1 Generalized Staircase Transformations

An examination of the proofs of our results for the examples in the paper leads us to generalize the class of staircase transformations to what we term generalized staircase transformations. These transformations share the properties of staircases necessary for above proofs, allowing us to show mixing on them as well. It is indeed possible to extend the following argument to what would be called generalized polynomial staircase transformations. The interested reader is encouraged to attempt to combine the argument below with the argument for polynomial staircase transformations.

Definition 1. *Let $\{r_n\}$ be a nondecreasing unbounded sequence and $\{a_{n,i}\}_{\{\ell_n\}}$ be a dynamical sequence of nonnegative integers where $\{\ell_n\}$ is a nondecreasing (possibly bounded) sequence such that $\frac{\ell_n}{r_n} \to 0$ as $n \to \infty$. Define the dynamical sequence $\{s_{n,i}\}_{\{r_n\}}$ by $s_{n,i} = a_{n,i \mod \ell_n} + \lceil \frac{i}{\ell_n} \rceil$. The rank one transformation with spacer sequence $\{s_{n,i}\}_{\{r_n\}}$ is a* ***generalized staircase transformation****.*

1.1 A Generalized Staircase Transformation

We produce an example of a restricted growth generalized staircase transformation. By the coming theorem (Theorem 1.3), this transformation is a mixing transformation. Define the sequences $\{r_n\}$ and $\{\ell_n\}$ by $r_n = n(n+1)$ and $\ell_n = n+1$. Then, $\frac{\ell_n}{r_n} = \frac{1}{n} \to 0$ as $n \to \infty$. Let $\{a_{n,i}\}_{\{\ell_n\}}$ be the dynamical sequence given by $a_{n,i} = \binom{n}{i}$, where $\binom{n}{i}$ is the ith term in the nth row of Pascal's triangle defining the binomial coeffecients. Define the dynamical sequence $\{s_{n,i}\}_{\{r_n\}}$ by $s_{n,i} = a_{n,i \mod \ell_n} + \lceil \frac{i}{\ell_n} \rceil$. Let T be the rank one transformation with spacer sequence given by $\{s_{n,i}\}_{\{r_n\}}$ and cut sequence given by $\{r_n\}$. By construction, T is a generalized staircase transformation. Note that

$$\begin{aligned}\sum_{i=0}^{r_n-1} s_{n,i} &= \sum_{i=0}^{r_n-1} a_{n,i \mod \ell_n} + \left\lceil \frac{i}{\ell_n} \right\rceil = \sum_{i=0}^{n(n+1)-1} a_{n,i \mod (n+1)} + \left\lceil \frac{i}{n+1} \right\rceil \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n} a_{n,i(n+1)+j} + \left\lceil \frac{i(n+1)+j}{n+1} \right\rceil = \sum_{i=0}^{n-1}\sum_{j=0}^{n} a_{n,j} + i \\ &= n\sum_{j=0}^{n}\binom{n}{j} + \frac{n(n-1)}{2} = n2^n + \frac{n(n-1)}{2} \leq (n+1)2^n.\end{aligned}$$

Let $\{h_n\}$ denote the sequence of heights for T and note that $h_{n+1} \geq r_n h_n$ and so $h_{n+1} \geq \prod_{i=0}^{n} r_i = \prod_{i=0}^{n} i(i+1) = (n+1)n!^2$. Therefore,

$$\frac{1}{h_n}\sum_{i=0}^{r_n-1} s_{n,i} \leq \frac{(n+1)2^n}{n(n-1)!^2} = \frac{2^n}{(n-1)!^2}.$$

Observe that $\frac{2^{n+1}}{n!^2}\frac{(n-1)!^2}{2^n} = \frac{2}{n^2} \to 0$ as $n \to \infty$ and so $\frac{2^n}{(n-1)!^2} \to 0$ as $n \to \infty$. Thus, T has restricted growth (and is finite measure-preserving).

1.2 Power Uniform Ergodicity

To show that generalized staircases are power uniform ergodic transformations, we will show that the spacer sequence for such a transformation is totally ergodic with respect to any totally ergodic transformation as in the previous sections. First, we will show the total ergodicity of these transformations by showing they have a mixing height sequence.

Proposition 1.1. *Let T be a generalized staircase transformation. Then T is power uniform ergodic.*

Proof. Let T be a generalized staircase transformation with spacer sequence $\{s_{n,i}\}_{\{r_n\}}$ defined by a strictly increasing sequence $\{\ell_n\}$ and dynamical sequence $\{a_{n,i}\}_{\{\ell_n\}}$. Denote the partial sum dynamical sequences of the spacer sequence by $\{s^{(k)}_{n,i}\}_{\{r^{(k)}_n\}}$. First, we observe that for any n and $0 \leq i < r_n - \ell_n$,

$$s_{n,i+\ell_n} = a_{n,(i+\ell_n) \mod \ell_n} + \left\lceil \frac{i+\ell_n}{\ell_n} \right\rceil = a_{n,i \mod \ell_n} + \left\lceil \frac{i}{\ell_n} \right\rceil + 1.$$

Then, for any n, k and $0 \leq i < r^{(k)}_n - \ell_n$,

$$s^{(k)}_{n,i+\ell_n} = \sum_{z=0}^{k-1} s_{n,i+\ell_n+z} = \sum_{z=0}^{k-1} \big(s_{n,i+z} + 1\big) = s^{(k)}_{n,i} + k.$$

Applying the above repeatedly for any $0 \leq z < r_n$, write $z = q\ell_n + t$ where $0 \leq q$ and $0 \leq t < \ell_n$, the term $s^{(k)}_{n,z} = s^{(k)}_{n,t} + qk$.

Note that for any measurable set B and any n, following a similar line of reasoning as in the proof of the Block Lemma,

$$\begin{aligned}
&\int |\frac{1}{r_n} \sum_{i=0}^{r_n-1} \chi_B \circ T^{-s_{n,i}} - \mu(B)|^2 d\mu \\
&\leq \int |(\frac{r_n}{\ell_n})^{-1} \sum_{i=0}^{\lceil \frac{r_n}{\ell_n} \rceil - 1} \frac{1}{\ell_n} \sum_{j=0}^{\ell_n-1} \chi_B \circ T^{-s_{n,i\ell_n+j}} - \mu(B)|^2 d\mu \\
&\qquad + \int |\frac{1}{r_n} \sum_{i=\lceil \frac{r_n}{\ell_n} \rceil}^{r_n-1} \chi_B \circ T^{-s_{n,i}} - \mu(B)|^2 d\mu \\
&\leq \frac{1}{\ell_n} \sum_{j=0}^{\ell_n-1} \int |(\frac{r_n}{\ell_n})^{-1} \sum_{i=0}^{\lceil \frac{r_n}{\ell_n} \rceil - 1} \chi_B \circ T^{-s_{n,j}-i} - \mu(B)|^2 d\mu + \frac{\ell_n}{r_n}\mu(B) \\
&= \frac{1}{\ell_n} \sum_{j=0}^{\ell_n-1} \int |(\frac{r_n}{\ell_n})^{-1} \sum_{i=0}^{\lceil \frac{r_n}{\ell_n} \rceil - 1} \chi_B \circ T^{-i} - \mu(B)|^2 d\mu + \frac{\ell_n}{r_n}\mu(B) \\
&= \int |(\frac{r_n}{\ell_n})^{-1} \sum_{i=0}^{\lceil \frac{r_n}{\ell_n} \rceil - 1} \chi_B \circ T^{-i} - \mu(B)|^2 d\mu + \frac{\ell_n}{r_n}\mu(B).
\end{aligned}$$

Thus, since $\frac{r_n}{\ell_n} \to \infty$ as $n \to \infty$ and since T is ergodic, the spacer sequence for T is ergodic with respect to T. Hence, T has a mixing sequence of heights.

Similar to the above reasoning, observe that for any measurable set B, any n and any k,

$$\int |\frac{1}{r_n^{(k)}} \sum_{i=0}^{r_n^{(k)}-1} \chi_B \circ T^{-s_{n,i}^{(k)}} - \mu(B)|^2 d\mu$$

$$\leq \int |(\frac{r_n^{(k)}}{\ell_n})^{-1} \sum_{i=0}^{\lceil \frac{r_n^{(k)}}{\ell_n} \rceil -1} \frac{1}{\ell_n} \sum_{j=0}^{\ell_n -1} \chi_B \circ T^{-s_{n,i\ell_n+j}^{(k)}} - \mu(B)|^2 d\mu$$

$$+ \int |\frac{1}{r_n^{(k)}} \sum_{i=\lceil \frac{r_n^{(k)}}{\ell_n} \rceil}^{r_n^{(k)}-1} \chi_B \circ T^{-s_{n,i}^{(k)}} - \mu(B)|^2 d\mu$$

$$\leq \frac{1}{\ell_n} \sum_{j=0}^{\ell_n-1} \int |(\frac{r_n^{(k)}}{\ell_n})^{-1} \sum_{i=0}^{\lceil \frac{r_n^{(k)}}{\ell_n} \rceil -1} \chi_B \circ T^{-s_{n,j}^{(k)}-ik} - \mu(B)|^2 d\mu + \frac{\ell_n}{r_n^{(k)}} \mu(B)$$

$$= \frac{1}{\ell_n} \sum_{j=0}^{\ell_n-1} \int |(\frac{r_n^{(k)}}{\ell_n})^{-1} \sum_{i=0}^{\lceil \frac{r_n^{(k)}}{\ell_n} \rceil -1} \chi_B \circ T^{-ik} - \mu(B)|^2 d\mu + \frac{\ell_n}{r_n^{(k)}} \mu(B)$$

$$= \int |(\frac{r_n^{(k)}}{\ell_n})^{-1} \sum_{i=0}^{\lceil \frac{r_n^{(k)}}{\ell_n} \rceil -1} \chi_B \circ T^{-ik} - \mu(B)|^2 d\mu + \frac{\ell_n}{r_n^{(k)}} \mu(B).$$

Since T has a mixing sequence of heights, T is totally ergodic because weak mixing implies total ergodicity, and so for each fixed k, the k-th partial sum dynamical sequence of the spacer sequence is ergodic with respect to T. Hence, T has a totally ergodic spacer sequence and is therefore power uniform ergodic by the Theorem in the paper. □

1.3 Mixing on Restricted Growth Generalized Staircases

Generalizing the rest of the proof that staircase transformations are mixing, we show that if a generalized staircase transformation has restricted growth then it is a mixing transformation. The method used here mirrors the method used above applied to the proof of mixing on power uniform ergodic staircase transformations.

Proposition 1.2. *Let T be a power uniform ergodic transformation. Then, for a generalized staircase transformation, the sequence of spacers is uniformly ergodic with respect to T.*

Proof. Let T be a power uniform ergodic transformation and $\{s_{n,i}\}_{\{r_n\}}$ the spacer sequence for a generalized staircase transformation defined by $\{\ell_n\}$ and

$\{a_{n,i}\}_{\{\ell_n\}}$. Denote the family of partial sum dynamical sequences of the spacer sequence by $\{s^{(k)}_{n,i}\}_{\{r^{(k)}_n\}}$. Let $\{k_n\}$ be any sequence of positive integers such that

$$0 < \liminf_{n\to\infty} \frac{k_n}{r_n} \leq \limsup_{n\to\infty} \frac{k_n}{r_n} < 1.$$

First, note that $\frac{r_n - k_n}{r_n} = 1 - \frac{k_n}{r_n}$, so

$$0 < \liminf_{n\to\infty} \frac{r_n - k_n}{r_n} \leq \limsup_{n\to\infty} \frac{r_n - k_n}{r_n} < 1$$

which means that

$$1 < \liminf_{n\to\infty} \frac{r_n}{r_n - k_n} \leq \limsup_{n\to\infty} \frac{r_n}{r_n - k_n} < \infty.$$

Hence,

$$\frac{\ell_n}{r_n^{(k_n)}} = \frac{\ell_n}{r_n - k_n} = \frac{\ell_n}{r_n}\frac{r_n}{r_n - k_n} \to 0$$

since $\frac{\ell_n}{r_n} \to 0$ as $n \to \infty$.

Then, as in our proof that generalized staircases are power uniform ergodic,

$$\begin{aligned}
&\int |\frac{1}{r_n^{(k_n)}} \sum_{i=0}^{r_n^{(k_n)}-1} \chi_B \circ T^{-s^{(k_n)}_{n,i}} - \mu(B)|^2 d\mu \\
&\leq \int |(\frac{r_n^{(k_n)}}{\ell_n})^{-1} \sum_{i=0}^{\lceil \frac{r_n^{(k_n)}}{\ell_n} \rceil - 1} \frac{1}{\ell_n} \sum_{j=0}^{\ell_n - 1} \chi_B \circ T^{-s^{(k_n)}_{n,i\ell_n+j}} - \mu(B)|^2 d\mu \\
&\qquad + \int |\frac{1}{r_n^{(k_n)}} \sum_{i=\lceil \frac{r_n^{(k_n)}}{\ell_n} \rceil}^{r_n^{(k_n)}-1} \chi_B \circ T^{-s^{(k_n)}_{n,i}} - \mu(B)|^2 d\mu \\
&\leq \frac{1}{\ell_n} \sum_{j=0}^{\ell_n-1} \int |(\frac{r_n^{(k_n)}}{\ell_n})^{-1} \sum_{i=0}^{\lceil \frac{r_n^{(k_n)}}{\ell_n} \rceil - 1} \chi_B \circ T^{-s^{(k_n)}_{n,j} - ik} - \mu(B)|^2 d\mu + \frac{\ell_n}{r_n^{(k_n)}} \mu(B) \\
&= \frac{1}{\ell_n} \sum_{j=0}^{\ell_n-1} \int |(\frac{r_n^{(k_n)}}{\ell_n})^{-1} \sum_{i=0}^{\lceil \frac{r_n^{(k_n)}}{\ell_n} \rceil - 1} \chi_B \circ T^{-ik_n} - \mu(B)|^2 d\mu + \frac{\ell_n}{r_n^{(k_n)}} \mu(B) \\
&= \int |(\frac{r_n^{(k_n)}}{\ell_n})^{-1} \sum_{i=0}^{\lceil \frac{r_n^{(k_n)}}{\ell_n} \rceil - 1} \chi_B \circ T^{-ik_n} - \mu(B)|^2 d\mu + \frac{\ell_n}{r_n^{(k_n)}} \mu(B).
\end{aligned}$$

Hence, since T is power uniform ergodic and since $\frac{\ell_n}{r_n^{(k_n)}} \to 0$ as $n \to \infty$, the dynamical sequence $\{s_{n,i}\}_{\{r_n\}}$ is uniformly ergodic with respect to T. □

We are now in the position to show mixing.

Theorem. *Every restricted growth generalized staircase transformation is a mixing transformation.*

Proof. Let T be a restricted growth generalized staircase transformation. Then, by Proposition 1.1, T is a power uniform ergodic transformation. By Proposition 1.2, the spacer sequence for T is then uniformly ergodic with respect to T. Hence, by the Main Theorem, T is a mixing transformation. □