0.1. Total Ergodicity on Rank One Transformations. This an alternate proof to the same theorem found in the paper. It relies on spectral theory and eigenvalue criteria rather than measure theory and distribution arguments.

Theorem 1. *Let T be a rank one transformation with (possibly normalized) nonpathological spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ such that for all fixed positive integers $L > 1$,*

$$\limsup_{n\to\infty} \frac{1}{r_n} \#\{j \in \mathbb{Z}_{r_n-1} : L \text{ divides } s_{n,j+1} - s_{n,j}\} < 1.$$

Then T is a totally ergodic transformation.

Lemma 0.1. *Let $\{s_{n,j}\}_{\{r_n\}}$ be a dynamical sequence of integers and let L be an integer greater than 1. Then*

$$\limsup_{n\to\infty} \Big|\frac{1}{r_n} \sum_{j=0}^{r_n-1} e^{2\pi i \frac{1}{L} s_{n,j}}\Big| = 1$$

if and only if there exists $b \in \mathbb{Z}_L$ such that

$$\limsup_{n\to\infty} \frac{1}{r_n} \#\{j \in \mathbb{Z}_{r_n} : s_{n,j} \mod L = b\} = 1.$$

Proof. Let $\{s_{n,j}\}_{\{r_n\}}$ and L be as above. Define the dynamical sequence of integers $\{a_{n,j}\}_{\{r_n\}}$ by $a_{n,j} = s_{n,j} \mod L$ for all positive integers n and all $j \in \mathbb{Z}_{r_n}$. Note that for all such n and j,

$$e^{2\pi i \frac{1}{L} s_{n,j}} = e^{2\pi i \frac{1}{L}(\lfloor \frac{s_{n,j}}{L} \rfloor + a_{n,j})} = e^{2\pi i \lfloor \frac{s_{n,j}}{L} \rfloor} e^{2\pi i \frac{1}{L} a_{n,j}} = e^{2\pi i \frac{1}{L} a_{n,j}}.$$

Consider the sets of integers $B_{n,b}$ for all positive integers n and all $b \in \mathbb{Z}_L$ and their densities $D_{n,b}$ given by

$$B_{n,b} = \{j \in \mathbb{Z}_{r_n} : a_{n,j} = b\} \quad \text{and} \quad D_{n,b} = \frac{1}{r_n} \# B_{n,b}.$$

Assume that there does not exist $b \in \mathbb{Z}_L$ such that $D_{n,b} \to 1$ along some subsequence of $n \to \infty$. Then there exists $\delta > 0$ such that for each sufficiently large positive integer n there exists distinct $c_n, d_n \in \mathbb{Z}_L$ such that $D_{n,c_n} \geq \delta$ and $D_{n,d_n} \geq \delta$. Hence, for each sufficiently large positive integer n, using the triangle inequality and that $cos\frac{2\pi a}{L} \leq cos\frac{2\pi}{L}$ for all $a \in \mathbb{Z}_L$, $a \neq 0$, we have

$$\begin{aligned}
\Big|\frac{1}{r_n} \sum_{j=0}^{r_n-1} e^{2\pi i \frac{1}{L} s_{n,j}}\Big| &= \Big|\sum_{b=0}^{L-1} D_{n,b} e^{2\pi i \frac{b}{L}}\Big| \\
&\leq \Big|D_{n,c_n} e^{2\pi i \frac{c_n}{L}} + D_{n,d_n} e^{2\pi i \frac{d_n}{L}}\Big| + (1 - D_{n,c_n} - D_{n,d_n}) \\
&= \big(D_{n,c_n}^2 + D_{n,d_n}^2 + 2 D_{n,c_n} D_{n,d_n} cos \frac{2\pi |c_n - d_n|}{L}\big)^{\frac{1}{2}} + (1 - D_{n,c_n} - D_{n,d_n}) \\
&\leq \big((D_{n,c_c} + D_{n,d_n})^2 - 2 D_{n,c_c} D_{n,d_n} (1 - cos\frac{2\pi}{L})\big)^{\frac{1}{2}} + (1 - D_{n,c_c} - D_{n,d_n}).
\end{aligned}$$

For any real numbers α and β such that $0 < \beta < \alpha \leq 1$, we see that $(\alpha - \beta)^{\frac{1}{2}} \leq \alpha^{\frac{1}{2}} - \frac{\beta}{2}$ (details are left to the reader). Thus, continuing from above, for each positive integer n,

$$\begin{aligned}
&\Big|\frac{1}{r_n}\sum_{j=0}^{r_n-1} e^{2\pi i \frac{1}{L} s_{n,j}}\Big| \\
&\qquad \leq (D_{n,c_n} + D_{n,d_n}) - D_{n,c_n} D_{n,d_n}(1 - cos\frac{2\pi}{L}) + (1 - D_{n,c_n} - D_{n,d_n}) \\
&\qquad = 1 - D_{n,c_n} D_{n,d_n}(1 - cos\frac{2\pi}{L}) \leq 1 - \delta^2(1 - cos\frac{2\pi}{L}).
\end{aligned}$$

Since $\delta > 0$ and L is fixed,

$$\limsup_{n\to\infty} \Big|\frac{1}{r_n}\sum_{j=0}^{r_n-1} e^{2\pi i \frac{1}{L} s_{n,j}}\Big| \leq 1 - \delta^2(1 - cos\frac{2\pi}{L}) < 1.$$

For the converse, assume there exists $b \in \mathbb{Z}_L$ and a strictly increasing sequence of positive integers $\{a_n\}$ such that for any $\epsilon > 0$ there exists a positive integer N such that for all integers $n \geq N$, $\frac{1}{r_{a_n}}\#\{j \in \mathbb{Z}_{r_{a_n}} : s_{a_n,j} \mod L = b\} > 1 - \epsilon$. Then, for each $n \geq N$,

$$\Big|\frac{1}{r_{a_n}}\sum_{j=0}^{r_{a_n}-1} e^{2\pi i \frac{1}{L} s_{a_n,j}} - e^{2\pi i \frac{b}{L}}\Big| \leq \frac{1}{r_{a_n}}\#\{j \in \mathbb{Z}_{r_{a_n}} : s_{a_n,j} \mod L \neq b\} < \epsilon.$$

Thus, $\frac{1}{r_{a_n}}\sum_{j=0}^{r_{a_n}-1} e^{2\pi i \frac{1}{L} s_{a_n,j}} \to e^{2\pi i \frac{b}{L}}$ as $n \to \infty$ so $\Big|\frac{1}{r_{a_n}}\sum_{j=0}^{r_{a_n}-1} e^{2\pi i \frac{1}{L} s_{a_n,j}}\Big| \to \Big|e^{2\pi i \frac{b}{L}}\Big| = 1$ as $n \to \infty$. ☐

Proof. (of Theorem 1) Let T and $\{s_{n,j}\}_{\{r_n\}}$ be as above and let $\{h_n\}$ be the height sequence for T. Suppose that T is not totally ergodic. Then there exists an integer $L > 1$ such that $e^{2\pi i \frac{1}{L}}$ is an eigenvalue of T. A theorem in [Na98] (section 16.13) tells us that a complex number z is an eigenvalue of T if and only if

$$\sum_{n=1}^{\infty}\Big(1 - \Big|\frac{1}{r_n}\sum_{j=0}^{r_n-1} z^{jh_n + \sum_{a=0}^{j} s_{n,a}}\Big|^2\Big) < \infty.$$

Hence, using the notation for partial sums of dynamical sequences,

$$\lim_{n\to\infty} \Big|\frac{1}{r_n}\sum_{j=0}^{r_n-1} e^{2\pi i \frac{1}{L}(jh_n + s_{n,a}^{(j+1)})}\Big| = 1.$$

Applying Lemma 0.1, we have that there exists $b \in \mathbb{Z}_L$ such that

$$\limsup_{n\to\infty} \frac{1}{r_n}\#\{j \in \mathbb{Z}_{r_n} : (jh_n + s_{n,a}^{(j+1)}) \mod L = b\} = 1.$$

Observe that for each positive integer n,

$$\begin{aligned}&\frac{1}{r_n}\#\{j \in \mathbb{Z}_{r_n-1} : \big(jh_n + s_{n,a}^{(j+1)}\big) \text{ and } \big((j+1)h_n + s_{n,a}^{(j+2)}\big) \mod L = b\} \\ &\qquad \geq \frac{r_n - 1}{r_n} - \frac{2}{r_n}\#\{j \in \mathbb{Z}_{r_n} : \big(jh_n + s_{n,a}^{(j+1)}\big) \mod L \neq b\}\end{aligned}$$

which approaches 1 along some subsequence as $n \to \infty$ since $r_n \to \infty$ as $n \to \infty$ and $\frac{1}{r_n}\#\{j \in \mathbb{Z}_{r_n} : \big(jh_n + s_{n,a}^{(j+1)}\big) \mod L \neq b\} \to 0$ along some subsequence as $n \to \infty$. For any $j \in \mathbb{Z}_{r_n-1}$, if $\big(jh_n + s_{n,a}^{(j+1)}\big)$ and $\big((j+1)h_n + s_{n,a}^{(j+2)}\big) \mod L = b$ then $(h_n + s_{n,j+1}) \mod L = 0$ since $\big((j+1)h_n + s_{n,a}^{(j+2)}\big) - \big(jh_n + s_{n,a}^{(j+1)}\big) = h_n + s_{n,j+1}$. Thus, letting $\mathbb{Z}_N^+$ denote the set of positive integers less than N,

$$\limsup_{n\to\infty} \frac{1}{r_n}\#\{j \in \mathbb{Z}_{r_n}^+ : (h_n + s_{n,j}) \mod L = 0\} = 1.$$

Applying the same argument again using that $(h_n + s_{n,j+1}) - (h_n + s_{n,j}) = s_{n,j+1} - s_{n,j}$ for all positive integers n and all $j \in \mathbb{Z}_{r_n-1}$,

$$\limsup_{n\to\infty} \frac{1}{r_n}\#\{j \in \mathbb{Z}_{r_n-1}^+ : (s_{n,j+1} - s_{n,j}) \mod L = 0\} = 1.$$

Since $r_n \to \infty$ as $n \to \infty$, this contradicts our hypothesis. □

References

[Na98] M. Nadkarni, **Spectral Theory of Dynamical Systems**, Birkhäuser (1998).