Math 31B Midterm #2 Solutions

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1. For each of the following integrals, compute the integral or show that it diverges.

$$\int_0^1 2x\ln(x)dx$$

Solution. Since the function $x\ln(x)$ is not well-defined at 0 this is an improper integral and so

$$\int_0^1 2x\ln(x)dx = \lim_{R\to 0^+} \int_R^1 2x\ln(x)dx$$

To evaluate that integral we proceed by parts and set

$$u = \ln(x) \qquad dv = 2xdx$$

and so

$$du = \frac{dx}{x} \qquad v = x^2$$

and therefore

$$\begin{aligned}
\int_R^1 2x\ln(x)dx &= \int_R^1 udv \\
&= uv\big|_{x=R}^{x=1} - \int_R^1 vdu \\
&= x^2\ln(x)\big|_{x=R}^{x=1} - \int_R^1 x^2\frac{dx}{x} \\
&= 1^2\ln(1) - R^2\ln(R) - \int_R^1 xdx \\
&= 0 - R^2\ln(R) - \frac{1^2}{2} + \frac{R^2}{2} \\
&= -\frac{1}{2} + \frac{1}{2}R^2 - R^2\ln(R)
\end{aligned}$$

Taking the limit as $R \to 0^+$ we find that

$$\lim_{R\to 0^+} \frac{1}{2}R^2 = 0$$

and by L'Hopital's Rule (since $R^2 \ln(R) = 0(-\infty)$ is indeterminate that

$$\lim_{R \to 0^+} R^2 \ln(R) = \lim_{R \to 0^+} \frac{\ln(R)}{R^{-2}} = \lim_{R \to 0^+} \frac{R^{-1}}{-2R^{-3}} = \lim_{R \to 0^+} \frac{R^2}{-2} = 0$$

and plugging this back in

$$\int_0^1 2x \ln(x) dx = -\frac{1}{2} + 0 - 0 = -\frac{1}{2}$$

□

$$\int_0^\infty \frac{x dx}{x^2 + 1}$$

Solution. When $x \geq 1$ we see that

$$\frac{x}{x^2+1} \geq \frac{x}{2x^2} = \frac{1}{2x}$$

and by the Theorem on improper integrals of powers

$$\int_1^\infty \frac{dx}{2x} = \frac{1}{2} \int_1^\infty \frac{dx}{x} = \infty$$

which is to say it diverges. Then by the Convergence Test

$$\int_1^\infty \frac{x dx}{x^2+1} = \infty$$

also diverges.

Since the integrand is nonnegative on $x \geq 0$

$$\int_0^\infty \frac{x dx}{x^2+1} \geq \int_1^\infty \frac{x dx}{x^2+1} = \infty$$

diverges. □

2. Compute the following sums:

$$\sum_{n=0}^\infty \frac{2 + 3^{2n} - 2^{n+2}}{10^n}$$

Solution. Using the linearity of sums

$$\sum_{n=0}^{\infty} \frac{2+3^{2n}-2^{n+2}}{10^n} = 2\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n + \sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n + 4\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n$$

since by the geometric series formula each of these is summable so

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{2+3^{2n}-2^{n+2}}{10^n} &= 2\frac{1}{1-\frac{1}{10}} + \frac{1}{1-\frac{9}{10}} - 4\frac{1}{1-\frac{1}{5}} \\ &= 2\frac{10}{9} + 10 - 4\frac{5}{4} = \frac{65}{9} \end{aligned}$$

□

$$\sum_{n=2}^{\infty} \frac{2}{n(n+2)}$$

Solution. First we realize that

$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$$

and therefore the series is telescoping and so

$$\begin{aligned} \sum_{n=2}^{N} \frac{2}{n(n+2)} &= \sum_{n=2}^{N} \frac{1}{n} - \frac{1}{n+2} \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8} + \cdots + \frac{1}{N} - \frac{1}{N+2} \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{N+1} - \frac{1}{N+2} \end{aligned}$$

Taking the limit $N \to \infty$ we obtain $\frac{5}{6}$.

□

3. Compute the following integral:

$$\int \frac{5x^2+4x+3}{x^3+x^2+x} dx$$

Solution. First we factor the denominator as $x(x^2+x+1)$ and find a partial fraction expansion

$$\frac{5x^2+4x+3}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

for some A and B and C. Clearing denominators we obtain

$$5x^2 + 4x + 3 = A(x^2 + x + 1) + (Bx + C)x = (A + B)x^2 + (A + C)x + A$$

and therefore

$$A + B = 5 \qquad A + C = 4 \qquad A = 3$$

which means $B = 2$ and $C = 1$.

Therefore

$$\begin{aligned} \int \frac{5x^2 + 4x + 3}{x^3 + x^2 + x} dx &= \int \frac{3}{x} + \frac{2x + 1}{x^2 + x + 1} dx \\ &= 3 \ln |x| + \int \frac{du}{u} \\ &= 3 \ln |x| + \ln |u| + C \\ &= 3 \ln |x| + \ln |x^2 + x + 1| + C \end{aligned}$$

□

4. Let $T_n(x)$ be the Taylor Polynomials around $a = \pi$ for the function

$$f(x) = \sin(x) + 2\cos(x)$$

(a) Compute $T_4(x)$ (you may leave factors of $x - \pi$).
(b) Find n such that $|f(\pi - 1) - T_n(\pi - 1)| \leq 10^{-3}$.

Solution. First we need to compute derivatives:

$$\begin{aligned} f(x) &= \sin(x) + 2\cos(x) \\ f'(x) &= \cos(x) - 2\sin(x) \\ f''(x) &= -\sin(x) - 2\cos(x) \\ f'''(x) &= -\cos(x) + 2\sin(x) \\ f^{(4)}(x) &= \sin(x) + 2\cos(x) \end{aligned}$$

And then we plug in $a = \pi$:

$$f(\pi) = -2 \quad f'(\pi) = -1 \quad f''(\pi) = 2 \quad f'''(\pi) = 1 \quad f^{(4)}(\pi) = -2$$

and so

$$\begin{aligned} T_4(x) &= f(\pi) + f'(\pi)(x - \pi) + \frac{f''(\pi)}{2}(x - \pi)^2 + \frac{f'''(\pi)}{3!}(x - \pi)^3 + \frac{f^{(4)}(\pi)}{4!}(x - \pi)^4 \\ &= -2 - (x - \pi) + \frac{2}{2}(x - \pi)^2 + \frac{1}{6}(x - \pi)^3 - \frac{2}{24}(x - \pi)^4 \\ &= -2 - (x - \pi) + (x - \pi)^2 + \frac{1}{6}(x - \pi)^3 - \frac{1}{12}(x - \pi)^4 \end{aligned}$$

The Error Bound for $T_n(x)$ around π is

$$|f(x) - T_n(x)| \leq K\frac{|x-\pi|^{n+1}}{(n+1)!}$$

where K is such that $|f^{(n+1)}(u)| \leq K$ for all u between x and π. In our case we have $x = \pi - 1$ and for any u

$$|f^{(n+1)}(u)| \leq 3$$

since it looks like either a sin or a cos and then twice the other. So we can use $K = 3$. Then

$$|f(\pi-1) - T_n(\pi-1)| \leq 3\frac{(\pi-\pi+1)^{n+1}}{(n+1)!} = \frac{3}{(n+1)!}$$

and therefore if we take $n = 11$ we will certainly be less than 10^{-3}. □

5. Compute the arc length of the curve $f(x) = -\ln(\cos(x))$ over $[0, \frac{\pi}{4}]$.

Solution. First we notice that

$$f'(x) = -\frac{-\sin(x)}{\cos(x)} = \tan(x)$$

and therefore

$$\begin{aligned}\sqrt{1+f'(x)^2} &= \sqrt{1+\tan^2(x)} \\ &= \sqrt{\sec^2(x)} = \sec(x)\end{aligned}$$

The arc length formula is

$$\begin{aligned}L &= \int_a^b \sqrt{1+f'(x)^2}dx \\ &= \int_0^{\frac{\pi}{4}} \sec(x)dx \\ &= \ln(\sec(\frac{\pi}{4}) + \tan(\frac{\pi}{4})) - \ln(\sec(0) + \tan(0)) \\ &= \ln(\sqrt{2}+1) - \ln(1+0) = \ln(\sqrt{2}+1)\end{aligned}$$

□