

Measure-Theoretically Mixing Subshifts of Minimal Word Complexity

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Abstract We resolve a long-standing open question on the relationship between measure-theoretic dynamical complexity and symbolic complexity by establishing the exact word complexity at which measure-theoretic strong mixing manifests:

For every superlinear $f : \mathbb{N} \rightarrow \mathbb{N}$, i.e. $f(q)/q \rightarrow \infty$, there exists a subshift admitting a (strongly) mixing of all orders probability measure with word complexity p such that $p(q)/f(q) \rightarrow 0$.

For a subshift with word complexity p which is non-superlinear, i.e. $\liminf p(q)/q < \infty$, every ergodic probability measure is partially rigid.

Introduction

Among measure-theoretic dynamical properties of measure-preserving transformations, strong mixing of all orders is the ‘most complex’: every finite collection of measurable sets tends asymptotically toward independence, necessarily implying a significant amount of randomness. Despite this, ‘low complexity’ mixing transformations exist—there are mixing transformation with zero entropy—raising the question of how deterministic a mixing transformation can be.

Word complexity, the number $p(q)$ of distinct words of length q appearing in the language of the subshift, provides a more fine-grained means of quantifying complexity in the zero entropy setting, leading to the question of how low the word complexity of a mixing transformation can be.

Ferenczi [Fer95] initially conjectured that mixing transformations’ word complexity should be superpolynomial but quickly refuted this himself [Fer96] showing that the staircase transformation, proven mixing by Adams [Ada98], has quadratic word complexity. Recent joint work of the author and R. Pavlov and S. Rodock [CPR22] exhibited subshifts admitting mixing measures with word complexity functions which are subquadratic but superlinear by more than a logarithm. We exhibit subshifts admitting mixing measures with complexity arbitrarily close to linear:

Theorem A. For every $f : \mathbb{N} \rightarrow \mathbb{N}$ which is superlinear, $f(q)/q \rightarrow \infty$, there exists a subshift, admitting a strongly mixing probability measure, with word complexity p such that $p(q)/f(q) \rightarrow 0$.

Our examples, which we call quasi-staircase transformations, are mixing rank-one transformations hence mixing of all orders [Kal84], [Ryz93]. We establish their word complexity is optimal:

Theorem B. Every subshift of non-superlinear word complexity, $\liminf p(q)/q < \infty$, equipped with an ergodic probability measure is partially rigid hence not strongly mixing,

Non-superlinear complexity subshifts are conjugate to S -adic shifts (Donoso, Durand, Maass and Petite [DDMP21]). Named by Vershik and the subject of a well-known conjecture of Host, S -adic subshifts are quite structured (see e.g. [Ler12] for more information on S -adicity).

Our work may be viewed as saying there is a sharp divide in ‘measure-theoretic complexity’, precisely at superlinear word complexity, between highly structured and highly complicated: as soon as the word complexity is ‘large enough’ to escape the S -adic structure and partial rigidity, there is already ‘enough room’ for (strong) mixing of all orders.

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Cyr and Kra established that superlinear complexity is the dividing line for a subshift admitting countably many ergodic measures: there exists subshifts with complexity arbitrarily close to linear which admit uncountably many ergodic measures [CK20b] and non-superlinear complexity implies at most countably many [CK19], [Bos85]. Our work implies that in the non-superlinear case, the at most countably measures are all partially rigid (with a uniform rigidity constant). Their result, like ours, indicates that superlinear word complexity is the line at which complicated measure-theoretic phenomena can manifest.

Beyond the structure imposed by S -adicity, linear complexity subshifts are known to be structured in various ways (e.g. [CFPZ19], [CK20a], [DDMP16], [DOP21], [PS22a], [PS22b]). Our work indicates there is no hope for similar phenomena in any superlinear setting.

1 Definitions and preliminaries

1.1 Symbolic dynamics

Definition 1.1. A **subshift** on the finite set \mathcal{A} is any subset $X \subset \mathcal{A}^{\mathbb{Z}}$ which is closed in the product topology and shift-invariant: for all $x = (x_n)_{n \in \mathbb{Z}} \in X$ and $k \in \mathbb{Z}$, the translate $(x_{n+k})_{n \in \mathbb{Z}}$ of x by k is also in X .

Definition 1.2. A **word** is any element of \mathcal{A}^ℓ for some ℓ , the **length** of w , written $\|w\|$. A word w is a **subword** of a word or biinfinite sequence x if there exists k so that $w_i = x_{i+k}$ for all $1 \leq i \leq \|w\|$. A word u is a **prefix** of w when $u_i = w_i$ for $1 \leq i \leq \|u\|$ and a word v is a **suffix** of w when $v_i = w_{i+\|w\|-\|v\|}$ for $1 \leq i \leq \|v\|$.

For words v, w , we denote by vw their concatenation—the word obtained by following v immediately by w . We write such concatenations with product or exponential notation, e.g. $\prod_i w_i$ or 0^n .

Definition 1.3. The **language** of a subshift X is $\mathcal{L}(X) = \{w : w \text{ is a subword of some } x \in X\}$.

Definition 1.4. The **word complexity function** of a subshift X over \mathcal{A} is the function $p_X : \mathbb{N} \rightarrow \mathbb{N}$ defined by $p_X(q) = |\mathcal{L}(X) \cap \mathcal{A}^q|$, the number of words of length q in the language of X .

When X is clear from context, we suppress the subscript and just write $p(n)$.

For subshifts on the alphabet $\{0, 1\}$, we consider:

Definition 1.5. The set of **right-special** words is $\mathcal{L}^{RS}(X) = \{w \in X : w0, w1 \in \mathcal{L}(X)\}$.

Cassaigne [Cas97] showed the well-known: $p(q) = p(m) + \sum_{\ell=m}^{q-1} |\{w \in \mathcal{L}^{RS} : \|w\| = \ell\}|$ for $m < q$.

1.2 Ergodic theory

Definition 1.6. A **transformation** T is a measurable map on a standard Borel or Lebesgue measure space (Y, \mathcal{B}, μ) that is measure-preserving: $\mu(T^{-1}B) = \mu(B)$ for all $B \in \mathcal{B}$.

Definition 1.7. Two transformations T on (Y, \mathcal{B}, μ) and T' on (Y', \mathcal{B}', μ') are **measure-theoretically isomorphic** when there exists a bijective map ϕ between full measure subsets $Y_0 \subset Y$ and $Y'_0 \subset Y'$ where $\mu(\phi^{-1}A) = \mu'(A)$ for all measurable $A \subset Y'_0$ and $(\phi \circ T)(y) = (T' \circ \phi)(y)$ for all $y \in Y_0$.

Definition 1.8. A transformation T is **ergodic** when $A = T^{-1}A$ implies that $\mu(A) = 0$ or $\mu(A^c) = 0$.

Theorem 1.9 (Mean Ergodic Theorem). *If T is ergodic and on a finite measure space and $f \in L^2(Y)$,*

$$\lim_{n \rightarrow \infty} \int \left| \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^{-i} - \int f \, d\mu \right| d\mu = 0$$

Definition 1.10. A transformation T is **mixing** when for all $A, B \in \mathcal{B}$, $\mu(T^n A \cap B) \rightarrow \mu(A)\mu(B)$.

1.3 Rank-one transformations

A **rank-one transformation** is a transformation T constructed by “cutting and stacking”. Here Y represents a (possibly infinite) interval, \mathcal{B} is the induced σ -algebra from \mathbb{R} , and μ is Lebesgue measure. We give a brief description, referring the reader to [FGH⁺21] or [Sil08] for more details.

The transformation is defined inductively on larger and larger portions of the space through Rohlin towers or **columns**, denoted C_n . Each column C_n consists of **levels** $I_{n,j}$ where $0 \leq j < h_n$ is the height of the level within the column. All levels $I_{n,j}$ in C_n are intervals with the same length, $\mu(I_n)$, and the total number of levels in a column is the **height** of the column, denoted by h_n . The transformation T is defined on all levels $I_{n,j}$ except the top one I_{n,h_n-1} by sending each $I_{n,j}$ to $I_{n,j+1}$ using the unique order-preserving affine map.

Start with $C_1 = [0, 1)$ with height $h_1 = 1$. To obtain C_{n+1} from C_n , we require a **cut sequence**, $\{r_n\}$ such that $r_n \geq 1$ for all n . Make r_n vertical cuts of C_n to create $r_n + 1$ **subcolumns** of equal width. Denote a **sublevel** of C_n by $I_{n,j}^{[i]}$ where $0 \leq i < h_n$ is the height of the level within that column, and i represents the position of the subcolumn, where $i = 0$ represents the leftmost subcolumn and $i = r_n$ is the rightmost subcolumn. After cutting C_n into subcolumns, add extra intervals called **spacers** on top of each subcolumn to function as levels of the next column. The **spacer sequence**, $\{s_{n,i}\}$ such that $0 \leq i \leq r_n$ and $s_{n,i} \geq 0$, specifies how many sublevels to add above each subcolumn. Spacers are the same width as the sublevels, act as new levels in the column C_{n+1} , and are taken to be the leftmost intervals in $[1, \infty)$ not in C_n . After the spacers are added, stack the subcolumns with their spacers right on top of left, i.e. so that $I_{n,0}^{[i+1]}$ is directly above $I_{n,h_n-1}^{[i]}$. This gives the next column, C_{n+1} .

Each column C_n defines T on $\bigcup_{j=0}^{h_n-2} I_{n,j}$ and the partially defined map T on C_{n+1} agrees with that of C_n , extending the definition of T to a portion of the top level of C_n where it was previously undefined. Continuing this process gives the sequence of columns $\{C_1, \dots, C_n, C_{n+1}, \dots\}$ and T is then the limit of the partially defined maps.

Though this construction could result in Y being an infinite interval with infinite Lebesgue measure, Y has finite measure if and only if $\sum_n \frac{1}{r_n h_n} \sum_{i=0}^{r_n} s_{n,i} < \infty$, see [CS10]. All rank-one transformations we define satisfy this condition, and for convenience we renormalize so that $Y = [0, 1)$. Every rank-one transformation is ergodic and invertible.

The reader should be aware that we are making r_n cuts and obtaining $r_n + 1$ subcolumns (following Ferenczi [Fer96]), while other papers (e.g. [Cre21]) use r_n as the number of subcolumns.

1.4 Symbolic models of rank-one transformations

For a rank-one transformation defined as above, we define a subshift $X(T)$ on the alphabet $\{0, 1\}$ which is measure-theoretically isomorphic to T :

Definition 1.11. The **symbolic model** $X(T)$ of a rank-one transformation T is given by the sequence of words: $B_1 = 0$ and

$$B_{n+1} = B_n 1^{s_{n,0}} B_n 1^{s_{n,1}} \dots B_n 1^{s_{n,r_n}} = \prod_{i=0}^{r_n} B_n 1^{s_{n,i}}$$

and $X(T)$ is the set of all biinfinite sequences such that every subword is a subword of some B_n .

The words B_n are a symbolic coding of the column C_n : 0 represents C_1 and 1 represents the spacers. There is a natural measure associated to $X(T)$:

Definition 1.12. The **empirical measure** for a symbolic model $X(T)$ of a rank-one transformation T is the measure ν defined by, for each word w ,

$$\nu([w]) = \lim_{n \rightarrow \infty} \frac{|\{1 \leq j \leq \|B_n\| - \|w\| : B_n[j, j+\|w\|] = w\}|}{\|B_n\| - \|w\|}$$

Danilenko [Dan16] (combined with [dJ77] and [Kal84]) proved that the symbolic model $X(T)$ of a rank-

one subshift, equipped with its empirical measure, is measure-theoretically isomorphic to the cut-and-stack construction (see [AFP17]; see [FGH⁺21] for the full generality including odometers).

Due to this isomorphism, we move back and forth between rank-one and symbolic model terminology as needed and write $\mathcal{L}(T)$ for the language of $X(T)$.

2 Quasi-staircase transformations

Definition 2.1. Given nondecreasing sequences of integers $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ tending to infinity such that $c_1 \geq 1$ and $c_{n+1} \geq c_n + b_n$, a **quasi-staircase transformation** is a rank-one transformation with cut sequence $r_n = a_n b_n$ and spacer sequence $s_{n,t} = c_n + \lfloor \frac{t}{a_n} \rfloor$ for $0 \leq t < r_n$ and $s_{n,r_n} = 0$.

The symbolic representation of a quasi-staircase is $B_1 = 0$ and

$$B_{n+1} = \left(\prod_{i=0}^{b_n-1} (B_n 1^{c_n+i})^{a_n} \right) B_n$$

The height sequence of a quasi-staircase is $h_1 = 1$ and $h_{n+1} = (a_n b_n + 1)h_n + a_n b_n c_n + \frac{1}{2} a_n b_n (b_n - 1)$.

2.1 Quasi-staircase right-special words

Lemma 2.2. *Let $01^z 0 \in \mathcal{L}(T)$. Then there are unique n and i with $0 \leq i < b_n$ such that $z = c_n + i$. $01^{c_n+i} 0$ is not a subword of B_m for $m \leq n$ and every occurrence of $01^{c_n+i} 0$ is as a suffix of $1^{c_{n+1}} (\prod_{j=0}^{i-1} (B_n 1^{c_n+j})^{a_n}) (B_n 1^{c_n+i})^q 0$ for some $1 \leq q \leq a_n$ (adopting the convention that \prod_0^{-1} is the empty word).*

Proof. As every B_n begins and ends with 0, the only such words are of the form $01^{c_n+i} 0$. Since $c_{n+1} \geq c_n + b_n$, such n and i are unique. This also gives that 1^{c_n} is not a subword of B_n .

The word $01^{c_n+i} 0$ only occurs inside B_{n+1} due to $c_{n+1} \geq c_n + b_n$, and only as part of the $(B_n 1^{c_n+i})^{a_n}$ in its construction, and B_{n+1} is always preceded by $1^{c_{n+1}}$ \square

Proposition 2.3. *If $w \in \mathcal{L}^{RS}(T)$ then at least one of the following holds:*

- (i) $w = 1^{\|w\|}$
- (ii) w is a suffix of $1^{c_n+i-1} (B_n 1^{c_n+i})^{a_n}$ for some n and $0 \leq i < b_n$
- (iii) w is a suffix of $1^{c_n+b_n-1} B_n 1^{c_n}$ for some n
- (iv) $w = 1^{c_n} (B_n 1^{c_n})^{a_n}$

Proof. Let $w \in \mathcal{L}^{RS}(T)$. Since $c_1 \geq 1$, the word $00 \notin \mathcal{L}(T)$ so w does not end in 0. If $w = 1^{\|w\|}$ then w is of form (i) so from here on, assume that w contains at least one 0.

Let $z \geq 1$ such that w has 01^z as a suffix. Then $w0$ has $01^z 0$ as a suffix so $z = c_n + i$ for some unique $n \geq 1$ and $0 \leq i < b_n$ by Lemma 2.2. As $w0$ has $01^{c_n+i} 0$ as a suffix, $w0$ shares a suffix with the word $1^{c_{n+1}} (\prod_{j=0}^{i-1} (B_n 1^{c_n+j})^{a_n}) (B_n 1^{c_n+i})^q 0$ for some $1 \leq q \leq a_n$.

First consider the case when $i > 0$. If w is a suffix of $1^{c_n+i-1} (B_n 1^{c_n+i})^{a_n}$ then it is of form (ii) so we need only consider w that have $01^{c_n+i-1} (B_n 1^{c_n+i})^q$ as a suffix. For such w , the word $w1$ has the suffix $01^{c_n+i-1} (B_n 1^{c_n+i})^{q-1} B_n 1^{c_n+i+1}$ but that word is only in $\mathcal{L}(T)$ if $q-1 = a_n$ which is impossible.

Now consider the case when $i = 0$, i.e. $z = c_n$. If w is a suffix of $1^{c_n-1} (B_n 1^{c_n})^{a_n}$ then it is of form (ii) so we may assume that w has $1^{c_n-1} (B_n 1^{c_n})^q$ as a strict suffix for some $1 \leq q \leq a_n$. Since $B_n 1^{c_n}$ is always preceded by 1^{c_n} (possibly as part of some $1^{c_{n+1}+i}$ or 1^{c_n+i}), w cannot have $01^{c_n-1} B_n 1^{c_n}$ as a subword so w has $1^{c_n} (B_n 1^{c_n})^q$ as a suffix for some $1 \leq q \leq a_n$.

Take q maximal so that w has $1^{c_n} (B_n 1^{c_n})^q$ as a suffix.

Consider first when w has $1^{c_n}(B_n 1^{c_n})^{a_n}$ as a suffix, i.e. when $q = a_n$. If $w = 1^{c_n}(B_n 1^{c_n})^{a_n}$ then it is of form (iv). If w has $01^{c_n}(B_n 1^{c_n})^{a_n}$ as a suffix then $w0 \notin \mathcal{L}(T)$ as $0(1^{c_n} B_n)^{a_n} 1^{c_n} 0 \notin \mathcal{L}(T)$. If w has $11^{c_n}(B_n 1^{c_n})^{a_n}$ as a suffix then $w1$ has $1^{c_n+1}(B_n 1^{c_n})^{a_n-1} B_n 1^{c_n+1}$ as a suffix but that is not in $\mathcal{L}(T)$.

So we may assume $q < a_n$. Since $1^{c_n}(B_n 1^{c_n})^q$ is then of form (ii), we may assume $1^{c_n}(B_n 1^{c_n})^q$ is a strict suffix of w .

Consider when w has $01^{c_n}(B_n 1^{c_n})^q$ as a suffix. As $01^{c_n}(B_n 1^{c_n})^q$ only appears as a suffix of $B_n 1^{c_n}(B_n 1^{c_n})^q$ and that word is always preceded by 1^{c_n} (possibly as part of some 1^{c_n+1+i}), w then shares a suffix with $1^{c_n}(B_n 1^{c_n})^{q+1}$. As q is maximal, then w is a suffix of $1^{c_n-1}(B_n 1^{c_n})^{q+1}$ and, as $q < a_n$, this means w is of form (ii).

We are left with the case when w has $1^{c_n+1}(B_n 1^{c_n})^q$ as a suffix for some $1 \leq q < a_n$. If $q \geq 2$ then $w1$ has $1^{c_n+1}(B_n 1^{c_n})^{q-1} B_n 1^{c_n+1}$ as a suffix but that is not in $\mathcal{L}(T)$ for $q-1 \geq 1$. So we are left with the situation when w shares a suffix with $1^{c_n+1} B_n 1^{c_n}$. So $w0$ shares a suffix with $1^{c_n+1} B_n 1^{c_n} 0$ which must share a suffix with $1^{c_n+1} B_n 1^{c_n} 0$, meaning that w shares a suffix with $1^{c_n+1} B_n 1^{c_n}$. If w is a suffix of $1^{c_n+b_n-1} B_n 1^{c_n}$ then it is of form (iii). If not then w has the suffix $1^{c_n+b_n} B_n 1^{c_n}$ so $w1$ has suffix $1^{c_n+b_n} B_n 1^{c_n+1}$ which is not in $\mathcal{L}(T)$ since $B_n 1^{c_n+1}$ is always preceded by $B_n 1^{c_n}$ or $B_n 1^{c_n+1}$. \square

Lemma 2.4. $1^\ell \in \mathcal{L}^{RS}(T)$ for all ℓ .

Proof. For n such that $\ell < c_n$, as the word $1^{c_n} B_n$ is a subword of B_{n+1} , so are $1^{\ell+1}$ and $1^\ell 0$ since $\ell < c_n$ and B_n starts with 0. \square

Lemma 2.5. If w is a suffix of $1^{c_n}(B_n 1^{c_n})^{a_n}$ then $w \in \mathcal{L}^{RS}(T)$.

Proof. B_{n+2} has $1^{c_n+1} B_{n+1} = 1^{c_n+1-c_n} 1^{c_n} B_{n+1}$ as a subword which has $1^{c_n}(B_n 1^{c_n})^{a_n} B_n$ as a subword which gives $1^{c_n}(B_n 1^{c_n})^{a_n} 0 \in \mathcal{L}(T)$. B_{n+1} has $(B_n 1^{c_n})^{a_n} B_n 1^{c_n+1}$ as a prefix which has suffix $1^{c_n}(B_n 1^{c_n})^{a_n-1} B_n 1^{c_n+1}$ and that word is $1^{c_n}(B_n 1^{c_n})^{a_n} 1$ giving $1^{c_n-1}(B_n 1^{c_n})^{a_n} 1 \in \mathcal{L}(T)$. \square

Lemma 2.6. If w is a suffix of $1^{c_n+i-1}(B_n 1^{c_n+i})^{a_n}$ for $0 < i < b_n$ then $w \in \mathcal{L}^{RS}(T)$.

Proof. B_{n+1} has $1^{c_n+i-1}(B_n 1^{c_n+i})^{a_n} B_n$ as a subword which gives $1^{c_n+i-1}(B_n 1^{c_n+i})^{a_n} 0 \in \mathcal{L}(T)$. When $i < b_n - 1$, B_{n+1} has $(1^{c_n+i} B_n)^{a_n} 1^{c_n+i+1}$ as a subword which gives $1^{c_n+i-1}(B_n 1^{c_n+i})^{a_n} 1 \in \mathcal{L}(T)$; when $i = b_n - 1$, B_{n+2} has the subword $(1^{c_n+b_n-1} B_n)^{a_n} 1^{c_n+1}$ so $1^{c_n+b_n-2}(B_n 1^{c_n+b_n-1})^{a_n} 1^{c_n+1-c_n-b_n+1} \in \mathcal{L}(T)$ so $1^{c_n+b_n-2}(B_n 1^{c_n+b_n-1})^{a_n} 1 \in \mathcal{L}(T)$ as $c_{n+1} \geq c_n + b_n$. \square

Lemma 2.7. If w is a suffix of $1^{c_n+b_n-1} B_n 1^{c_n}$ then $w \in \mathcal{L}^{RS}(T)$.

Proof. B_{n+2} has $B_{n+1} 1^{c_n+1} B_{n+1}$ as a subword which has $B_{n+1} 1^{c_n+1} B_n 1^{c_n} B_n$ as a prefix, and that word has $1^{c_n+b_n-1} B_n 1^{c_n} 0$ as a subword since $c_n + b_n - 1 < c_{n+1}$. Also B_{n+2} has $B_{n+1} 1^{c_n+1}$ as a subword which has $1^{c_n+b_n-1} B_n 1^{c_n+1}$ as a suffix which then has $1^{c_n+b_n-1} B_n 1^{c_n} 1$ as a subword. \square

2.2 The level- n complexity functions

Definition 2.8. For a word w , define the **tail length** $z(w)$ such that $w = u01^{z(w)}$ for some (possibly empty) word u with the conventions that $z(1^{\|w\|}) = \infty$ and $z(u0) = 0$.

Definition 2.9. For $1 \leq n < \infty$, the set of **level- n generating words** is

$$W_n = \{w \in \mathcal{L}^{RS}(T) : c_n \leq z(w) < c_{n+1}\}$$

Proposition 2.10. $\mathcal{L}^{RS}(T) = \{1^\ell : \ell \in \mathbb{N}\} \sqcup \bigsqcup_{n=1}^{\infty} W_n$.

Proof. $\{c_n\}$ is strictly increasing so the W_n are disjoint. Lemma 2.4 says $1^\ell \in \mathcal{L}^{RS}(T)$ for all ℓ and as every word in W_n has 0 as a subword, these are disjoint from the W_n . If $z(w) < c_1$ then $w0 \notin \mathcal{L}(T)$ by Lemma 2.2 so all right-special words with 0 as a subword are in some W_n . \square

Definition 2.11. The level- n complexity is $p_n(q) = |\{w \in W_n : \|w\| < q\}|$.

By definition, $p_n(\ell + 1) - p_n(\ell) = |\{w \in W_n : \|w\| = \ell\}|$.

Proposition 2.12. The complexity function p satisfies $p(q) = 1 + q + \sum_{n=1}^{\infty} p_n(q)$.

Proof. Using Proposition 2.10 and that $p(\ell + 1) - p(\ell) = |\{w \in \mathcal{L}^{RS} : \|w\| = \ell\}|$,

$$\begin{aligned} p(q) - p(1) &= \sum_{\ell=1}^{q-1} (p(\ell + 1) - p(\ell)) = \sum_{\ell=1}^{q-1} |\{w \in \mathcal{L}^{RS}(T) : \|w\| = \ell\}| \\ &= \sum_{\ell=1}^{q-1} \left(\sum_{n=1}^{\infty} |\{w \in W_n : \|w\| = \ell\}| + |\{1^\ell\}| \right) = \sum_{\ell=1}^{q-1} \left(\sum_{n=1}^{\infty} (p_n(\ell + 1) - p_n(\ell)) + 1 \right) \\ &= \sum_{n=1}^{\infty} \left(\sum_{\ell=1}^{q-1} (p_n(\ell + 1) - p_n(\ell)) \right) + q - 1 = \sum_{n=1}^{\infty} (p_n(q) - p_n(1)) + q - 1 \end{aligned}$$

All words in W_n have length at least $1 + c_n > 1$ so $p_n(1) = 0$. The claim follows as $p(1) = 2$. \square

2.3 Counting quasi-staircase words

Lemma 2.13. If $w \in W_n$ then exactly one of the following holds:

- (i) w is a suffix of $1^{c_n+i-1}(B_n 1^{c_n+i})^{a_n}$ and $\|w\| > c_n + i$ for some $0 \leq i < b_n$;
- (ii) w is a suffix of $1^{c_n+b_n-1} B_n 1^{c_n}$ and $\|w\| > h_n + 2c_n$; or
- (iii) $w = 1^{c_n}(B_n 1^{c_n})^{a_n}$

Proof. The only words in Proposition 2.3 which have $c_n \leq z(w) < c_{n+1}$ are of the stated forms; Lemmas 2.5, 2.6 and 2.7 state that these words are in $\mathcal{L}^{RS}(T)$. The forms do not overlap due to the restriction on $\|w\|$ in form (ii). \square

Lemma 2.14. Fix $0 \leq i < b_n$. For $c_n + i < \ell < a_n h_n + (a_n + 1)(c_n + i)$ there is exactly one word in W_n of form (i) for that value of i ; for ℓ not in that range, there are no words of form (i) for that i in W_n .

Proof. For $w \in W_n$ of form (i), $w = u 1^{c_n+i}$ where u is a nonempty suffix of $1^{c_n+i-1}(B_n 1^{c_n+i})^{a_n-1} B_n$. The word u is unique if it exists which is exactly when $c_n + i = \|1^{c_n+i}\| < \|w\| \leq \|1^{c_n+i-1}(B_n 1^{c_n+i})^{a_n}\| = a_n h_n + (a_n + 1)(c_n + i) - 1$. \square

Lemma 2.15. For $h_n + 2c_n < \ell < h_n + 2c_n + b_n$ there is exactly one word in W_n of form (ii); for ℓ not in that range, there are no words of form (ii) in W_n .

Proof. To be of that form, $w = u 1^{c_n}$ where u is a nonempty suffix of $1^{c_n+b_n-1} B_n$ that has 1^{c_n+1} as a prefix. The word u is unique if it exists and it exists exactly when $h_n + 2c_n + 1 = \|1^{c_n+1} B_n 1^{c_n}\| \leq \|w\| \leq \|1^{c_n+b_n-1} B_n 1^{c_n}\| = h_n + 2c_n + b_n - 1$. \square

Lemma 2.16. If $\ell \leq c_n$ then $p_n(\ell + 1) - p_n(\ell) = 0$.

Proof. Every $w \in W_n$ has subwords 1^{c_n} and 0 so $\|w\| \geq c_n + 1$. \square

Lemma 2.17. If $c_n < \ell < c_n + b_n$ then $p_n(\ell + 1) - p_n(\ell) = \ell - c_n$.

Proof. Lemma 2.14 applies for $0 \leq i < \ell - c_n$ but not for $\ell - c_n \leq i < b_n$. Lemma 2.15 does not apply. \square

Lemma 2.18. If $c_n + b_n \leq \ell \leq h_n + 2c_n$ then $p_n(\ell + 1) - p_n(\ell) = b_n$.

Proof. Lemma 2.14 applies for all $0 \leq i < b_n$ and Lemma 2.15 does not apply. \square

Lemma 2.19. *If $h_n + 2c_n < \ell < h_n + 2c_n + b_n$ then $p_n(\ell + 1) - p_n(\ell) = b_n + 1$.*

Proof. Lemma 2.14 applies for all $0 \leq i < b_n$ and Lemma 2.15 applies. \square

Lemma 2.20. *If $h_n + 2c_n + b_n \leq \ell < a_n h_n + (a_n + 1)c_n$ then $p_n(\ell + 1) - p_n(\ell) = b_n$.*

Proof. Lemma 2.14 applies for all $0 \leq i < b_n$ and Lemma 2.15 does not apply. \square

Lemma 2.21. $p_n(a_n h_n + (a_n + 1)c_n + 1) - p_n(a_n h_n + (a_n + 1)c_n) = b_n + 1$.

Proof. Lemma 2.14 applies for all $0 \leq i < b_n$ and Lemma 2.15 does not apply. Lemma 2.13 form (iii) gives one additional word in W_n . \square

Lemma 2.22. *If $a_n h_n + (a_n + 1)c_n + 1 < \ell < a_n h_n + (a_n + 1)(c_n + b_n - 1)$ then $p_n(\ell + 1) - p_n(\ell) \leq b_n$.*

Proof. Lemma 2.14 applies for some subset of $0 \leq i < b_n$ and Lemma 2.15 does not apply. \square

Lemma 2.23. $p(a_n h_n + (a_n + 1)c_n + (a_n + 1)(b_n - 1)) - p(a_n h_n + (a_n + 1)c_n) = \frac{1}{2}(a_n + 1)b_n(b_n - 1) + 1$.

Proof. For each $0 \leq i < b_n$, Lemma 2.14 applies for $\ell = a_n h_n + (a_n + 1)c_n + y$ exactly when $0 \leq y < (a_n + 1)i$, therefore there are a total of $(a_n + 1)\frac{1}{2}b_n(b_n - 1)$ words in W_n of the enclosed lengths from Lemma 2.14. Lemma 2.15 does not apply and Lemma 2.13 form (iii) gives one additional word. \square

Lemma 2.24. *If $a_n h_n + (a_n + 1)(c_n + b_n - 1) \leq \ell$ then $p_n(\ell + 1) - p_n(\ell) = 0$.*

Proof. Neither Lemma 2.14 nor 2.15 apply. \square

2.4 Bounding the complexity of quasi-staircases

Since $p_n(\ell + 1) - p_n(\ell) = 0$ for $\ell \geq a_n h_n + (a_n + 1)(c_n + b_n - 1)$, we define:

Definition 2.25. The **post-productive sequence** is

$$m_n = a_n h_n + (a_n + 1)(c_n + b_n - 1)$$

Lemma 2.26. $p_n(m_n) = h_{n+1} - h_n$

Proof. By Lemma 2.16, $p_n(c_n) = \sum_{\ell=0}^{c_n-1} (p_n(\ell + 1) - p_n(\ell)) = 0$.

By Lemma 2.17, $p_n(c_n + b_n) - p_n(c_n) = \sum_{\ell=c_n}^{c_n+b_n-1} (\ell - c_n) = \frac{1}{2}b_n(b_n - 1)$.

By Lemma 2.18, $p_n(h_n + 2c_n + 1) - p_n(c_n + b_n) = (h_n + c_n + 1 - b_n)b_n$.

By Lemma 2.19, $p_n(h_n + 2c_n + b_n) - p_n(h_n + 2c_n + 1) = (b_n + 1)(b_n - 1)$.

By Lemma 2.20, $p_n(a_n h_n + (a_n + 1)c_n) - p_n(h_n + 2c_n + b_n) = ((a_n - 1)h_n + (a_n - 1)c_n - b_n)b_n$.

By Lemma 2.23, $p_n(m_n) - p(a_n h_n + (a_n + 1)c_n) = \frac{1}{2}(a_n + 1)b_n(b_n - 1) + 1$. Therefore

$$\begin{aligned} p_n(m_n) &= \frac{1}{2}b_n(b_n - 1) + (h_n + c_n + 1 - b_n)b_n + (b_n + 1)(b_n - 1) \\ &\quad + ((a_n - 1)h_n + (a_n - 1)c_n - b_n)b_n + \frac{1}{2}(a_n + 1)b_n(b_n - 1) + 1 \\ &= a_n b_n h_n + a_n b_n c_n + \frac{1}{2}a_n b_n(b_n - 1) + b_n(b_n - 1) + b_n - b_n^2 + b_n^2 - 1 - b_n^2 + 1 = h_{n+1} - h_n \quad \square \end{aligned}$$

Definition 2.27. For $q \in \mathbb{N}$ define

$$\alpha(q) = \max\{n : m_n \leq q\} \quad \text{and} \quad \beta(q) = \min\{n : q < c_{n+1}\}$$

Lemma 2.28. $\alpha(q) \leq \beta(q)$

Proof. If $\beta(q) \leq \alpha(q) - 1$ then $m_{\alpha(q)} \leq q < c_{\beta(q)+1} \leq c_{\alpha(q)-1+1} = c_{\alpha(q)} < m_{\alpha(q)}$ is impossible. \square

Lemma 2.29. If $q < c_n$ then $p_n(q) = 0$. If $c_n \leq q < m_n$ then $p_n(q) \leq (q - c_n + 1)b_n$. If $m_n \leq q$ then $p_n(q) = h_{n+1} - h_n$.

Proof. Lemma 2.16 gives $p_n(\ell + 1) - p_n(\ell) = 0$ for $0 \leq \ell < c_n$. Lemmas 2.17, 2.18, 2.19, 2.20 and 2.22 all give $p_n(\ell + 1) - p_n(\ell) \leq b_n$ for $c_n \leq \ell < m_n$ except for Lemma 2.19 which gives $p_n(\ell + 1) - p_n(\ell) = b_n + 1$ for exactly $b_n - 1$ values of ℓ and Lemma 2.21 which gives one additional word. Then, for $c_n \leq q < m_n$,

$$p_n(q) = \sum_{\ell=0}^{q-1} (p_n(\ell + 1) - p_n(\ell)) = \sum_{\ell=0}^{c_n-1} 0 + \sum_{\ell=c_n}^{q-1} (p_n(\ell + 1) - p_n(\ell)) \leq (q - c_n)b_n + b_n$$

Lemma 2.24 says $p_n(\ell + 1) - p_n(\ell) = 0$ for $\ell \geq m_n$ so when $q \geq m_n$, $p_n(q) = p_n(m_n)$ and Lemma 2.26 gives the final statement. \square

Proposition 2.30. $p(q) \leq q \left(2 + \sum_{n=\alpha(q)}^{\beta(q)} b_n \right)$ for all q .

Proof. For n such that $\beta(q) < n$, by Lemma 2.16, $p_n(q) = 0$. Proposition 2.12 and Lemma 2.29 give, using that $h_1 = 1$ so $1 + \sum_{n=1}^{\alpha(q)} (h_{n+1} - h_n) = h_{\alpha(q)+1}$,

$$\begin{aligned} p(q) &= q + 1 + \sum_{n=1}^{\alpha(q)} p_n(q) + \sum_{n=\alpha(q)+1}^{\beta(q)} p_n(q) + \sum_{n=\beta(q)+1}^{\infty} p_n(q) \\ &\leq q + 1 + \sum_{n=1}^{\alpha(q)} (h_{n+1} - h_n) + \sum_{n=\alpha(q)+1}^{\beta(q)} (q - c_n + 1)b_n + 0 \leq q + h_{\alpha(q)+1} + \sum_{n=\alpha(q)+1}^{\beta(q)} qb_n \end{aligned}$$

and
$$\begin{aligned} h_{\alpha(q)+1} &= h_{\alpha(q)} + b_{\alpha(q)}(a_{\alpha(q)}h_{\alpha(q)} + a_{\alpha(q)}c_{\alpha(q)} + \frac{1}{2}a_{\alpha(q)}(b_{\alpha(q)} - 1)) \\ &\leq h_{\alpha(q)} + b_{\alpha(q)}m_{\alpha(q)} \leq m_{\alpha(q)}(1 + b_{\alpha(q)}) \leq q(1 + b_{\alpha(q)}) \end{aligned} \quad \square$$

3 Quasi-staircase complexity arbitrarily close to linear

Proposition 3.1. Let $\{d_n\}$ be a nondecreasing sequence of integers such that $d_n \rightarrow \infty$ and $d_1 = d_2 = 1$ and $d_{n+1} - d_n \in \{0, 1\}$ and $d_{n+1} - d_n$ does not take the value 1 for consecutive n .

Let $\{b_n\}$ be a nondecreasing sequence of integers such that $b_n \rightarrow \infty$ and $b_1 = 3$ and $b_n \leq n + 2$.

Set $a_n = 2n + 2$. Set $c_1 = 1$ and for $n > 1$,

$$c_n = \begin{cases} m_n - d_n & \text{when } d_n = d_{n-1} \\ c_{n-1} + b_{n-1} & \text{when } d_n = d_{n-1} + 1 \end{cases}$$

Then $\{a_n\}, \{b_n\}, \{c_n\}$ define a quasi-staircase transformation with $\frac{a_n b_n^2}{h_n} \rightarrow 0$ and $\sum \frac{c_n + b_n}{h_n} < \infty$.

Proof. Since $r_n = a_n b_n$, we have $6n + 6 \leq r_n \leq (2n + 2)(n + 2)$. Then $\prod_{j=1}^{n-1} (r_j + 1) \geq n!$ so $h_n \geq \prod_{j=1}^{n-1} (r_j + 1) \geq n!$ so $\sum \frac{b_n}{h_n} < \infty$ and $\frac{a_n b_n^2}{h_n} \rightarrow 0$.

For n such that $d_n = d_{n-1} + 1$, we have $c_n = c_{n-1} + b_{n-1}$ and $d_{n-1} - d_{n-2} = 0$ since $\{d_n\}$ never increases for two consecutive values, so $c_{n-1} = m_{n-1-d_{n-1}}$. As $n - d_n = n - 1 - d_{n-1}$, then $c_n = m_{n-1-d_{n-1}} + b_{n-1} = m_{n-d_n} + b_{n-1}$. So $m_{n-d_n} \leq c_n \leq m_{n-d_n} + b_{n-1}$ for all n .

Since $b_n \geq 3$, we have $r_n \leq \frac{1}{2}r_n(b_n - 1)$. As $b_n \leq a_n + 1$ and $a_n + 1 \leq a_n b_n$,

$$m_n = a_n h_n + (a_n + 1)c_n + r_n + b_n - a_n - 1 \leq (a_n b_n + 1)h_n + a_n b_n c_n + \frac{1}{2}r_n(b_n - 1) = h_{n+1}$$

and therefore, since $d_n \geq 1$ so $n - d_n + 1 \leq n$,

$$c_n \leq m_{n-d_n} + b_{n-1} \leq h_{n-d_n+1} + b_{n-1} \leq h_n + b_{n-1} \leq 2h_n$$

meaning that, as $r_n + b_n \leq h_n$,

$$m_n = (a_n + 1)c_n + a_n h_n + r_n + b_n - a_n - 1 \leq 2(a_n + 1)h_n + a_n h_n + h_n \leq 3(a_n + 1)h_n$$

We now claim that $c_{n+1} \geq c_n + b_n$ for all n . The case when $c_n = m_{n-d_n}$, which occurs when $d_n = d_{n-1}$, is all we need to check. Since $d_2 = d_1 = 1$, we have $c_2 = m_1 \geq c_1 + b_1$. Since $d_n \leq \frac{n}{2}$, we have $a_{n-d_{n-1}} = 2(n - d_n - 1) + 2 \geq 2(\frac{n}{2} - 1) + 2 = n$. As $b_n \leq n + 2$ and $b_n \geq 3$,

$$\begin{aligned} c_n - c_{n-1} - b_{n-1} &\geq m_{n-d_n} - (m_{n-d_{n-1}-1} + b_{n-2}) - b_{n-1} = m_{n-d_n} - m_{n-d_{n-1}} - b_{n-2} - b_{n-1} \\ &\geq a_{n-d_n} h_{n-d_n} - 3(a_{n-d_{n-1}} + 1)h_{n-d_{n-1}} - n - (n + 1) \\ &\geq a_{n-d_n} a_{n-d_{n-1}} b_{n-d_{n-1}} h_{n-d_{n-1}} - 3(a_{n-d_{n-1}} + 1)h_{n-d_{n-1}} - 2n - 1 \\ &= (a_{n-d_n} a_{n-d_{n-1}} b_{n-d_{n-1}} - 3(a_{n-d_{n-1}} + 1))h_{n-d_{n-1}} - 2n - 1 \\ &\geq (3a_{n-d_n} a_{n-d_{n-1}} - 3a_{n-d_{n-1}} - 3)h_{n-d_{n-1}} - 2n - 1 \\ &\geq 3a_{n-d_{n-1}}(a_{n-d_n} - 2)h_{n-d_{n-1}} - 2n - 1 \geq 3n(n - 2) - 2n - 1 > 0 \end{aligned}$$

for $n \geq 3$. Then $c_{n+1} \geq c_n + b_n$ for all n so $\{c_n\}$, $\{a_n\}$, $\{b_n\}$ define a quasi-staircase transformation.

Now observe that

$$\sum_n \frac{c_n}{h_n} \leq \sum_n \frac{m_{n-d_n} + b_{n-1}}{h_n} \leq 3 \sum_n \frac{(a_{n-d_n} + 1)h_{n-d_n}}{h_n} + \sum_n \frac{b_n}{h_n}$$

and the second sum converges as shown at the start of the proof.

As $h_n \geq h_{n-d_n} \prod_{j=n-d_n}^{n-1} (r_j + 1)$,

$$\begin{aligned} \sum_n \frac{(a_{n-d_n} + 1)h_{n-d_n}}{h_n} &\leq \sum_n \frac{a_{n-d_n} + 1}{\prod_{j=n-d_n}^{n-1} (r_j + 1)} = \sum_n \frac{a_{n-d_n} + 1}{r_{a_{n-d_n}} + 1} \frac{1}{\prod_{j=n-d_n+1}^{n-1} (r_j + 1)} \\ &\leq \sum_n \frac{1}{(r_{n-d_n+1} + 1)^{d_n-1}} = \sum_n (r_{n-d_n+1} + 1)^{1-d_n} \end{aligned}$$

Since $r_{n-d_n+1} \geq 2(n - d_n + 1) \geq 2(n - n/2 + 1) \geq n$, we have $(r_{n-d_n+1} + 1)^{1-d_n} \leq n^{1-d_n}$. Then, as $d_n \geq 3$ eventually, $\sum (r_{n-d_n+1} + 1)^{1-d_n} \leq \sum n^{1-d_n} < \infty$. \square

Lemma 3.2. *If $f : \mathbb{N} \rightarrow \mathbb{N}$ is any function such that $f(q) \rightarrow \infty$ then there exists $g : \mathbb{N} \rightarrow \mathbb{N}$ which is nondecreasing such that $g(1) = 1$ and $g(q) \leq f(q)$ and $g(q+2) - g(q) \leq 1$ for all q and $g(q) \rightarrow \infty$.*

Proof. Set $f^*(q) = \inf_{q' \geq q} f(q')$. Then $f^*(q) \rightarrow \infty$ and $f^*(q)$ is nondecreasing and $f^*(q) \leq f(q)$ for all q . Set $g(1) = 1 \leq f^*(1)$. For $n \geq 0$, set $g(2n+2) = g(2n+1)$ and for $n \geq 1$ set

$$g(2n+1) = g(2n) + \begin{cases} 1 & \text{when } f^*(2n+1) > f^*(2n-1) \\ 0 & \text{otherwise} \end{cases}$$

Then g is nondecreasing and $g(q+2) - g(q) \leq 1$ for all q . Since f^* is integer-valued, if $f^*(2n+1) - f^*(2n-1) \neq 0$ then $f^*(2n+1) - f^*(2n-1) \geq 1$. Then $g(2n+1) - g(2n-1) \leq f^*(2n+1) - f^*(2n-1)$

so for all n we have

$$g(2n+1) = g(1) + \sum_{m=1}^n (g(2m+1) - g(2m-1)) \leq f^*(1) + \sum_{m=1}^n (f^*(2m+1) - f^*(2m-1)) = f^*(2n+1)$$

so, as $g(2n+2) = g(2n+1) \leq f^*(2n+1) \leq f^*(2n+2)$, we have $g(q) \leq f^*(q) \leq f(q)$ for all q . If $g(q) \leq C$ for all q then $f^*(2n+1) = f^*(2n-1)$ eventually, contradicting that $f^*(q) \rightarrow \infty$. Therefore $g(q) \rightarrow \infty$. \square

Theorem 3.3. *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be any function such that $f(q) \rightarrow \infty$. There exists a quasi-staircase transformation with $\frac{a_n b_n^2}{h_n} \rightarrow 0$ and $\sum \frac{c_n + b_n}{h_n} < \infty$ with complexity satisfying $\frac{p(q)}{qf(q)} \rightarrow 0$.*

Proof. By Lemma 3.2, we may assume f is nondecreasing and that $f(n+2) - f(n) \leq 1$ for all n . Then $f(n+1) - f(n) \in \{0, 1\}$ and is never 1 for two consecutive values. We may also assume $f(1) = 1$.

Set $d_1 = d_2 = 1$ and $d_n = \lfloor \sqrt[3]{f(n)} \rfloor$ for $n > 2$. Then $d_n \rightarrow \infty$ is nondecreasing. Also $d_{n+1} - d_n \in \{0, 1\}$ and is never 1 for two consecutive values.

Set $b_n = 3$ for all n such that $\sqrt[3]{f(n)} < 3$ and $b_n = \lfloor \sqrt[3]{f(n)} \rfloor$ for n such that $\sqrt[3]{f(n)} \geq 3$. Then $b_n \rightarrow \infty$ is nondecreasing and $b_n \leq f(n) + 2 \leq n + 2$ as $f(n) \leq n$ since $f(1) = 1$ and $f(n+2) - f(n) \leq 1$ imply $f(n) \leq 1 + \frac{n}{2}$.

Take the quasi-staircase transformation from Proposition 3.1 with defining sequences $\{a_n\}$ and $\{c_n\}$.

Since $0 \leq d_{n+1} - d_n \leq 1$, the sequence $n - d_n$ is nondecreasing and attains every value in \mathbb{N} . For each q , let n_q be the largest n such that $m_{n-d_n} \leq q$. Then $q < m_{n_q+1-d_{n_q+1}}$ so $n_q + 1 - d_{n_q+1} > n_q - d_{n_q}$ and so $1 > d_{n_q+1} - d_{n_q}$ meaning that $d_{n_q+1} = d_{n_q}$. Therefore $c_{n_q+1} = m_{n_q+1-d_{n_q+1}} = m_{n_q-d_{n_q}+1}$.

So $\alpha(q) = n_q - d_{n_q}$ as $m_{n_q-d_{n_q}} \leq q < m_{n_q+1-d_{n_q+1}} = m_{n_q-d_{n_q}+1}$ and $\beta(q) \leq n_q$ since $q < m_{n_q-d_{n_q}+1} = c_{n_q+1}$. By Proposition 2.30, since $q \geq n_q$ and f is nondecreasing to infinity and $n_q \rightarrow \infty$,

$$\begin{aligned} \frac{p(q)}{qf(q)} &\leq \frac{2 + \sum_{n=\alpha(q)}^{\beta(q)} b_n}{f(q)} \leq \frac{2 + \sum_{n=n_q-d_{n_q}}^{n_q} b_n}{f(q)} \leq \frac{2 + (d_{n_q} + 1)b_{n_q}}{f(n_q)} \\ &\leq \frac{2 + (\sqrt[3]{f(n_q)} + 1)\sqrt[3]{f(n_q)}}{f(n_q)} = \frac{2}{f(n_q)} + \frac{1}{\sqrt[3]{f(n_q)}} + \frac{1}{(\sqrt[3]{f(n_q)})^2} \rightarrow 0 \end{aligned} \quad \square$$

4 Mixing for quasi-staircase transformations

Proposition 4.1. *Let T be a quasi-staircase transformation given by $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ with height sequence $\{h_n\}$ such that $\sum \frac{c_n + b_n}{h_n} < \infty$. Then T is on a finite measure space.*

Proof. Writing S_n for the spacers added above the n^{th} column,

$$\mu(S_n) = (c_n r_n + \frac{1}{2} r_n (b_n - 1)) \mu(I_{n+1}) = \left(c_n \frac{r_n}{r_n + 1} + \frac{1}{2} \frac{r_n (b_n - 1)}{r_n + 1} \right) \mu(I_n) \leq \frac{c_n + b_n}{h_n} \mu(C_n)$$

and therefore $\mu(C_{n+1}) = \mu(C_n) + \mu(S_n) \leq (1 + \frac{c_n + b_n}{h_n}) \mu(C_n)$. Then $\mu(C_{n+1}) \leq \prod_{j=1}^n (1 + \frac{c_j + b_j}{h_j}) \mu(C_1)$, meaning that $\log(\mu(C_{n+1})) \leq \log(\mu(C_1)) + \sum_{j=1}^n \log(1 + \frac{c_j + b_j}{h_j})$. As $\frac{c_n + b_n}{h_n} \rightarrow 0$, since $\log(1 + x) \approx x$ for $x \approx 0$, $\lim_n \log(\mu(C_{n+1})) \lesssim \log(\mu(C_1)) + \sum_{j=1}^{\infty} \frac{c_j + b_j}{h_j} < \infty$. \square

For the remainder of this section, all transformations are on probability spaces.

Recall that a sequence $\{t_n\}$ is **mixing** when for all measurable sets A and B , $\mu(T^{t_n} A \cap B) \rightarrow \mu(A)\mu(B)$.

Definition 4.2. The **independence discrepancy** of two sets A and B is

$$\lambda_B(A) = \mu(A \cap B) - \mu(A)\mu(B)$$

So $\{t_n\}$ is mixing when $\lambda_B(T^{t_n}A) \rightarrow 0$ for all measurable A and B . The following is left to the reader:

Lemma 4.3. *If A and A' are disjoint then*

$$\lambda_B(A \sqcup A') = \lambda_B(A) + \lambda_B(A') \quad \text{and} \quad |\lambda_B(A)| \leq \mu(A)$$

and, writing $\chi_B(x) = \mathbb{1}_B(x) - \mu(B)$, for $n \in \mathbb{Z}$, $\lambda_B(T^n A) = \int_A \chi_B \circ T^n d\mu$.

For a rank-one transformation T , a sequence $\{t_n\}$ is **rank-one uniform mixing** when for every union of levels B , $\sum_{j=0}^{h_n-1} |\lambda_B(T^{t_n} I_{n,j})| \rightarrow 0$. Rank-one uniform mixing for a sequence implies mixing for that sequence [CS04] Proposition 5.6.

Lemma 4.4. *Let T be a rank-one transformation, B a union of levels in some column C_N and $n \geq N$. Then for any $0 \leq j < h_n$ and $0 \leq i \leq r_n$,*

$$\lambda_B(I_{n,j}^{[i]}) = \frac{1}{r_n + 1} \lambda_B(I_{n,j})$$

Proof. Since B is a union of levels in C_N , either $I_{n,j} \subseteq B$ or $I_{n,j} \cap B = \emptyset$. If $I_{n,j} \subseteq B$ then $\mu(I_{n,j}^{[i]} \cap B) = \mu(I_{n,j}^{[i]}) = \frac{1}{r_n+1} \mu(I_{n,j}) = \frac{1}{r_n+1} \mu(I_{n,j} \cap B)$ and if $I_{n,j} \cap B = \emptyset$ then $\mu(I_{n,j}^{[i]} \cap B) = 0 = \frac{1}{r_n+1} \mu(I_{n,j} \cap B)$. \square

Lemma 4.5. *Let T be a quasi-staircase transformation. Then for any n and $0 \leq \ell < b_n$ and $k, i \geq 0$ such that $i + k \leq a_n$ and any $j \geq k\ell$,*

$$T^{k(h_n+c_n)} I_{n,j}^{[\ell a_n+i]} = I_{n,j-k\ell}^{[\ell a_n+i+k]}$$

Proof. There are $c_n + \lfloor \frac{\ell a_n+i}{a_n} \rfloor = c_n + \ell$ spacers above $I_{n,j}^{[\ell a_n+i]}$ so $T^{h_n+c_n} I_{n,j}^{[\ell a_n+i]} = I_{n,j-\ell}^{[\ell a_n+i+1]}$. Since $i+k \leq a_n$, there are also $c_n + \ell$ spacers above each $I_{n,j-v\ell}^{[\ell a_n+i+v]}$ for $1 \leq v < k$ so applying $T^{h_n+c_n}$ repeated k times, the claim follows. \square

Proposition 4.6. *Let T be a quasi-staircase transformation and $k \in \mathbb{N}$. If T^k is ergodic then $\{k(h_n+c_n)\}$ is rank-one uniform mixing.*

Proof. Let B be a union of levels in C_N . Write $\chi_B(x) = \mathbb{1}_B(x) - \mu(B)$ so $\int \chi_B d\mu = 0$. Since $\frac{b_n}{h_n} \rightarrow 0$ and $a_n \rightarrow \infty$ there exists N such that for all $n \geq N$ we have $k < a_n$ and $kb_n < h_n$. By Lemma 4.5 and then Lemma 4.4, for $kb_n \leq j < h_n$,

$$\begin{aligned} |\lambda_B(T^{k(h_n+c_n)} I_{n,j})| &= \left| \sum_{\ell=0}^{b_n-1} \sum_{i=0}^{a_n-1} \lambda_B(T^{k(h_n+c_n)} I_{n,j}^{[\ell a_n+i]}) + \lambda_B(T^{k(h_n+c_n)} I_{n,j}^{[r_n]}) \right| \\ &\leq \left| \sum_{\ell=0}^{b_n-1} \sum_{i=0}^{a_n-k-1} \lambda_B(T^{k(h_n+c_n)} I_{n,j}^{[\ell a_n+i]}) \right| + (b_n k + 1) \mu(I_{n+1}) \\ &= \left| \sum_{\ell=0}^{b_n-1} \sum_{i=0}^{a_n-k-1} \lambda_B(I_{n,j-k\ell}^{[\ell a_n+i+k]}) \right| + (b_n k + 1) \mu(I_{n+1}) \\ &= \left| \sum_{\ell=0}^{b_n-1} \sum_{i=0}^{a_n-k-1} \frac{1}{r_n+1} \lambda_B(I_{n,j-k\ell}) \right| + \frac{b_n k + 1}{r_n+1} \mu(I_n) \\ &= \left| \frac{1}{r_n+1} \sum_{\ell=0}^{b_n-1} \sum_{i=0}^{a_n-k-1} \lambda_B(T^{-k\ell} I_{n,j}) \right| + \frac{b_n k + 1}{r_n+1} \mu(I_n) \\ &= \frac{a_n - k}{r_n+1} \left| \sum_{\ell=0}^{b_n-1} \lambda_B(T^{-k\ell} I_{n,j}) \right| + \frac{b_n k + 1}{r_n+1} \mu(I_n) \leq \frac{1}{b_n} \left| \sum_{\ell=0}^{b_n-1} \lambda_B(T^{-k\ell} I_{n,j}) \right| + \frac{k+1}{a_n} \mu(I_n) \end{aligned}$$

$$= \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n-1} \int_{I_{n,j}} \chi_B \circ T^{-k\ell} d\mu \right| + \frac{k+1}{a_n} \mu(I_n) \leq \int_{I_{n,j}} \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n-1} \chi_B \circ T^{-k\ell} \right| d\mu + \frac{k+1}{a_n} \mu(I_n)$$

Therefore

$$\begin{aligned} \sum_{j=0}^{h_n-1} |\lambda_B(T^{k(h_n+c_n)} I_{n,j})| &\leq \sum_{j=kb_n}^{h_n-1} |\lambda_B(T^{k(h_n+c_n)} I_{n,j})| + kb_n \mu(I_n) \\ &\leq \sum_{j=kb_n}^{h_n-1} \left(\int_{I_{n,j}} \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n-1} \chi_B \circ T^{-k\ell} \right| d\mu + \frac{k+1}{a_n} \mu(I_n) \right) + kb_n \mu(I_n) \\ &\leq \int \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n-1} \chi_B \circ T^{-k\ell} \right| d\mu + \frac{k+1}{a_n} + \frac{kb_n}{h_n} \rightarrow 0 \end{aligned}$$

since T^k is ergodic, $b_n \rightarrow \infty$, $a_n \rightarrow \infty$ and $\frac{b_n}{h_n} \rightarrow 0$. \square

Proposition 4.7 ([CPR22] Proposition A.13). *Let T be a rank-one transformation and $\{c_n\}$ a sequence such that $\frac{c_n}{h_n} \rightarrow 0$. If $k \in \mathbb{N}$ and $\{q(h_n + c_n)\}$ is rank-one uniform mixing for each $q \leq k+1$ and $\{t_n\}$ is a sequence such that $h_n + c_n \leq t_n < (q+1)(h_n + c_n)$ for all n then $\{t_n\}$ is mixing.*

Proposition 4.8 ([CPR22] Proposition A.16). *Let T be a rank-one transformation and $\{c_n\}$ a sequence such that $\frac{c_n}{h_n} \rightarrow 0$. If $\{q(h_n + c_n)\}$ is rank-one uniform mixing for each fixed q and $k_n \rightarrow \infty$ is such that $\frac{k_n}{n} \leq 1$ then $\int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi \circ T^{-jk_n} \right| d\mu \rightarrow 0$.*

Lemma 4.9. *Let T be a quasi-staircase transformation. Let $n > 0$ and $0 \leq x < b_n$ and $0 \leq q < a_n$.*

If $0 \leq \ell < b_n - x$ and $0 \leq i < a_n - q$ and $j \geq \frac{1}{2}a_n x(x-1) + qx + ix + \ell(xa_n + q)$ then

$$T^{(xa_n+q)(h_n+c_n)} I_{n,j}^{[\ell a_n+i]} = I_{n,j-\frac{1}{2}a_n x(x-1)-qx-ix-\ell(xa_n+q)}^{[(\ell+x)a_n+i+q]}$$

Proof. If $x = 0$ then Lemma 4.5 applied with q in place of k gives the claim. So we can write

$$xa_n + q = (a_n - i) + (x-1)a_n + (q+i)$$

and assume all three terms on the right are nonnegative.

Using Lemma 4.5,

$$T^{(a_n-i)(h_n+c_n)} I_{n,j}^{[\ell a_n+i]} = I_{n,j-(a_n-i)\ell}^{[\ell a_n+i+a_n-i]} = I_{n,j-(a_n-i)\ell}^{[(\ell+1)a_n]}$$

Now observe that, by Lemma 4.5 with 0 as i and a_n as k , for any $0 \leq v < x$ and any $a_n x \leq z < h_n$,

$$T^{a_n(h_n+c_n)} I_{n,z}^{[va_n]} = I_{n,z-a_nv}^{[(v+1)a_n]}$$

so applying that $x-1$ times for $v = \ell+1, \ell+2, \dots, \ell+x-1$,

$$T^{(x-1)a_n(h_n+c_n)} I_{n,j-(a_n-i)\ell}^{[(\ell+1)a_n]} = I_{n,j-(a_n-i)\ell-(x-1)\ell a_n-\frac{1}{2}x(x-1)a_n}^{[(\ell+x)a_n]}$$

since $\sum_{v=\ell+1}^{\ell+x-1} v = \frac{1}{2}(\ell+x)(\ell+x-1) - \frac{1}{2}\ell(\ell+1) = (x-1)\ell + \frac{1}{2}x(x-1)$. Then applying Lemma 4.5 one final time with $q+i$ in place of k ,

$$\begin{aligned} T^{(q+i)(h_n+c_n)} I_{n,j-(a_n-i)\ell-(x-1)\ell a_n-\frac{1}{2}x(x-1)a_n}^{[(\ell+x)a_n]} &= I_{n,j-(a_n-i)\ell-(x-1)\ell a_n-\frac{1}{2}x(x-1)a_n-(x+\ell)(q+i)}^{[(\ell+x)a_n+q+i]} \\ &= I_{n,j-x\ell a_n-\frac{1}{2}x(x-1)a_n-xi-xq-\ell q}^{[(\ell+x)a_n+q+i]} \end{aligned} \quad \square$$

Lemma 4.10. *Let T be a quasi-staircase transformation. Let $n > 0$ and $0 \leq x < b_n$ and $0 \leq q < a_n$.*

If $0 \leq \ell < b_n - x - 1$ and $a_n - q \leq i < a_n$ and $j \geq \frac{1}{2}a_n x(x+1) + q(x+1) + i(x+1) + \ell(xa_n + 1)$ then

$$T^{(xa_n+q)(h_n+c_n)} I_{n,j}^{[\ell a_n+i]} = I_{n,j-\frac{1}{2}a_n x(x+1)-(q+i-a_n)(x+1)-\ell(xa_n+q)}^{[(\ell+x)a_n+i+q]}$$

Proof. The same proof as Lemma 4.9 except we write $xa_n + q = (a_n - i) + xa_n + (q + i - x)$. \square

Proposition 4.11. *Let T be a quasi-staircase transformation such that $\frac{a_n b_n^2}{h_n} \rightarrow 0$. Let $\{t_n\}$ be a sequence such that $h_n + c_n \leq t_n < h_{n+1} + c_{n+1}$. Then $\{t_n\}$ is mixing.*

Proof. As T is ergodic, Proposition 4.6 with $k = 1$ gives that $\{h_n + c_n\}$ is rank-one uniform mixing, hence mixing, so T is totally ergodic. Then Proposition 4.6 gives that for each fixed k the sequence $\{k(h_n + c_n)\}$ is rank-one uniform mixing.

Since $h_{n+1} + c_{n+1} - (r_n + 1)(h_n + c_n) = \frac{1}{2}r_n(b_n - 1) - c_n + c_{n+1} \geq 0$, if $t_n \geq (r_n + 1)(h_n + c_n)$ then

$$\begin{aligned} \sum_{j=0}^{h_{n+1}-1} |\lambda_B(T^{t_n} I_{n+1,j})| &\leq \sum_{j=h_{n+1}+c_{n+1}-t_n}^{h_{n+1}-1} |\lambda_B(T^{t_n} I_{n+1,j})| + (h_{n+1} + c_{n+1} - t_n)\mu(I_{n+1}) \\ &\leq \sum_{j=0}^{t_n-c_{n+1}-1} |\lambda_B(T^{h_{n+1}+c_{n+1}} I_{n+1,j})| + \frac{\frac{1}{2}r_n(b_n - 1) - c_n + c_{n+1}}{h_{n+1}} \rightarrow 0 \end{aligned}$$

since $\{h_{n+1} + c_{n+1}\}$ is rank-one uniform mixing and $\frac{r_n b_n + c_{n+1}}{h_{n+1}} \leq \frac{b_n}{h_n} + \frac{c_{n+1}}{h_{n+1}} \rightarrow 0$.

So we may assume $t_n < (r_n + 1)(h_n + c_n)$ and therefore write $t_n = k_n(h_n + c_n) + z_n$ for $1 \leq k_n \leq r_n$ and $0 \leq z_n < h_n + c_n$. By Proposition 4.7, if $\{k_n\}$ is bounded along a subsequence then $\{t_n\}$ is mixing along that subsequence, so we may assume that $k_n \rightarrow \infty$.

For $0 \leq j < h_n - z_n$, we have $T^{t_n}(I_{n,j}) = T^{k_n(h_n+c_n)}(I_{n,j+z_n})$ and for $h_n + c_n - z_n \leq j < h_n$,

$$T^{t_n}(I_{n,j}) = T^{t_n+j}(I_{n,0}) = T^{k_n(h_n+c_n)+z_n+j}(I_{n,0}) = T^{(k_n+1)(h_n+c_n)}(I_{n,j+z_n-h_n-c_n})$$

Let B be a union of levels C_N . For $n \geq N$,

$$\begin{aligned} \sum_{j=0}^{h_n-1} |\lambda_B(T^{t_n} I_{n,j})| &\leq \sum_{j=0}^{h_n-z_n-1} |\lambda_B(T^{t_n} I_{n,j})| + c_n \mu(I_n) + \sum_{j=h_n-z_n+c_n}^{h_n-1} |\lambda_B(T^{t_n} I_{n,j})| \\ &\leq \sum_{j=0}^{h_n-z_n-1} |\lambda_B(T^{k_n(h_n+c_n)} I_{n,j+z_n})| + c_n \mu(I_n) + \sum_{j=h_n+c_n-z_n}^{h_n-1} |\lambda_B(T^{(k_n+1)(h_n+c_n)} I_{n,j+z_n-(h_n+c_n)})| \\ &\leq \sum_{x=0}^{h_n-1} |\lambda_B(T^{k_n(h_n+c_n)} I_{n,x})| + c_n \mu(I_n) + \sum_{x=0}^{h_n-1} |\lambda_B(T^{(k_n+1)(h_n+c_n)} I_{n,x})| \end{aligned}$$

We now show the sum on the left tends to zero; the same argument with $k_n + 1$ in place of k_n gives the same for the right sum. As $c_n \mu(I_n) \rightarrow 0$, this will complete the proof.

Write $k_n = x_n a_n + q_n$ for $0 \leq q_n < a_n$ and $0 \leq x_n < b_n$.

Begin by observing that

$$\sum_{j=0}^{h_n-1} |\lambda_B(T^{k_n(h_n+c_n)} I_{n,j})| \leq \sum_{j=0}^{h_n-1} \left| \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=0}^{a_n-1} \lambda_B(T^{k_n(h_n+c_n)} I_{n,j}^{[\ell a_n+i]}) \right| + 2a_n h_n \mu(I_{n+1}) \quad (\star)$$

$$+ \sum_{j=0}^{h_n-1} \left| \sum_{\ell=b_n-x_n+1}^{b_n-1} \sum_{i=0}^{a_n-1} \lambda_B(T^{k_n(h_n+c_n)} I_{n,j}^{[\ell a_n+i]}) \right| \quad (\star\star)$$

We handle the sum $(\star\star)$ first and return to the sum in (\star) shortly.

For $0 \leq \ell < b_n$ and $0 \leq i < a_n$, we have that

$$I_{n,0}^{[\ell a_n + i]} = T^{(\ell a_n + i)(h_n + c_n)} I_{n, \frac{1}{2}\ell(\ell-1)a_n + i\ell}^{[0]}$$

since $\frac{1}{2}\ell(\ell-1)a_n + i\ell \leq a_n b_n^2 + a_n b_n < h_n$ as $\frac{a_n b_n^2}{h_n} \rightarrow 0$.

For $b_n - x_n + 1 \leq \ell < b_n$ and $0 \leq i < a_n$, since $x + \ell \geq b_n + 1$,

$$\begin{aligned} k_n(h_n + c_n) + (\ell a_n + i)(h_n + c_n) &= (x_n a_n + q_n + \ell a_n + i)(h_n + c_n) \\ &\geq (b_n a_n + a_n)(h_n + c_n) \\ &= (b_n a_n + 1)h_n + b_n a_n c_n + (a_n - 1)h_n + a_n c_n \geq h_{n+1} \end{aligned}$$

since $\frac{1}{2}a_n b_n(b_n - 1) \leq h_n$. Also,

$$\begin{aligned} k_n(h_n + c_n) + (\ell a_n + i)(h_n + c_n) + \frac{1}{2}\ell(\ell-1)a_n + i\ell \\ &= ((x_n + \ell)a_n + q_n + i)(h_n + c_n) + \frac{1}{2}\ell(\ell-1)a_n + i\ell \\ &\leq 2b_n a_n(h_n + c_n) + \frac{1}{2}b_n(b_n - 1)a_n + a_n b_n < 2h_{n+1} \end{aligned}$$

Since a sublevel in I_n is a level in I_{n+1} , we then have that, as $\{h_{n+1}\}$ is rank-one uniform mixing,

$$\sum_{j=0}^{h_n-1} \sum_{\ell=b_n-x_n+1}^{b_n-1} \sum_{i=0}^{a_n-1} |\lambda_B(T^{k_n(h_n+c_n)} I_{n,j}^{[\ell a_n + i]})| \leq \sum_{y=0}^{h_{n+1}-1} |\lambda_B(T^{h_{n+1}} I_{n+1,y})| \rightarrow 0$$

As $2a_n h_n \mu(I_{n+1}) \leq \frac{2a_n h_n}{h_{n+1}} \leq \frac{2}{b_n} \rightarrow 0$, it remains only to show that the sum in (\star) tends to zero.

Observe that

$$\begin{aligned} \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=0}^{a_n-1} \lambda_B(T^{k_n(h_n+c_n)} I_{n,j}^{[\ell a_n + i]}) &= \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=0}^{a_n-q_n-1} \lambda_B(T^{k_n(h_n+c_n)} I_{n,j}^{[\ell a_n + i]}) \quad (\dagger) \\ &\quad + \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=a_n-q_n}^{a_n-1} \lambda_B(T^{k_n(h_n+c_n)} I_{n,j}^{[\ell a_n + i]}) \quad (\ddagger) \end{aligned}$$

First, we address (\dagger) : set $y_n = \frac{1}{2}a_n x_n(x_n - 1) + q_n x_n$. For $i < a_n - q_n$ and $\ell < b_n - x_n - 1$, we have $y_n + i x_n + \ell k_n \leq 3a_n b_n^2$ so for $j \geq 3a_n b_n^2$, by Lemma 4.9 and Lemma 4.4,

$$\begin{aligned} \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=0}^{a_n-q_n-1} \lambda_B(T^{k_n(h_n+c_n)} I_{n,j}^{[\ell a_n + i]}) &= \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=0}^{a_n-q_n-1} \lambda_B(I_{n,j-y_n-i x_n-\ell k_n}^{[(\ell+x_n)a_n+i+q_n]}) \\ &= \frac{1}{r_n + 1} \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=0}^{a_n-q_n-1} \lambda_B(I_{n,j-y_n-i x_n-\ell k_n}) = \frac{1}{r_n + 1} \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=0}^{a_n-q_n-1} \lambda_B(T^{-\ell k_n - i x_n - y_n} I_{n,j}) \end{aligned}$$

Then, summing over all $3a_n b_n^2 \leq j < h_n$,

$$\begin{aligned} \sum_{j=3a_n b_n^2}^{h_n-1} \left| \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=0}^{a_n-q_n-1} \lambda_B(T^{k_n(h_n+c_n)} I_{n,j}^{[\ell a_n + i]}) \right| \\ &= \sum_{j=3a_n b_n^2}^{h_n-1} \left| \frac{1}{r_n + 1} \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=0}^{a_n-q_n-1} \lambda_B(T^{-\ell k_n - i x_n - y_n} I_{n,j} \cap B) \right| \\ &\leq \frac{b_n}{r_n + 1} \sum_{i=0}^{a_n-q_n-1} \sum_{j=0}^{h_n-1} \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n-x_n-2} \lambda_B(T^{-\ell k_n - i x_n - y_n} I_{n,j}) \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{b_n}{r_n + 1} \sum_{i=0}^{a_n - q_n - 1} \int \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n - x_n - 2} \chi_B \circ T^{-\ell k_n - i x_n - y_n} \right| d\mu \\
 &= \frac{b_n(a_n - q_n)}{r_n + 1} \int \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n - x_n - 2} \chi_B \circ T^{-\ell k_n} \right| d\mu \leq \int \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n - x_n - 2} \chi_B \circ T^{-\ell k_n} \right| d\mu \rightarrow 0
 \end{aligned}$$

by Proposition 4.8. So the sum over the levels with $3a_n b_n^2 \leq j < h_n$ for (†) tends to zero.

For (‡): set $y'_n = \frac{1}{2}a_n x_n(x_n + 1) + (q_n - a_n)(x_n + 1)$. By Lemma 4.10 and Lemma 4.4, for $j \geq 3a_n b_n^2$,

$$\begin{aligned}
 \sum_{\ell=0}^{b_n - x_n - 2} \sum_{i=a_n - q_n}^{a_n - 1} \lambda_B(T^{k_n(h_n + c_n)} I_{n,j}^{[\ell a_n + i]}) &= \sum_{\ell=0}^{b_n - x_n - 2} \sum_{i=a_n - q_n}^{a_n - 1} \lambda_B(I_{n,j - y'_n - i(x_n + 1) - \ell k_n}^{[(\ell + x_n)a_n + i + q_n]}) \\
 &= \frac{1}{r_n + 1} \sum_{\ell=0}^{b_n - x_n - 2} \sum_{i=a_n - q_n}^{a_n - 1} \lambda_B(I_{n,j - y'_n - i(x_n + 1) - \ell k_n}) \\
 &= \frac{1}{r_n + 1} \sum_{\ell=0}^{b_n - x_n - 2} \sum_{i=a_n - q_n}^{a_n - 1} \lambda_B(T^{-\ell k_n - i(x_n + 1) - y'_n} I_{n,j})
 \end{aligned}$$

Similar to the left sum, then

$$\begin{aligned}
 &\sum_{j=3a_n b_n^2}^{h_n - 1} \left| \sum_{\ell=0}^{b_n - x_n - 2} \sum_{i=a_n - q_n}^{a_n - 1} \lambda_B(T^{k_n(h_n + c_n)} I_{n,j}^{[\ell a_n + i]}) \right| \\
 &= \sum_{j=3a_n b_n^2}^{h_n - 1} \left| \frac{1}{r_n + 1} \sum_{\ell=0}^{b_n - x_n - 2} \sum_{i=a_n - q_n}^{a_n - 1} \lambda_B(T^{-\ell k_n - i(x_n + 1) - y'_n} I_{n,j}) \right| \\
 &\leq \frac{b_n}{r_n + 1} \sum_{i=a_n - q_n}^{a_n - 1} \int \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n - 1} \chi_B \circ T^{-\ell k_n - i(x_n + 1) - y'_n} \right| d\mu = \frac{b_n q_n}{r_n + 1} \int \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n} \chi_B \circ T^{-\ell k_n} \right| d\mu \rightarrow 0
 \end{aligned}$$

Since $3a_n b_n^2 / h_n \rightarrow 0$, this shows that the sum in (★) tends to zero, completing the proof. \square

Theorem 4.12. *Let T be a quasi-staircase transformation on a finite measure space such that $\frac{a_n b_n^2}{h_n} \rightarrow 0$. Then T is mixing.*

Proof. Let $\{t_m\}$ be any strictly increasing sequence. Set p_m such that $h_{p_m} + c_{p_m} \leq t_m < h_{p_m + 1} + c_{p_m + 1}$. Choose a subsequence $\{t_{m_j}\}$ such that p_{m_j} is strictly increasing. Define $\{q_n\}$ by setting $q_{p_{m_j}} = t_{m_j}$ and $q_n = h_n + c_n$ for n such that $n \neq p_{m_j}$ for any j . Proposition 4.11 gives $\{q_n\}$ is mixing so $\{t_{m_j}\}$ is. As every $\{t_m\}$ has a mixing subsequence, T is mixing. \square

5 Non-superlinear word complexity implies partial rigidity

Theorem 5.1. *Let X be a subshift with word complexity p such that $\liminf \frac{p(q)}{q} < \infty$. Then there exists a constant $\delta_X > 0$ such that every ergodic probability measure μ on X is at least δ_X -partially rigid.*

5.1 Word combinatorics

Notation 5.2. *For x a finite or infinite word and $-\infty \leq i < j \leq \infty$,*

$$x_{[i,j]} = \text{the subword of } x \text{ from position } i \text{ through position } j - 1$$

Notation 5.3. $[w] = \{x \in X : x_{[0, \|w\|]} = w\}$ for finite words w .

Notation 5.4. For a word v and $0 \leq q < \|v\|$, let $v^{q/\|v\|}$ be the suffix of v of length q .

Let $v^{n+q/\|v\|} = v^{q/\|v\|}v^n$ for $n \in \mathbb{N}$.

Definition 5.5. A word v is a **root** of w if there exists $q = p/\|v\|$ with $p \geq \|v\|$ such that $w = v^q$. The **minimal root** of w is the shortest suffix v which is a root of w .

Every word has a unique minimal root since it is a root of itself.

Lemma 5.6. If $uw = wv$ and $\|v\| \leq \|w\|$ then v is a root of w .

Proof. As w has v as a suffix, $w = w'v$. Then $uw'v = uw = wv = w'v$ so $uw' = w'v$. If $\|w'\| \geq \|v\|$, repeat this process until it terminates at $w = w''v^n$ with $\|w''\| < \|v\|$. Then $uw'' = w''v$ so w'' is a suffix of v . \square

Lemma 5.7. If $uv = vu$ then $u = v_0^t$ and $v = v_0^s$ for some word v_0 and $t, s \in \mathbb{N}$.

Proof. If $\|u\| = \|v\|$ then $uv = vu$ immediately implies $u = v$. Let

$$V = \{(u, v) : uv = vu, \|v\| < \|u\|, \text{ there is no word } v_0 \text{ with } u = v_0^t \text{ and } v = v_0^s \text{ for } s, t \in \mathbb{N}\}$$

and suppose $V \neq \emptyset$. Let $(u, v) \in V$ such that $\|u\|$ is minimal. As $\|u\| > \|v\|$, $uv = vu$ implies $u = vu' = u''v$ for some nonempty words u', u'' . Then $vu'v = uv = vu = vu''v$ so $u' = u''$ and $vu' = u'v$. If $\|u'\| = \|v\|$ then $u' = v$ so $u = v^2$ contradicting that $(u, v) \in V$.

Consider when $\|u'\| < \|v\|$. Since $\|u'\| < \|u\|$ and $\|v\| < \|u\|$, the minimality of $\|u\|$ implies that $(v, u') \notin V$. Then $v = v_0^n$ and $u' = v_0^m$ for some word v_0 and $n, m \in \mathbb{N}$. So $u = v_0^{n+m}$ meaning $(u, v) \notin V$. When $\|v\| < \|u'\|$, we have $(u', v) \notin V$ so $u' = v_0^n$ and $u = v_0^{n+m}$. So $V = \emptyset$. \square

Lemma 5.8. If u and v are both roots of a word w and uv is a suffix of w and $\|v\| < \|u\|$ then there exists a suffix v_0 of v such that $u = v_0^n$ and $v = v_0^m$ for some $n, m \in \mathbb{N}$.

In particular, if v is the minimal root of w and u is a root of w and uv is a suffix of w then u is a multiple of v , i.e. there exists $n \in \mathbb{N}$ such that $u = v^n$.

Proof. Writing u' and v' for the appropriate suffixes of u and v , we have $w = u'u^t = v'v^q$ for some $t, q \in \mathbb{N}$. Then $u = u_0v^a$ for some proper (possibly empty) suffix u_0 of v and $1 \leq a \leq q$. So $u'(u_0v^a)^t = v'v^q$ meaning that $u'(u_0v^a)^{t-1}u_0 = v'v^{q-a}$. As $t \geq 2$, $\|v'v^{q-a}\| = \|u'(u_0v^a)^{t-1}u_0\| \geq \|u_0v^a u_0\| \geq \|vu_0\|$ so, as u_0 is a suffix of v , then $v'v^{q-a}$ has u_0v as a suffix. This means $vu_0 = u_0v$ so Lemma 5.7 gives v_0 such that $v = v_0^n$ and $u_0 = v_0^m$ so $u = v_0^{m+an}$. If v is the minimal root then $v = v_0$ since v_0 is a root of w . \square

Lemma 5.9. Let w be a word with minimal root v . If $0 \leq i \leq \frac{1}{2}\|w\|$ and $T^i[w] \cap [w] \neq \emptyset$ then i is a multiple of $\|v\|$.

Proof. Let u be the prefix of B of length i and v_0 be the suffix of B of length i . For $x \in T^i[w] \cap [w]$, then $x_{[-i, \|w\|)} = uw = wv_0$. By Lemma 5.6, then v_0 is a root of w . As $\|v_0\| = i \leq \frac{1}{2}\|w\|$, w has v_0v_0 as a suffix. By Lemma 5.8, since v is the minimal root then v_0 is a multiple of v . \square

5.2 Language analysis

Proposition 5.10. There exists $C, k > 0$, depending only on X , and $\ell_n \rightarrow \infty$ and, for each n , at most C words $B_{n,j}$ so that $X_0 = \{x \in X : \text{every finite subword of } x \text{ is a subword of a concatenation of the } B_{n,j}\}$ has measure one.

Let $h_{n,j} = \|B_{n,j}\|$. Then $\max_j h_{n,j} \leq k\ell_n$ and $\min_j h_{n,j} \rightarrow \infty$. Let

$$W_{B_{n,j}} = W_{n,j} = \{x \in X_0 : x \text{ can be written as a concatenation such that } x_{[0, h_{n,j})} = B_{n,j}\} \subseteq [B_{n,j}]$$

There exists $c_{n,j} \leq k\ell_n$ such that the sets $T^i W_{n,j}$ are disjoint over $0 \leq i < c_{n,j}$.

For j such that $h_{n,j} > \frac{1}{2}\ell_n$, $c_{n,j} \geq \frac{1}{2}\ell_n$.

For j such that $h_{n,j} \leq \frac{1}{2}\ell_n$, $c_{n,j} = h_{n,j}$. For such j , also $W_{n,j} = T^{\ell_n}[B_{n,j}^{\ell_n/h_{n,j}} B_{n,j}]$ and $B_{n,j}$ is the minimal root of $B_{n,j}^{\ell_n/h_{n,j}} B_{n,j}$.

If $x \in T^{h_{n,j}} W_{n,j} \cap W_{n,j'}$ for $j \neq j'$ and $h_{n,j'} \leq \frac{1}{2}\ell_n$ then $x_{(-\infty,0)}$ has $B_{n,j'}^{\ell_n/h_{n,j'}}$ as a suffix and does not have $B_{n,j'}^{\ell_n/h_{n,j'}} B_{n,j'}$ as a suffix.

Proof. Since $\liminf \frac{p(q)}{q} < \infty$, [Bos85] Theorem 2.2 gives a constant k and $\ell_n \rightarrow \infty$ such that $p(\ell_n + 1) - p(\ell_n) \leq k$ and $p(\ell_n) \leq k\ell_n$. We perform an analysis similar to Ferenczi [Fer96] Proposition 4.

Let G_q be the Rauzy graphs: the vertices are the words of length q in $\mathcal{L}(X)$ and the directed edges are from words w to w' such that $wa = bw'$ for some letters a and b and we label the edge with the letter a . As μ is ergodic, exactly one strongly connected component has measure one and the rest have measure zero so we may assume G_q is strongly connected.

Let V_q^{RS} be the set of all vertices with more than one outgoing edge, i.e. the right-special vertices. Let \mathcal{B}_q be the set of all paths from some $v \in V_q^{RS}$ to some $v' \in V_q^{RS}$ that do not pass through any $v'' \in V_q^{RS}$. Then every $v \in V_q$ is necessarily along such a path so every word of length q in $\mathcal{L}(X)$ is a subword of some concatenation of labels of paths in \mathcal{B}_q .

The labels of the paths between right-special vertices are nested: \mathcal{B}_{q+1} is necessarily a concatenation of paths in \mathcal{B}_q since words corresponding to elements of V_{q+1}^{RS} necessarily have right-special suffixes. There are therefore recursion formulas defining \mathcal{B}_{q+1} in terms of \mathcal{B}_q though we do not make use of this fact.

Writing $\text{outdeg}(v)$ for the number of outgoing edges of a vertex, $\sum_{v \in V_{\ell_n}^{RS}} (\text{outdeg}(v) - 1) = p(\ell_n + 1) - p(\ell_n) \leq k$ meaning that $|V_{\ell_n}^{RS}| \leq k$ and therefore $\sum_{v \in V_{\ell_n}^{RS}} \text{outdeg}(v) \leq 2k$. Therefore $|\mathcal{B}_{\ell_n}| \leq 2k$. No path in \mathcal{B}_{ℓ_n} properly contains a cycle so $\|B\| \leq p(\ell_n) \leq k\ell_n$ for any label B of a path in \mathcal{B}_{ℓ_n} .

Let \mathcal{B}_n^g be the set of all concatenations of paths in \mathcal{B}_{ℓ_n} of total length at least $\frac{3}{2}\ell_n$ and at most $k\ell_n$ not properly containing any cycles. As such a path contains no cycle properly, it has at most $|\mathcal{B}_{\ell_n}| \leq 2k$ segments from some vertex in $V_{\ell_n}^{RS}$ to another, so there are at most $K = \sum_{t=1}^{2k} (2k)^t$ such paths.

Let \mathcal{B}_n^c be the set of all concatenations of paths in \mathcal{B}_{ℓ_n} of total length less than $\frac{3}{2}\ell_n$ which are simple cycles. Then $|\mathcal{B}_n^c| \leq K$ as each path has at most $2k$ segments and at most $2k$ choices for each segment. Every biinfinite concatenation of paths in \mathcal{B}_{ℓ_n} is necessarily a concatenation of paths in $\mathcal{B}_n^g \cup \mathcal{B}_n^c$.

Let B be the label of a path in \mathcal{B}_n^g and let v be its minimal root. Suppose that $\|v\| < \frac{1}{2}\ell_n$. Then the vertex at which the path corresponding to B ends is the word $v^{\ell_n/\|v\|}$ as it must be a suffix of B . Let B' such that $B = B'v$. Then $\|B'\| = \|B\| - \|v\| \geq \frac{3}{2}\ell_n - \|v\| > \ell_n$. Then the path corresponding to B reaches its final vertex twice as B' has suffix $v^{\ell_n/\|v\|}$ corresponding to that vertex. This means the path properly contains a cycle which is a contradiction. So all labels of paths in \mathcal{B}_n^g have minimal root of length at least $\frac{1}{2}\ell_n$. By Lemma 5.9, then $T^i W_{n,j} \cap W_{n,j} \neq \emptyset$ for $0 < i \leq \frac{1}{2}\|B\|$ only when i is a multiple of $\|v\|$. Set $c_{n,j} = \min(\|v\|, \frac{1}{2}\|B\|) \geq \frac{1}{2}\ell_n$ and then $T^i W_{n,j}$ are disjoint over $0 \leq i < c_{n,j}$.

Let B be the label of a simple cycle beginning and ending at the word w . Since B is the label of a path beginning at w , every appearance of B as a label in $x \in X$ is preceded by w , i.e. $W_B \subseteq T^{\ell_n}[wB]$. Since B either has w as a suffix or B is a root of w by Lemma 5.6, B is a root of wB . Let v be the minimal root of wB and write $B = B'v$. Then wB' has v as a root and $\|wB'\| = \ell_n + \|B'\|$ so wB' has suffix $v^{\ell_n/\|v\|}$. If B' is nonempty then the path corresponding to B passes through its final vertex before the path ends, contradicting that it is a simple cycle. So $B = v$ is the minimal root of wB .

Then Lemma 5.9 implies that $T^i W_B \cap W_B \neq \emptyset$ for $0 < i \leq \frac{1}{2}\|wB\|$ only when i is a multiple of $\|B\|$. So if $\|B\| > \frac{1}{2}\ell_n$ then set $c_{n,j} = \min(\|B\|, \frac{1}{2}\|wB\|) > \frac{1}{2}\ell_n$. If $\|B\| \leq \frac{1}{2}\ell_n$, set $c_{n,j} = \|B\|$. For such B , since $W_B \subseteq T^{\ell_n}[wB]$, we have that every occurrence of B as a label of a path is preceded by $w = B^{\ell_n/\|B\|}$. Moreover, if $x_{[-\ell_n, \|B\|)} = wB$ then $x_{[0, \|B\|)}$ is the label of a path beginning at the vertex w and ending at w so $x \in W_B$.

For $x \in W_B$, if $x_{(-\infty, 0)}$ has $B^{\ell_n/\|B\|} B$ as a suffix then the path reaches w prior to the final B in that suffix. As no word B' appearing in the concatenation is the label of a path properly containing

a cycle, this means the word preceding $x_{[0, \|B\|]} = B$ in x must be B , i.e. $x \in T^{\ell_n + \|B\|} [B^{\ell_n / \|B\|} B]$ so $x \in T^{\|B\|} W_B \cap W_B$ and $x \notin T^{\|B'\|} W_{B'} \cap W_B$ for every $B' \neq B$ as the path for B' does not properly contain a cycle.

Let $\mathcal{B}_n^* = \mathcal{B}_n^g \cup \mathcal{B}_n^c$. Then $|\mathcal{B}_n^*| \leq 2K = C$ for all n and every biinfinite word is a concatenation of labels of paths in \mathcal{B}_n^* . Let \mathcal{R}_n be the set of all labels of paths in \mathcal{B}_n^* .

Let $\mathcal{D}_M = \{B : \|B\| \leq M \text{ and } B \in \mathcal{R}_n \text{ infinitely often}\}$. Then $|\mathcal{D}_M| < \infty$ as there only finitely many words of length at most M (as non-superlinear complexity implies finite alphabet rank [DDMP21]). Let X_M be the set of $x \in X$ such that for infinitely many n , x cannot be written as a concatenation of labels in \mathcal{B}_n^* without using at least one label in \mathcal{D}_M .

For $x \in X_M$, there exist infinitely many t such that x has $B_t^{r_t}$ as a subword for some $B_t \in \mathcal{D}_M$ and $r_t \rightarrow \infty$ (since the label B_t is preceded by the word $B_t^{\lfloor \ell_n / (\|B_t\|) \rfloor}$). As $|\mathcal{D}_M| < \infty$, there exists B such that $B_t = B$ infinitely often. Then B^{r_t} is a subword of x for $r_t \rightarrow \infty$ meaning x is periodic. Therefore $\bigcup_M X_M \subseteq \{\text{periodic words}\}$ so $\mu(\bigcup_M X_M) = 0$ as μ is ergodic hence nonatomic and a periodic word of positive measure would be an atom (there are at most countably many periodic words).

Define $\{B_{n,j}\}$ to be the set of all labels of paths in \mathcal{B}_n^* which are in $\mathcal{R}_n \setminus \bigcup_M \mathcal{D}_M$. If $\liminf_n \min_j \|B_{n,j}\| < \infty$ then $B_{n,j} = B$ for some fixed B infinitely often (as there are only finitely many words of up to some fixed length). But then $B \in \mathcal{D}_{\|B\|}$, a contradiction, so $\lim_n \min_j \|B_{n,j}\| = \infty$. As $X_0 = X \setminus \bigcup_M X_M$, we have $\mu(X_0) = 1$. \square

5.3 Measure-theoretic analysis

Definition 5.11. Let $C_{n,j} = \bigcup_{i=0}^{h_{n,j}-1} T^i W_{n,j}$.

Definition 5.12. For j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, let

$$\begin{aligned} Z_{n,j} &= [B_{n,j}^{\ell_n/h_{n,j}} B_{n,j}] \setminus T^{h_{n,j}} [B_{n,j}^{\ell_n/h_{n,j}} B_{n,j}] \\ &= \{x \in X : x_{[0, \ell_n + h_{n,j}]} = B_{n,j}^{\ell_n/h_{n,j}} B_{n,j} \text{ and } x_{[-h_{n,j}, \ell_n]} \neq B_{n,j}^{\ell_n/h_{n,j}} B_{n,j}\} \end{aligned}$$

Proposition 5.13. For j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, the sets $T^{ah_{n,j}} Z_{n,j}$ are disjoint over $0 \leq a \leq \lfloor \frac{\ell_n}{h_{n,j}} \rfloor$.

Proof. For $0 \leq a < b \leq \lfloor \frac{\ell_n}{h_{n,j}} \rfloor$ and $x \in T^{ah_{n,j}} Z_{n,j} \cap T^{bh_{n,j}} Z_{n,j}$, writing $z = \ell_n - \lfloor \frac{\ell_n}{h_{n,j}} \rfloor h_{n,j}$, we would have $x_{[z-(a+1)h_{n,j}, z-ah_{n,j}]} \neq B_{n,j}$ but $x_{[z-bh_{n,j}, z]} = B_{n,j}^b$ which is impossible. \square

Proposition 5.14. For j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, the sets $T^i Z_{n,j}$ are disjoint over $0 \leq i < c_{n,j}$.

Proof. Lemma 5.9 as $B_{n,j}$ is the minimal root of $B_{n,j}^{\ell_n/h_{n,j}} B_{n,j}$ and $c_{n,j} \leq \frac{1}{2}\ell_n < \frac{1}{2}\|B_{n,j}^{\ell_n/h_{n,j}} B_{n,j}\|$. \square

Definition 5.15. For j such that $\|B_{n,j}\| > \frac{1}{2}\ell_n$, let, for $0 \leq i < c_{n,j}$,

$$I_{n,j,i} = T^i W_{n,j}$$

and for j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, let, for $0 \leq i < c_{n,j}$,

$$I_{n,j,i} = T^i \left(\bigcup_{a=0}^{\lfloor \frac{\ell_n}{h_{n,j}} \rfloor} T^{ah_{n,j}} Z_{n,j} \right)$$

As T is measure-preserving, $\mu(I_{n,j,i}) = \mu(I_{n,j,0})$ for all n, j and $0 \leq i < c_{n,j}$.

Definition 5.16. Let $\tilde{C}_{n,j} = \bigsqcup_{i=0}^{c_{n,j}-1} I_{n,j,i}$. For j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, let $\hat{C}_{n,j} = \bigsqcup_{i=0}^{h_{n,j}-1} T^i W_{n,j}$.

Proposition 5.17. For j such that $\|B_{n,j}\| > \frac{1}{2}\ell_n$, we have $\mu(\tilde{C}_{n,j}) \geq \frac{1}{2k}\mu(C_{n,j})$.

Proof. $\mu(C_{n,j}) \leq h_{n,j}\mu(W_{n,j}) = h_{n,j}\mu(I_{n,j,0}) = \frac{h_{n,j}}{c_{n,j}}\mu(\tilde{C}_{n,j}) \leq \frac{k\ell_n}{\frac{1}{2}\ell_n}\mu(\tilde{C}_{n,j}) = 2k\mu(\tilde{C}_{n,j})$. \square

Proposition 5.18. $\lim_n \max_j \{\mu(I_{n,j,0})\} = 0$.

Proof. For j such that $\|B_{n,j}\| > \frac{1}{2}\ell_n$, we have $1 \geq \mu(\tilde{C}_{n,j}) = c_{n,j}\mu(I_{n,j,0}) \geq \frac{1}{2}\ell_n\mu(I_{n,j,0})$ and $\ell_n \rightarrow \infty$. For j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, we have $1 \geq \mu(\tilde{C}_{n,j}) = h_{n,j}\mu(I_{n,j,0})$ and $\min_j h_{n,j} \rightarrow \infty$. \square

Proposition 5.19. $T^{h_{n,j}}W_{n,j} \subseteq \bigcup_{j'} W_{n,j'}$ and $X_0 = \bigcup_j C_{n,j}$.

Proof. Every $x \in X_0$ is a concatenation of words of the form $B_{n,j}$ so every occurrence of $B_{n,j}$ is followed immediately by some $B_{n,j'}$ and $x_{[0,\infty)} = uB_1B_2 \cdots$ for some u a suffix of some $B_{n,j}$ and $B_\ell \in \{B_{n,j}\}$. \square

Proposition 5.20. Let $E \subseteq W_{n,j}$. Then there exists j' such that $\mu(T^{h_{n,j}}E \cap W_{n,j'}) \geq \frac{1}{C}\mu(E)$.

Proof. $T^{h_n}E = T^{h_n}E \cap T^{h_{n,j}}W_{n,j} \subseteq T^{h_n}E \cap \bigcup_{j'} W_{n,j'}$ and there are at most C choices of j' . \square

Lemma 5.21. $\mu(W_{n,j}) \geq \frac{1}{k\ell_n}\mu(\tilde{C}_{n,j})$.

Proof. For j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, by Proposition 5.10, $T^{-\ell_n}W_{n,j} = [B_{n,j}^{\ell_n/h_{n,j}}B_{n,j}] \supseteq Z_{n,j}$ so

$$\mu(W_{n,j}) \geq \mu(Z_{n,j}) = \frac{1}{\lfloor \frac{\ell_n}{h_{n,j}} \rfloor + 1} \mu(I_{n,j,0}) \geq \frac{1}{\frac{\ell_n}{h_{n,j}}} \frac{1}{h_{n,j}} \mu(\tilde{C}_{n,j}) = \frac{1}{\ell_n} \mu(\tilde{C}_{n,j})$$

and for j such that $\|B_{n,j}\| > \frac{1}{2}\ell_n$, we have $\mu(W_{n,j}) = \frac{1}{c_{n,j}}\mu(\tilde{C}_{n,j}) \geq \frac{1}{k\ell_n}\mu(\tilde{C}_{n,j})$ since $c_{n,j} \leq k\ell_n$. \square

Proposition 5.22. If $\mu(T^{h_{n,j}}W_{n,j} \cap W_{n,j'}) \geq \delta\mu(W_{n,j'})$ for $j \neq j'$ then $\mu(\tilde{C}_{n,j'}) \geq \frac{1}{2k}\delta\mu(\tilde{C}_{n,j''})$.

Proof. For j' such that $h_{n,j'} < \frac{1}{2}\ell_n$, Proposition 5.10 states that, as $j \neq j'$, for $x \in T^{h_{n,j}}W_{n,j} \cap W_{n,j'}$, the word $x_{(-\infty,0)}$ has $B_{n,j'}^{\ell_n/h_{n,j'}}$ as a suffix but does not have $B_{n,j'}^{\ell_n/h_{n,j'}}B_{n,j'}$ as a suffix. Therefore $T^{-\ell_n}(T^{h_{n,j}}W_{n,j} \cap W_{n,j'}) \subseteq [B_{n,j'}^{\ell_n/h_{n,j'}}B_{n,j'}] \setminus T^{h_{n,j'}}[B_{n,j'}^{\ell_n/h_{n,j'}}B_{n,j'}] = Z_{n,j'}$. This means that $\mu(Z_{n,j'}) \geq \mu(T^{h_{n,j}}W_{n,j} \cap W_{n,j'}) \geq \delta\mu(W_{n,j'})$ so

$$\begin{aligned} \mu(\tilde{C}_{n,j'}) &= h_{n,j'}\mu(I_{n,j',0}) = h_{n,j'} \left(\lfloor \frac{\ell_n}{h_{n,j'}} \rfloor + 1 \right) \mu(Z_{n,j'}) \geq h_{n,j'} \frac{\ell_n}{h_{n,j'}} \delta\mu(W_{n,j'}) \\ &\geq \ell_n \delta \frac{1}{c_{n,j'}} \mu(\tilde{C}_{n,j''}) \geq \ell_n \delta \frac{1}{k\ell_n} \mu(\tilde{C}_{n,j''}) = \delta \frac{1}{k} \mu(\tilde{C}_{n,j''}) \end{aligned}$$

For j' such that $h_{n,j'} > \frac{1}{2}\ell_n$, using Lemma 5.21 and that $\mu(W_{n,j'}) \geq \delta\mu(W_{n,j''})$,

$$\mu(\tilde{C}_{n,j'}) = c_{n,j'}\mu(W_{n,j'}) \geq c_{n,j'}\delta\mu(W_{n,j''}) \geq c_{n,j'}\delta \frac{1}{k\ell_n} \mu(\tilde{C}_{n,j''}) \geq \frac{\ell_n}{2} \delta \frac{1}{k\ell_n} \mu(\tilde{C}_{n,j''}) = \frac{1}{2k} \delta \mu(\tilde{C}_{n,j''}) \quad \square$$

Proposition 5.23. For j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, we have $\mu(T^{h_{n,j}}I_{n,j,0} \cap I_{n,j,0}) \geq \frac{1}{2}\mu(I_{n,j,0})$.

Proof.

$$\mu(T^{h_{n,j}} I_{n,j,0} \cap I_{n,j,0}) \geq \mu\left(\prod_{a=1}^{\lfloor \frac{\ell_n}{h_{n,j}} \rfloor} T^{ah_{n,j}} Z_{n,j}\right) = \left\lfloor \frac{\ell_n}{h_{n,j}} \right\rfloor \mu(Z_{n,j}) = \frac{\left\lfloor \frac{\ell_n}{h_{n,j}} \right\rfloor}{\left\lfloor \frac{\ell_n}{h_{n,j}} \right\rfloor + 1} \mu(I_{n,j,0}) \geq \frac{1}{2} \mu(I_{n,j,0}) \quad \square$$

5.4 Partial rigidity

We employ ideas similar to Danilenko's [Dan16] proof that exact finite rank implies partial rigidity:

Proposition 5.24. *If there exists $\delta > 0$ and j_n and $t_n \rightarrow \infty$ with $\mu(\tilde{C}_{n,j_n}) \geq \delta$ (or $\mu(\hat{C}_{n,j_n}) \geq \delta$ when applicable) and $\mu(T^{t_n} I_{n,j_n} \cap I_{n,j_n}) \geq \delta \mu(I_{n,j_n})$ then (X, μ) is $\frac{1}{2} \delta^2$ -partially rigid.*

Proof. Let $A = W_{N,J}$ for some fixed N and J . Define $\alpha_n = \{0 \leq i < c_{n,j_n} - h_{N,J} : I_{n,j_n,i} \subseteq A\}$.

For j_n such that $h_{n,j_n} > \frac{1}{2} \ell_n$, if $x \in I_{n,j_n,i} \cap W_{N,J}$ then $x_{[-i, -i+h_{n,j_n})} = B_{n,j_n}$ and $x_{[0, h_{N,J})} = B_{N,J}$ meaning that $(B_{n,j_n})_{[i, i+h_{N,J})} = B_{N,J}$. This implies that $T^i W_{n,j_n} \subseteq W_{N,J}$ provided $i < h_{n,j_n} - h_{N,J}$.

For j_n such that $h_{n,j_n} \leq \frac{1}{2} \ell_n$, if $x \in I_{n,j_n,i} \cap W_{N,J}$ then $x_{[-i, -i+\ell_n/h_{n,j_n})} = B_{n,j_n}^{\ell_n/h_{n,j_n}}$ and $x_{[0, h_{N,J})} = B_{N,J}$ so $(B_{n,j_n}^{\ell_n/h_{n,j_n}})_{[i, i+h_{N,J})} = B_{N,J}$ which implies $I_{n,j_n,i} \subseteq W_{N,J}$ provided $i < h_{n,j_n} - h_{N,J}$.

Therefore $(|\alpha_n| + h_{N,J}) \mu(I_{n,j_n,0}) \geq \mu(A \cap \tilde{C}_{n,j_n}) \geq |\alpha_n| \mu(I_{n,j_n,0})$. Likewise, if $\|B_{n,j_n}\| \leq \frac{1}{2} \ell_n$ then $(|\alpha_n| + h_{N,J}) \mu(W_{n,j_n}) \geq \mu(A \cap \tilde{C}_{n,j_n}) \geq |\alpha_n| \mu(W_{n,j_n})$ using $\alpha_n = \{0 \leq i < h_{n,j_n} - h_{N,J} : T^i W_{n,j_n} \subseteq A\}$.

For $m < c_{n,j_n}$, $\mu(T^m \tilde{C}_{n,j_n} \Delta \tilde{C}_{n,j_n}) \leq 2m \mu(I_{n,j_n,0})$, (and likewise $\mu(T^m \hat{C}_{n,j_n} \Delta \hat{C}_{n,j_n}) \leq 2m \mu(W_{n,j_n})$ when applicable) therefore

$$\int |\mathbb{1}_{\tilde{C}_{n,j_n}} \circ T^{-m} - \mathbb{1}_{\tilde{C}_{n,j_n}}|^2 d\mu = 2\mu(\tilde{C}_{n,j_n}) - 2\mu(T^m \tilde{C}_{n,j_n} \cap \tilde{C}_{n,j_n}) \leq 2m \mu(I_{n,j_n,0})$$

Therefore for $M < c_{n,j_n}$,

$$\begin{aligned} \left| \frac{1}{M} \sum_{m=1}^M \mu(T^{-m} A \cap \tilde{C}_{n,j_n}) - \mu(A \cap \tilde{C}_{n,j_n}) \right| &= \left| \frac{1}{M} \sum_{m=1}^M \mu(A \cap T^m \tilde{C}_{n,j_n}) - \mu(A \cap \tilde{C}_{n,j_n}) \right| \\ &\leq \frac{1}{M} \sum_{m=1}^M |\mu(A \cap T^m \tilde{C}_{n,j_n}) - \mu(A \cap \tilde{C}_{n,j_n})| \leq \frac{1}{M} \sum_{m=1}^M \int_A |\mathbb{1}_{\tilde{C}_{n,j_n}} \circ T^{-m} - \mathbb{1}_{\tilde{C}_{n,j_n}}| d\mu \\ &\leq \frac{1}{M} \sum_{m=1}^M \left(\int |\mathbb{1}_{\tilde{C}_{n,j_n}} \circ T^{-m} - \mathbb{1}_{\tilde{C}_{n,j_n}}|^2 d\mu \right)^{1/2} \leq \frac{1}{M} \sum_{m=1}^M \sqrt{2m \mu(I_{n,j_n,0})} \leq \sqrt{2M \mu(I_{n,j_n,0})} \end{aligned}$$

The mean ergodic theorem gives M such that $\int \left| \frac{1}{M} \sum_{m=1}^M \mathbb{1}_A \circ T^m - \mu(A) \right|^2 d\mu < (\frac{1}{4} \delta \mu(A))^2$ so

$$\begin{aligned} \left| \frac{1}{M} \sum_{m=1}^M \mu(T^{-m} A \cap \tilde{C}_{n,j_n}) - \mu(A) \mu(\tilde{C}_{n,j_n}) \right| &= \left| \int_{\tilde{C}_{n,j_n}} \frac{1}{M} \sum_{m=1}^M \mathbb{1}_A \circ T^m - \mu(A) d\mu \right| \\ &\leq \int_{\tilde{C}_{n,j_n}} \left| \frac{1}{M} \sum_{m=1}^M \mathbb{1}_A \circ T^m - \mu(A) \right| d\mu \leq \left(\int \left| \frac{1}{M} \sum_{m=1}^M \mathbb{1}_A \circ T^m - \mu(A) \right|^2 d\mu \right)^{1/2} < \frac{1}{4} \delta \mu(A) \end{aligned}$$

For n large enough that $c_{n,j_n} > M$ and $\sqrt{2M \mu(I_{n,j_n,0})} < \frac{1}{4} \delta \mu(A)$ (Proposition 5.18 states $\mu(I_{n,j_n,0}) \rightarrow 0$) then $|\mu(A \cap \tilde{C}_{n,j_n}) - \mu(A) \mu(\tilde{C}_{n,j_n})| < \frac{1}{2} \delta \mu(A)$. Then

$$\begin{aligned} \mu(T^{t_n} A \cap A) &\geq \mu(T^{t_n} (A \cap \tilde{C}_{n,j_n}) \cap (A \cap \tilde{C}_{n,j_n})) \geq \sum_{i \in \alpha_n} \mu(T^{t_n} T^i I_{n,j_n,0} \cap T^i I_{n,j_n,0}) \\ &= |\alpha_n| \mu(T^{t_n} I_{n,j_n,0} \cap I_{n,j_n,0}) \geq |\alpha_n| \delta \mu(I_{n,j_n,0}) \geq \delta (\mu(A \cap \tilde{C}_{n,j_n}) - h_{N,J} \mu(I_{n,j_n,0})) \\ &> \delta \left(\mu(A) \mu(\tilde{C}_{n,j_n}) - \frac{1}{2} \delta \mu(A) \right) - \delta h_{N,J} \mu(I_{n,j_n,0}) \end{aligned}$$

$$\geq \delta \left(\mu(A)\delta - \frac{1}{2}\delta\mu(A) \right) - \delta h_{N,J}\mu(I_{n,j_n,0}) = \frac{1}{2}\delta^2\mu(A) - \delta h_{N,J}\mu(I_{n,j_n,0})$$

with the same applying to \hat{C}_{n,j_n} when applicable. Therefore for fixed N and J and $0 \leq i < h_{N,J}$,

$$\liminf \mu(T^{t_n} T^i W_{N,J} \cap T^i W_{N,J}) = \liminf \mu(T^{t_n} W_{N,J} \cap W_{N,J}) \geq \frac{1}{2}\delta^2\mu(W_{N,J}) = \frac{1}{2}\delta^2\mu(T^i W_{N,J})$$

and since the sets $T^i W_{N,J}$ generate the Borel algebra, μ is $\frac{1}{2}\delta^2$ -partially rigid. \square

Proof of Theorem 5.1. We aim to apply Proposition 5.24. Set $\delta = \frac{1}{4k^2 C^{C+1}}$ which depends only on X .

There exists a_0 such that $\mu(C_{n,a_0}) \geq \frac{1}{C}$ since $X_0 = \bigcup_j C_{n,j}$. If $\|B_{n,a_0}\| \leq \frac{1}{2}\ell_n$ then $\mu(\hat{C}_{n,a_0}) = \mu(C_{n,a_0}) \geq \frac{1}{C}$ and Proposition 5.23 implies $\mu(T^{h_{n,a_0}} I_{n,a_0,0} \cap I_{n,a_0,0}) \geq \frac{1}{2}\mu(I_{n,a_0,0})$ so take $t_n = h_{n,a_0}$ and $j_n = a_0$.

Now consider when $\|B_{n,a_0}\| > \frac{1}{2}\ell_n$ so Proposition 5.17 implies $\mu(\tilde{C}_{n,a_0}) \geq \frac{1}{2k}\mu(C_{n,a_0}) \geq \frac{1}{2kC}$.

By Proposition 5.20, there exists a_1 such that $\mu(T^{h_{n,a_0}} W_{n,a_0} \cap W_{n,a_1}) \geq \frac{1}{C}\mu(W_{n,a_0})$. If $a_1 = a_0$ then $\mu(\tilde{C}_{n,a_1}) = \mu(\tilde{C}_{n,a_0}) \geq \frac{1}{2kC}$ and if $a_1 \neq a_0$ then Proposition 5.22 implies $\mu(\tilde{C}_{n,a_1}) \geq \frac{1}{2k}\mu(\tilde{C}_{n,a_0}) \geq \frac{1}{4k^2 C}$.

Proposition 5.20 then says there exists a_2 such that

$$\mu(T^{h_{n,a_1}} (T^{h_{n,a_0}} W_{n,a_0} \cap W_{n,a_1}) \cap W_{n,a_2}) \geq \frac{1}{C}\mu(T^{h_{n,a_0}} W_{n,a_0} \cap W_{n,a_1}) \geq \frac{1}{C^2}\mu(W_{n,a_0})$$

and then Proposition 5.22 gives $\mu(\tilde{C}_{n,a_2}) \geq \frac{1}{C^2} \frac{1}{2k}\mu(\tilde{C}_{n,a_0}) \geq \frac{1}{4k^2 C^3}$.

Repeating this process, we obtain a_ℓ for $0 \leq \ell \leq C$ such that $\mu(\tilde{C}_{n,a_\ell}) \geq \frac{1}{4k^2 C^{\ell+1}} \geq \frac{1}{4k^2 C^{C+1}}$ and

$$\mu(W_{n,a_C} \cap \bigcap_{\ell=0}^{C-1} T^{\sum_{z=\ell}^{C-1} h_{n,a_z}} W_{n,a_\ell}) \geq \frac{1}{C^C}\mu(W_{n,a_0})$$

If any of the a_ℓ are such that $h_{n,a_\ell} \leq \frac{1}{2}\ell_n$ then Proposition 5.23 implies $\mu(T^{h_{n,a_\ell}} I_{n,a_\ell,0} \cap I_{n,a_\ell,0}) \geq \frac{1}{2}\mu(I_{n,a_\ell,0})$ so take $t_n = h_{n,a_\ell}$ and $j_n = a_\ell$.

If $h_{n,a_\ell} > \frac{1}{2}\ell_n$ for all $0 \leq \ell \leq C$ then, since there are at most C choices of j , for some $q < s$ we must have $a_q = a_s$ so setting $j_n = a_q$ and $t_n = \sum_{z=q}^{s-1} h_{n,a_z}$,

$$\mu(T^{t_n} I_{n,j_n,0} \cap I_{n,j_n,0}) = \mu(T^{\sum_{z=q}^{s-1} h_{n,a_z}} W_{n,a_q} \cap W_{n,a_s}) \geq \mu(W_{n,a_C} \cap \bigcap_{\ell=0}^{C-1} T^{\sum_{z=\ell}^{C-1} h_{n,a_z}} W_{n,a_\ell}) \geq \frac{1}{C^C}\mu(W_{n,a_0})$$

$$\begin{aligned} \text{As } \mu(W_{n,a_0}) &= \mu(I_{n,a_0,0}) = \frac{1}{c_{n,a_0}}\mu(\tilde{C}_{n,a_0}) \geq \frac{1}{h_{n,a_0}} \frac{1}{2kC} \geq \frac{1}{k\ell_n} \frac{1}{2kC} \geq \frac{1}{k\ell_n} \frac{1}{2kC}\mu(\tilde{C}_{n,j_n}) \\ &= \frac{1}{k\ell_n} \frac{1}{2kC} c_{n,j_n} \mu(I_{n,j_n,0}) \geq \frac{1}{k\ell_n} \frac{1}{2kC} \frac{\ell_n}{2} \mu(I_{n,j_n,0}) = \frac{1}{4k^2 C} \mu(I_{n,j_n,0}) \end{aligned}$$

we then have $\mu(T^{t_n} I_{n,j_n,0} \cap I_{n,j_n,0}) \geq \frac{1}{4k^2 C^{C+1}} \mu(I_{n,j_n,0})$.

In all cases, by Proposition 5.24, we have that (X, μ, T) is $\frac{1}{2}\delta^2$ -partially rigid. \square

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