot|_p$ is the p-adic valuation.

We will be interested in algebraic groups $\mathbf{G}$ over $\mathbb{Q}$ (or more generally over a number field K) and S as above to form $\mathbf{G}(\mathcal{O}_S)$ the algebraic group of S-integer-valued $\mathbf{G}$ elements.

C.6 Arithmetic Lattices

The notion of arithmetic lattices is fundamental to Margulis' classification theory of lattices in semisimple Lie groups and more general semisimple groups.

Definition C.18. Let G be a semisimple Lie group with trivial center and no compact factors and Γ a lattice in G. Then Γ is **arithmetic** when there exists an algebraic group $\mathbf{H}$ defined over $\mathbb{Q}$ and a surjective homomorphism $\phi : \mathbf{H}(\mathbb{R})^0 \to G$ (from the connected component of $\mathbf{H}(\mathbb{R})$ to G) such that the kernel of ϕ is compact and $\phi(\mathbf{H}(\mathbb{Z}) \cap \mathbf{H}(\mathbb{R})^0)$ is a lattice in G that is commensurate with Γ.

This notion is the natural notion of arithmeticity in the sense that it captures the largest class of lattices that can be constructed from the integer points of an algebraic group (the motivation for the terminology arithmetic).

C.6.1 Classification of Lattices

One construction of arithmetic lattices is clear: given an algebraic group simply take the $\mathbb{Z}$ points in the $\mathbb{R}$ group and (restricting to the connected component) you must have a lattice in a Lie group. Up to isogeny then all "obvious" lattices are the $\mathbb{Z}$ points, i.e. arithmetic.

After some time of no one being able to exhibit a non-arithmetic lattices in a Lie group, Selberg conjectured that in fact every lattice in a semisimple Lie group is arithmetic. Margulis eventually proved this, in his classification theorem: the Margulis Arithmeticity Theorem. We will return to this in greater detail later (see Chapter 12: The Margulis-Zimmer Conjecture).

C.6.2 S-Arithmetic Lattices

The starting off point for the notion of S-arithmetic lattices is a result of Borel generalizing that arithmetic lattices are in fact lattices: for S be a finite collection of prime numbers write $\mathbb{Z}_S$ to be the **S-integers**, the rational numbers whose denominators (in simplest form) contain only factors in S. Then

Theorem C.19 (Borel)**.** *Let $\mathbf{G}$ be an algebraic group defined over $\mathbb{Q}$ and let S be a finite collection of prime numbers. When $\mathbf{G}$ is connected and semisimple $\mathbf{G}(\mathbb{Z}_S)$ is a lattice in $\prod_{p\in S}\mathbf{G}(\mathbb{Q}_p)$.*

Such a lattice will be called **S-arithmetic**. We now generalize this as we did with arithmetic lattices to obtain the complete definition.

Let K be a global (number) field and V the set of (inequivalent) valuations on K and V_∞ the archimedean valuations in V. Write $|\cdot|_v$ for each $v \in V$ to mean the valuation of an element of K and write K_v for the completion of K under $v \in V$. Concretely, when $K = \mathbb{Q}$ write $\mathbb{Q}_p$ for the completion under the p-adic valuation $|\cdot|_p$.

Definition C.20. Let $S \subseteq V$ such that $V_\infty \subseteq S$ and let $\mathbf{G}$ be an absolutely simple, simply connected algebraic group defined over K. As usual write $\mathcal{O}_S$ for the ring of S-integers in K. Let G be an algebraic group such that $\prod_{v\in S}\mathbf{G}(K_v) \to G$ is an isogeny. Any subgroup of G commensurate with the image of $\mathbf{G}(\mathcal{O}_S)$, embedded diagonally into the product group, is called **S-arithmetic**.

Note that $\mathbf{G}(\mathcal{O}_S)$ will be a lattice in $\prod_{v\in S}\mathbf{G}(K_v)$ and therefore so will be the image in G. Any group commensurate with a lattice (meaning the intersection of the group with the lattice has finite index in both) is of course also a lattice.

Arithmetic lattices usually means S-arithmetic lattices for arbitrary finite sets S but some use the phrase S-arithmetic without specifying S and reserve the unadorned arithmetic to mean images of the usual ring of integers (that is, the case when $S = V_\infty$). We will refer to lattices arising from the usual ring of integers as **pure arithmetic** to avoid confusion when necessary. The pure arithmetic lattices are then those of the form $\mathbf{G}(\mathbb{Z})$ and isogenic images of it. In particular, $\mathrm{SL}_n(\mathbb{Z})$ and all its finite index subgroups are arithmetic. A more interesting example is that $\mathrm{SL}_n(\mathbb{Z}[1/p])$ is an S-arithmetic lattice where S contains the p-adic valuation and the archimedean ones: $S = \{p, \infty\}$.

C.7 The Margulis Arithmeticity Theorem

While the construction of lattices in rank one algebraic semisimple groups is easy and there are a variety of them, in the higher-rank case (the rank at least two) the only lattices that anyone could exhibit were of the arithmetic type defined above. Margulis in the 1970s proved the celebrated Arithmeticity Theorem:

Theorem C.21 (Margulis)**.** *Let* Γ *be an irreducible lattice in a higher-rank semisimple algebraic group* G*. Then* Γ *is arithmetic.*

This theorem is interpreted as saying that there is a finite index subgroup of Γ that is the isogenic image of $\mathbf{G}(\mathcal{O}_S)$ embedded diagonally in the almost direct product of $\mathbf{G}(K_v)$ for $v \in S$ where S is some set of valuations. Our final chapter makes heavy use of this fact in stating and making progress on the Margulis-Zimmer Conjecture. In fact, the statement of the conjecture requires the arithmeticity theorem in order to be formulated.

Index

Bibliography

[Arv81] William Arveson, *An invitation to C^*-algebras*, Springer-Verlag, 1981.

[BDV08] Bachir Bekka, Pierre De La Harpe, and Alain Valette, *Kazhdan's property* (T), Cambridge University Press, 2008.

[BK96] Howard Becker and Alexander S. Kechris, *The descriptive set theory of Polish group actions*, London Mathematical Society Lecture Note Series, Cambridge University Press, 1996.

[BL01] Hyman Bass and Alexander Lubotzky, *Tree lattices*, Progress in Mathematics, vol. 176, Birkhäuser, 2001.

[BM00] Marc Burger and Shahar Mozes, *Groups acting on trees: From local to global structure*, Publications Mathématiques de l'I.H.É.S. **92** (2000), 113–150.

[BS05] Uri Bader and Yehuda Shalom, *Factor and normal subgroup theorems for lattices in products of groups*, Inventiones Mathematicae **163** (2005), no. 2, 415–454.

[Che46] Claude Chevalley, *Theory of Lie groups*, Princeton University Press, 1946.

[CR09] Pierre-Emmanuel Caprace and Bertrand Rémy, *Simplicity and superrigidity of twin building lattices*, Inventiones Mathematicae **176** (2009), 169–221.

[Del77] Patrick Delorme, *1-cohomologie des représentations unitaires des groupes de Lie semi-simple et résolubles. Produits tensoriels continus et représentations*, Bull. Soc. Math. France **105** (1977), 289–323.

[FG10] Hillel Furstenberg and Eli Glasner, *Stationary dynamical systems*, Preprint (2010), arXiv:0910.4185 [math.DS].

[FHT08] Benson Farb, Chris Hruska, and Anne Thomas, *Problems on automorphism groups of nonpositively curved polyhedral complexes and their lattices*, Geometry, Topology and Rigidity (to appear) (2008).

[FMT07] Benson Farb, Shahar Mozes, and Anne Thomas, *Lattices in trees and higher dimensional complexes*, www.aimath.org/pggt/Lattices_in_Trees_and_Higher_Dimensional_Complexes, 2007.

[Fur63] Harry Furstenberg, *A Poisson formula for semi-simple Lie groups*, The Annals of Mathematics **77** (1963), no. 2, 335–386.

[Fur67] ———, *Poisson boundaries and envelopes of discrete groups*, Bulletin of the American Mathematical Society **73** (1967), no. 3, 350–356.

[Fur71] ______, *Random walks and discrete subgroups of Lie groups*, Advances in Probability and Related Topics **1** (1971), 1–63.

[Fur73] ______, *Boundary theory and stochastic processes on homogenous spaces*, Proceedings of the Symposium on Pure Mathematics, vol. XXVI, 1973, pp. 193–229.

[Fur02] Alex Furman, *Random walks on groups and random transformations*, Handbook of Dynamical Systems, vol. 1A, 2002, pp. 931–1014.

[Fur03] ______, *On minimal, strongly proximal actions of locally compact groups*, Israel Journal of Mathematics **136** (2003), 173–187.

[GKM08] Tsachik Gelander, Anders Karlsson, and Gregory A. Margulis, *Superrigidity, generalized harmonic maps and uniformly convex spaces*, Geometric and Functional Analysis **17** (2008), 1524–1550.

[Gla76] Schmuel Glasner, *Proximal flows*, Springer-Verlag, 1976.

[Gla98] Eli Glasner, *On minimal actions of Polish groups*, Topology and Its Applications **85** (1998), 119–125.

[Gla03] ______, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, vol. 101, American Mathematical Society, 2003.

[GTW05] Eli Glasner, Boris Tsirelson, and Benji Weiss, *The automorphism group of the Gaussian measure cannot act pointwise*, Israel Journal of Mathematics **148** (2005), 305–329.

[Gui80] Alain Guichardet, *Cohomologie des groups topologies et des algèbres de Lie*, 1980.

[Jaw91] Wojciech Jaworski, *Poisson and Furstenberg boundaries of random walks*, Comptes Rendus Mathematiques de l'Academie des Sciences **13** (1991), no. 6, 279–284.

[Jaw94] ______, *Strongly approximately transitive group actions, the Choquet-Deny theorem, and polynomial growth*, Pacific Journal of Mathematics **165** (1994), no. 1, 115–129.

[Jaw98] ______, *Random walks on almost connected locally compact groups: boundary and convergence*, Journal D'Analyse Mathematique **74** (1998), 235–273.

[Kai88] V.A. Kaimanovich, *Brownian motion on foliations: entropy, invariant measures, mixing*, Functional Analysis and Its Applications **22** (1988), no. 4, 326–328.

[Kai92] ______, *Discretization of bounded harmonic functions on Riemannian manifolds and entropy*, Proceedings of the International Conference on Potential Theory (1992), 213–223.

[Kec00] Alexander S. Kechris, *Descriptive dynamics*, Descriptive Set Theory and Dynamical Systems, London Mathematical Society Lecture Note Series, vol. 277, Cambridge University Press, 2000.

[Kle10] Bruce Kleiner, *A new proof of Gromov's theorem on groups of polynomial growth*, Journal of the American Mathematical Society **23** (2010), no. 3, 815–829.

[KV83] V.A. Kaimanovich and A.M. Vershik, *Random walks on discrete groups: boundary and entropy*, Annals of Probability **11** (1983), no. 3, 457–490.

[Liu94] Y.S. Liu, *Density of the commensurability groups of uniform tree lattices*, Journal of Algebra **165** (1994), 346–359.

[LMZ94] Alexander Lubotzky, Shahar Mozes, and Robert Zimmer, *Superrigidity for the commensurability group of tree lattices*, Comment. Math. Helv. **69** (1994), 523–548.

[LS84] Terry Lyons and Dennis Sullivan, *Function theory, random paths and covering spaces*, Journal of Differential Geometry **19** (1984), 299–323.

[Lub95] Alexander Lubotzky, *Tree-lattices and lattices in Lie groups*, Combinatorial and Geometric Group Theory, London Mathematical Society Lecture Notes Series, vol. 204, Cambridge University Press, 1995.

[Mar79] Gregory A. Margulis, *Finiteness of quotient groups of discrete subgroups*, Functional Analysis and Applications **13** (1979), 178–187.

[Mar91] ———, *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, 1991.

[Mil11] J.S. Milne, *Algebraic groups, Lie groups, and their arithmetic subgroups*, www.jmilne.org-/math/CourseNotes/ala.html, 2011.

[Moz98] Shahar Mozes, *Products of trees, lattices and simple groups*, Documenta Mathematica (1998), 571–582.

[NZ02] Amos Nevo and Robert Zimmer, *A structure theorem for actions of semisimple Lie groups*, Annals of Mathematics **155** (2002), 565–594.

[Raj03] C.R.E. Raja, *On heredity of strongly proximal actions*, Archivum Mathematicum (Brno) **39** (2003), 51–55.

[Rau77] Albert Raugi, *Fonctions harmoniques sur les groupes localement compacts à base dénombrable*, Mémoires de la S.M.F. **54** (1977), 5–118.

[Run02] Volker Runde, *Lectures on amenability*, Lecture Notes in Mathematics, vol. 1774, Springer, 2002.

[Sch80] G. Schlichting, *Operationen mit periodischen stabilisatoren*, Archiv der Math. **34** (1980), 97–99.

[Sha98] Yehuda Shalom, *Rigidity and cohomology of unitary representations*, International Mathematics Research Notices **16** (1998), 829–849.

[Sha00a] ———, *Rigidity of commensurators and irreducible lattices*, Inventiones Mathematicae **141** (2000), 1–54.

[Sha00b] ———, *Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group*, Annals of Mathematics **152** (2000), 113–182.

[Sha06] ———, *The algebraization of property* (T), International Congress of Mathematicians (Madrid), vol. 2, 2006.

[SW09] Yehuda Shalom and George A. Willis, *Commensurated subgroups of arithmetic groups, totally disconnected groups and adelic rigidity*, Preprint (2009), arXiv:0911.1966v1 [math.GR].

[Tit70] Jacques Tits, *Sur le groupe des automorphismes d'un arbre*, Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham) (1970), 188–211.

[Tza00] Kroum Tzanev, *C^*-algébres de Hecke at K-theorie*, Ph.D. thesis, Université Paris 7 – Denis Diderot, 2000.

[Tza03] ———, *Hecke C^*-algébres and amenability*, Journal of Operator Theory **50** (2003), 169–178.

[Var74] V.S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Springer, 1974.

[Zim84] Robert Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, 1984.