0.1. Details on Mixing Heights. Here we outline in detail the proof that the heights of a rank one transformation form a mixing sequence with respect to that transformation. These techniques are also fundamental in the main theorem. This is intended for readers wishing a more explicit explanation of how the proof works.

As in the paper, we will use the notation $C_n = \{I_{n,i}\}$ for the columns of the transformation T with i ranging up to $h_n - 1$ where $\{h_n\}$ are the heights. The sublevels of $I_{n,i}$ are denoted $I_{n,i}^{[j]}$ with j ranging up to $r_n - 1$ where $\{r_n\}$ is the sequence of cuts. The dynamical sequence $\{s_{n,j}\}_{\{r_n\}}$ will denote the spacer sequence.

First, we consider that

$$I_{n,i} = \bigcup_{j=0}^{r_n-1} I_{n,i}^{[j]}.$$

Applying T^{h_n},

$$T^{h_n}(I_{n,i}) = \bigcup_{j=0}^{r_n-1} T^{h_n}(I_{n,i}^{[j]}) = \bigcup_{j=0}^{r_n-2} T^{-s_{n,j}}(I_{n,i}^{[j+1]}) \cup T^{h_n}(I_{n,i}^{[r_n-1]})$$

because there are $s_{n,j}$ spacers above the jth column.

Now, if we assume that $i \geq \sup_j s_{n,j}$, then $T^{-s_{n,j}}(I_{n,i}^{[j+1]}) = I_{n,i-s_{n,j}}^{[j+1]}$ due to the nature of T as a stack.

Thus, for such an i and any level $I_{n,\ell}$,

$$\mu(T^{h_n}(I_{n,i}) \cap I_{n,\ell}) = \sum_{j=0}^{r_n-2} \mu(I_{n,i-s_{n,j}}^{[j+1]} \cap I_{n,\ell}) \pm \mu(I_{n,i}^{[r_n-1]}).$$

Now, since $I_{n,i-s_{n,j}}^{[j+1]}$ is a sublevel of $I_{n,i-s_{n,j}}$,

$$\mu(I_{n,i-s_{n,j}}^{[j+1]} \cap I_{n,\ell}) = \mu(I_{n,i-s_{n,j}}^{[j+1]}) \quad \text{if } i - s_{n,j} = \ell$$
$$\mu(I_{n,i-s_{n,j}}^{[j+1]} \cap I_{n,\ell}) = 0 \quad \text{if } i - s_{n,j} \neq \ell$$

Then, when $i - s_{n,j} = \ell$,

$$\mu(I_{n,i-s_{n,j}}^{[j+1]} \cap I_{n,\ell}) = \mu(I_{n,i-s_{n,j}}^{[j+1]}) = \frac{1}{r_n}\mu(I_{n,i-s_{n,j}}) = \frac{1}{r_n}\mu(I_{n,i-s_{n,j}} \cap I_{n,\ell})$$

and when $i - s_{n,j} \neq \ell$,

$$\mu(I_{n,i-s_{n,j}}^{[j+1]} \cap I_{n,\ell}) = 0 = \frac{1}{r_n}0 = \frac{1}{r_n}\mu(I_{n,i-s_{n,j}} \cap I_{n,\ell}).$$

Thus, we have that

$$\mu(T^{h_n}(I_{n,i}) \cap I_{n,\ell}) = \sum_{j=0}^{r_n-2} \frac{1}{r_n}\mu(I_{n,i-s_{n,j}} \cap I_{n,\ell}) \pm \mu(I_{n,i}^{[r_n-1]})$$

and, as $I_{n,i-s_{n,j}} = T^{-s_{n,j}}(I_{n,i})$,

$$\mu(T^{h_n}(I_{n,i}) \cap I_{n,\ell}) = \frac{1}{r_n}\sum_{j=0}^{r_n-2} \mu(T^{-s_{n,j}}(I_{n,i}) \cap I_{n,\ell}) \pm \mu(I_{n,i}^{[r_n-1]}).$$

Considering this equation over all the levels in C_n compared to some set B which is a union of levels in C_n, we achieve

$$\begin{aligned}
&\sum_{i=0}^{h_n-1} |\mu(T^{h_n}(I_{n,i}) \cap I_{n,\ell}) - \mu(I_{n,i})\mu(B)| \\
&\quad = \sum_{i=0}^{h_n-1} |\frac{1}{r_n}\sum_{j=0}^{r_n-1} \mu(T^{-s_{n,j}}(I_{n,i}) \cap I_{n,\ell}) - \mu(I_{n,i})\mu(B)| \pm 2\mu(I_{n,i}^{[r_n-1]}) \\
&\quad \leq \sum_{i=0}^{h_n-1} |\int_{I_{n,i}} \frac{1}{r_n}\sum_{j=0}^{r_n-1} \chi_B \circ T^{s_{n,j}} - \mu(B) d\mu| \pm \frac{2}{r_n}\mu(C_n) \\
&\quad \leq \int |\frac{1}{r_n}\sum_{j=0}^{r_n-1} \chi_B \circ T^{s_{n,j}} - \mu(B)| d\mu + \frac{2}{r_n}.
\end{aligned}$$

Hence, the ergodicity of $\{s_{n,j}\}_{\{r_n\}}$ with respect to T and $r_n \to 0$ as $n \to \infty$ imply that $\{h_n\}$ is mixing with respect to T.

As mentioned earlier, these techniques when applied to $\{k_n h_n\}$ form the basis of the main theorem. The restricted growth condition arises from the need to drop the bottom $\sup_j s_{n,j}^{(k)}$ levels under such sequences.