Stabilizers of Actions of Lattices in Products of Groups

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We prove that any ergodic nonatomic probability-preserving action of an irreducible lattice in a semisimple group, at least one factor being connected and higher-rank, is essentially free. This generalizes the result of Stuck and Zimmer [SZ94] that the same statement holds when the ambient group is a semisimple real Lie group and every simple factor is higher-rank.

We also prove a generalization of a result of Bader and Shalom [BS06] by showing that any probability-preserving action of a product of simple groups, at least one with property (T), which is ergodic for each simple subgroup is either essentially free or essentially transitive.

Our method involves the study of relatively contractive maps and the Howe-Moore property, rather than the relaying on algebraic properties of semisimple groups and Poisson boundaries, and introduces a generalization of the ergodic decomposition to invariant random subgroups of independent interest.

1 Introduction

Generalizing a particular case of Margulis' breakthrough work [Mar79] showing that irreducible lattices in higher-rank semisimple groups have no nontrivial infinite index normal subgroups, Nevo and Stuck and Zimmer [SZ94],[NZ99] showed that irreducible lattices in semisimple real Lie groups, each simple factor having higher-rank, admit no nonatomic actions that are not essentially free (if one takes the Bernoulli shift action of a lattice modulo a normal subgroup and treats it as an action of the lattice, the Nevo-Stuck-Zimmer result then recovers Margulis' result). However, the question of actions of lattices in semisimple groups in general (allowing p-adic and removing the higher-rank assumption) remained open.

Bader and Shalom [BS06], more recently, proved a normal subgroup theorem for irreducible lattices in products of simple nondiscrete groups: as with lattices in semisimple groups, the only nontrivial normal subgroups of an irreducible lattice in a product of simple nondiscrete groups are all of finite index. While the methods of Bader and Shalom do provide information about the actions of products of groups (specifically, they obtain that if a product of two groups, both with property (T), acts on a probability space in such a way that each simple factor acts ergodically then the action is either essentially free or essentially transitive), their methods do not yield information about the actions of lattices in products, leaving open the question about lattices in general semisimple groups.

Addressing this issue, the author and Peterson [CP13], introduced a new method for studying lattices in semisimple groups, based on the commensurator approach developed by the author and Shalom [CS12],[Cre11], and showed that actions of irreducible lattices in products of groups with the Howe-Moore property (in particular, semisimple groups), at least one with property (T), at least one totally disconnected and such that every connected (real) factor has property (T), also only admit essentially free actions on nonatomic probability spaces. However, the requirement of higher-rank (property (T)) remained.

Our purpose here is to present a new proof of the results of Nevo and Stuck and Zimmer and of Bader and Shalom (in particular, without making use of their factor theorems), and to make substantial progress on removing the higher-rank (property (T)) requirement. Unlike the methods in [SZ94] and [BS06], which focus on the Poisson boundary, and, in the case of Stuck and Zimmer, on algebraic properties of semisimple groups, we follow an approach much more in the spirit of that of [CP13] focusing on

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contractive spaces and the Howe-Moore property. Our work here, combined with the work of the author and Peterson in [CP13], yields the following:

Theorem (Corollary 9.7)**.** *Let $G = G_1, \times \cdots \times G_k$ be a product of at least two simple nondiscrete noncompact locally compact second countable groups with the Howe-Moore property, at least one with property (T) and such that if any of the G_j are connected then at least one connected G_j has property (T). Let $\Gamma < G$ be an irreducible lattice and let (X, ν) be a nonatomic ergodic probability-preserving Γ-space. Then $\Gamma \curvearrowright (X, \nu)$ has finite stabilizers.*

In particular, for semisimple groups we obtain:

Corollary (Corollary 9.9)**.** *Let G be a semisimple group with trivial center and no compact factors with at least one simple factor being a connected (real) Lie group with property (T) (of higher-rank for example). Let $\Gamma < G$ be an irreducible lattice and (X, ν) be a nonatomic ergodic measure-preserving Γ-space. Then $\Gamma \curvearrowright (X, \nu)$ is essentially free.*

Unlike the methods of [CP13], we do obtain information about the actions of the ambient groups. In particular, we sharpen the result of Bader and Shalom on actions of products of groups by removing the requirement that all groups have property (T) and instead only requiring one group to have it (we also remove the requirement that the groups without property (T) be simple):

Theorem (Corollary 7.7)**.** *Let G be a product of a simple locally compact second countable group with property (T) and an arbitrary locally compact second countable group and let (X, ν) be a faithful measure-preserving G-space that is ergodic for both groups. Then $G \curvearrowright (X, \nu)$ is either essentially free or essentially transitive.*

Moreover, when both groups do have property (T), our methods allow us to relax the requirement that each factor act ergodically:

Corollary (Corollary 9.5)**.** *Let G_j be locally compact second countable groups for $j = 1, \ldots, k$ with $k \geq 2$ each with property (T). Set $G = G_1 \times \cdots \times G_k$ and let (X, ν) be an ergodic measure-preserving G-space. Assume that there exist closed subgroups $H_j < G_j$ such that the spaces of $\prod_{\ell \neq j} G_\ell$-ergodic components is isomorphic to $(G_j/H_j, \mathrm{Haar})$ for each j and such that any nontrivial normal subgroup of H_j has finite index in H_j. Then either at least one $G_j \curvearrowright (X, \nu)$ essentially free or $G \curvearrowright (X, \nu)$ is essentially transitive.*

Even when the ambient group does not have any property (T) (for example, lattices in products of at least two rank-one groups), one can obtain some information about the action. If G is a product of at least two simple locally compact second countable groups or be a semisimple group with at least two factors and Γ is an irreducible lattice in G then any ergodic nonatomic probability-preserving action of Γ is either essentially free or is weakly amenable and any ergodic probability-preserving action of G that is ergodic for each simple factor of G is either essentially free or weakly amenable (Corollary 7.7 and Theorem 9.6 combined with the results in [CP13]). Note that weak amenability of the action implies that almost every stabilizer subgroup is coamenable (see Remark 2.24). We mention that the previous statement for semisimple real Lie groups (without any higher-rank assumption) and irreducible lattices in semisimple real Lie groups is implicit already in [SZ94].

1.1 Stabilizers of Actions and Random Subgroups

Invariant random subgroups are the natural setting for the study of stabilizers of probability-preserving actions. The study of stabilizers goes back at least to Moore, [AM66] Chapter 2, and Ramsay, [Ram71] Section 9 (see also Adams and Stuck [AS93] Section 4). Bergeron and Gaboriau [BG04] noticed the similarities between normal subgroups and invariant random subgroups and recently this has been the focus of much attention: [ABB$^+$11], [AGV12], [Bow12], [CP13], [DM12], [GS12], [Gri11], [TD12a], [TD12b], [Ver11], [Ver12]. Random subgroups play a key role in our technique, and we prove several results involving invariant random subgroups in general that are of independent interest.

Given a measure-preserving action of a group on a probability space, the pushforward of the measure to the space of closed subgroups obtained by mapping each point to its stabilizer subgroup gives rise to a conjugation-invariant probability measure on the space of subgroups, that is, gives rise to an invariant random subgroup. As shown in [AGV12] (for discrete groups) and [CP13] (for the locally compact case), every invariant random subgroup arises in this way. We generalize this result in two directions.

Firstly, we show that every quasi-invariant random subgroup (probability measure on the space of closed subgroups that is quasi-invariant under conjugation) arises as the stabilizers of a quasi-invariant action of the group. Secondly, we study the notion of subgroups of random subgroups (introduced in [CP13], we say that a random subgroup α is a subgroup of a random subgroup β when there exists a joining ρ of α and β such that for ρ-almost every (H, K) it holds that H is a subgroup of K). Given an equivariant map $(X, \nu) \to (Y, \eta)$ of G-spaces, the pushforward measure $\mathrm{stab}_*\nu$ is evidently a subgroup of $\mathrm{stab}_*\eta$. We show that such pairs of random subgroups always arise as the stabilizers of actions of quotient maps of G-spaces (Theorem 3.9).

Building on this, we introduce the notion of a free extension of an action by a random subgroup: given an action of a group G on a probability space (X, ν) and given a G-equivariant map φ from X to random subgroups of G such that $\varphi(x)$ is supported on subgroups of $\mathrm{stab}(x)$ almost everywhere, we show there exists an a G-space, the free extension of (X, ν) by φ, having stabilizers equal to $\varphi_*\nu$ such that (X, ν) is a quotient of this space in a canonical way. In particular, given a subgroup of $\mathrm{stab}_*\nu$, there is always an extension of (X, ν) having stabilizers given by that subgroup.

Crucial to our work here, we also introduce the notion of the quotient of a space by a random subgroup, a generalization of the ergodic decomposition for a normal subgroup. Recall that if N is a normal subgroup of G and (X, ν) is a G-space, one can define the space of N-ergodic components by considering the algebra of N-invariant functions. We generalize this to considering the "invariant functions for a random subgroup below (X, ν)" (defined precisely in section 4) and prove various universality properties of the quotient space. We then apply this quotienting procedure to actions of products of groups by considering the random subgroup obtained by taking the projections of the stabilizer groups to each factor. This functor, the product random subgroups functor (see section 4.6), allows us to study action of products of groups at a much finer level of detail than the ergodic decomposition functor used by Bader and Shalom.

1.2 The Contractive Factor Theorem for Products

The second major ingredient in our work is a factor theorem for actions of products of groups based on the notion of relatively contractive maps introduced in [CP13]. Our factor theorem generalizes the Bader-Shalom factor theorem and allows us to study actions of products of groups which are not necessarily ergodic when restricted to each factor:

Theorem (Theorem 6.1)**.** *Let $G = G_1 \times G_2$ be a product of two locally compact second countable groups and let $\mu_j \in P(G_j)$ be admissible probability measures for $j = 1, 2$. Set $\mu = \mu_1 \times \mu_2$.*

Let (B, β) be the Poisson boundary for (G, μ) and let (X, ν) be a measure-preserving G-space. Let (W, ρ) be a G-space such that there exist G-maps $\pi : (B \times X, \beta \times \nu) \to (W, \rho)$ and $\varphi : (W, \rho) \to (X, \nu)$ with $\varphi \circ \pi$ being the natural projection to X.

Let (W_1, ρ_1) be the space of G_2-ergodic components of (W, ρ) and let (W_2, ρ_2) be the space of G_1-ergodic components. Likewise, let (X_1, ν_1) and (X_2, ν_2) be the ergodic components of (X, ν) for G_2 and G_1, respectively.

Then (W, ρ) is G-isomorphic to the independent relative joining of $(W_1, \rho_1) \times (W_2, \rho_2)$ and (X, ν) over $(X_1, \nu_1) \times (X_2, \nu_2)$.

The factor theorem of Bader and Shalom requires that X_1 and X_2 be trivial and can be phrased as saying that in that case W is always isomorphic to the independent joining of $W_1 \times W_2$ and X. Their theorem follows from a careful study of properties of the Poisson boundary. Our theorem, on the other hand, only makes use of two (easy) properties of the Poisson boundary: that it contractive and that it an amenable space. Replacing the study of boundary dynamics, we make use of a result about uniqueness of relatively contractive joinings which may be of independent interest:

Theorem (Corollary 5.3)**.** *Let G be a locally compact second countable group and let (X, ν), (Y, η), (Z, ζ) and (W, ρ) be G-spaces such that the following diagram of G-maps commutes:*

$$\begin{array}{ccc} (W,\rho) & \longrightarrow & (X,\nu) \\ \downarrow & & \downarrow \\ (Y,\eta) & \longrightarrow & (Z,\zeta) \end{array}$$

If the vertical maps are relatively measure-preserving and the horizontal maps are relatively contractive then (W, ρ) is isomorphic to the independent relative joining of (X, ν) and (Y, η) over (Z, ζ).

This contractive factor theorem allows us to study actions of lattices in products of groups by inducing the action to the ambient group and considering intermediate factors. Since in general the induced action will not be ergodic for each factor, our theorem allows us to study lattices where the Bader-Shalom theorem does not. Combining the induced action with the "projected action" (see section 8.1) obtained by considering the stabilizers of the original action of the lattice and projecting them to each factor and then taking free extensions by the corresponding random subgroups, we obtain enough information about the stabilizers of the induced action to study the stabilizers of the original action.

1.3 Relaxing the Property (T) Requirement

The third major ingredient in our work is the use of a type of relative property (T), in the form of resolutions (introduced by de Cornulier [dC05]), to relax the requirement (present in all previous work on the subject of actions of semisimple groups and lattices) that every simple factor have property (T). In the work of Stuck and Zimmer, Bader and Shalom, and the author and Peterson, the requirement of the ambient group having property (T), and therefore the lattice also having property (T), was a necessary step in moving from knowing the equivalence relation of an action is amenable to knowing the action is essentially transitive.

We develop a new approach to the study of actions generating an amenable orbit equivalence relation when the group involved does not have property (T) but has "some" property (T) (in the case of products, one factor having property (T) and in the case of lattices, admitting a resolution which in turn comes from one factor having property (T)). As an example, we obtain the following statement:

Theorem (Theorem 7.6)**.** *Let $G = G_1 \times G_2$ be a product of two locally compact second countable groups such that G_2 has property (T). Let (X, ν) be an ergodic measure-preserving G-space such that $G \curvearrowright (X, \nu)$ weakly amenably and not essentially transitively. Let $\mathcal{H}$ be the subspace of $L^2(X, \nu)$ consisting of the G_2-invariant functions that are not G-invariant. Then there exists a sequence of almost invariant vectors in $\mathcal{H}$.*

The previous theorem immediately implies that if both G_1 and G_2 act ergodically and G acts weakly amenably then the action is essentially transitive. The same ideas allow us to conclude a similar result for actions of lattices in such products.

2 Preliminaries

2.1 G-Spaces and G-Maps

Definition 2.1. Let G be a locally compact second countable group. A **G-space** is a probability space (X, ν) equipped with an action of G such that ν is quasi-invariant under the action (the class of null sets is preserved by the action). This will be written $G \curvearrowright (X, \nu)$.

Definition 2.2. Let G be a locally compact group and $G \curvearrowright (X, \nu)$ a G-space. The **translate of ν by** $g \in G$ is the probability measure $g\nu$ defined by $g\nu(E) = \nu(g^{-1}E)$ for all measurable sets E. If $\mu \in P(G)$ is a probability measure on G, the **convolution of ν by μ** is the probability measure $\mu * \nu \in P(X)$ given by

$$\mu * \nu(E) = \int_G g\nu(E)\; d\mu(g) = \int_G \nu(g^{-1}E)\; d\mu(g).$$

Definition 2.3. Let G be a locally compact second countable group and (X, ν) a G-space. Then (X, ν) is **measure-preserving** when $g\nu = \nu$ for all $g \in G$. If $\mu \in P(G)$ is a probability measure on G such that $\mu * \nu = \nu$ then (X, ν) is **μ-stationary**.

Definition 2.4. Let G be a locally compact second countable group and let (X, ν) and (Y, η) be G-spaces. A measurable map $\pi : X \to Y$ such that $\pi_*\nu = \eta$ is a **G-map** when π is G-equivariant: $\pi(gx) = g\pi(x)$ for all $g \in G$ and almost every $x \in X$ (here $\pi_*\nu$ is the **pushforward measure** defined by, for E a measurable subset of Y, $\pi_*\nu(E) = \nu(\pi^{-1}(E))$).

Definition 2.5. Let G be a locally compact second countable group and $\pi : (X, \nu) \to (Y, \eta)$ a G-map of G-spaces. The **disintegration** of ν over η is the almost everywhere unique map $D_\pi : Y \to P(X)$ such that the support of $D_\pi(y)$ is contained in $\pi^{-1}(y)$ and that $\int_Y D_\pi(y)\, d\eta(y) = \nu$.

Definition 2.6. Let G be a locally compact second countable group and $\pi : (X, \nu) \to (Y, \eta)$ a G-map of G-spaces. Then π is **relatively measure-preserving** when the disintegration of ν over η via π, $D_\pi : Y \to P(X)$, is G-equivariant: $D_\pi(gy) = gD_\pi(y)$ for all $g \in G$ and almost every $y \in Y$.

We also need the following well-known characterization of relatively measure-preserving:

Theorem 2.7. *Let G be a locally compact second countable group and $\pi : (X, \nu) \to (Y, \eta)$ a G-map of G-spaces. Then π is relatively measure-preserving if and only if the Radon-Nikodym derivatives satisfy $\frac{dg\nu}{d\nu}(x) = \frac{dg\eta}{d\eta}(\pi(x))$ almost everywhere.*

Proof. By uniqueness of the Radon-Nikodym derivative,

$$\frac{dg\nu}{d\nu}(x) = \frac{dgD_\pi(g^{-1}\pi(x))}{dD_\pi(\pi(x))}(x)\frac{dg\eta}{d\eta}(\pi(x)).$$

Therefore $\frac{dg\nu}{d\nu}(x) = \frac{dg\eta}{d\eta}(\pi(x))$ if and only if $\frac{dgD_\pi(g^{-1}\pi(x))}{dD_\pi(\pi(x))}(x) = 1$ almost surely which says precisely that π is relatively measure-preserving. $\square$

Definition 2.8. Let G be a locally compact second countable group and (X, ν) a G-space. Let $\mathcal{F} \subseteq L^\infty(X, \nu)$ be a closed G-invariant subalgebra. A **point realization** or **Mackey point realization** of $\mathcal{F}$ is a G-space (Y, η) where there exists a G-map $\pi : (X, \nu) \to (Y, \eta)$ such that $\mathcal{F} = \{f \circ \pi : f \in L^\infty(Y, \eta)\}$.

Theorem 2.9 (Mackey [Mac62], [Mac66])**.** *Let G be a locally compact second countable group and (X, ν) a G-space. Let $\mathcal{F} \subseteq L^\infty(X, \nu)$ be a closed G-invariant subalgebra. Then there exists a point realization of $\mathcal{F}$.*

2.2 Stabilizers of Actions

Definition 2.10. Let G be a locally compact second countable group and (X, ν) a G-space. The **stabilizer subgroups** are $\mathrm{stab}(x) = \{g \in G : gx = x\}$ for each $x \in X$.

Definition 2.11. Let G be a locally compact second countable group and (X, ν) a G-space. Then $G \curvearrowright (X, \nu)$ is **essentially free** when $\mathrm{stab}(x) = \{e\}$ for almost every $x \in X$.

Definition 2.12. Let G be a locally compact second countable group and (X, ν) a G-space. Then $G \curvearrowright (X, \nu)$ is **faithful** when the kernel of the action is trivial.

Definition 2.13. Let G be a locally compact second countable group and (X, ν) a G-space. Then $G \curvearrowright (X, \nu)$ is **essentially transitive** when there exists $x_0 \in X$ such that $\nu(G \cdot x_0) = 1$ (a full measure orbit).

Definition 2.14. Let G be a locally compact second countable group and $\pi : (X, \nu) \to (Y, \eta)$ a G-map of G-spaces. Then π is **orbital** when $\mathrm{stab}(x) = \mathrm{stab}(\pi(x))$ almost everywhere.

2.3 Weakly Amenable Actions

Introduced by Zimmer [Zim77], the notion of weakly amenable actions will play a crucial role in our study of the stabilizers of actions of groups.

Definition 2.15. Let $G \curvearrowright (X, \nu)$ be a quasi-invariant action of a locally compact second countable group. Let E be a separable Banach space and write E_1^* for the unit ball in the dual of E. Let $\alpha : G \times X \to \mathrm{Iso}(E)$ be a cocycle. Denote the dual cocycle α^* by $\alpha^*(g, x) = (\alpha(g, x)^{-1})^*$. Let $A_x \subseteq E_1^*$ be a closed convex nonempty set for almost every x such that $\alpha^*(g, x)A_{gx} = A_x$. Consider the space

$$A = \bigsqcup_x \{x\} \times A_x \subseteq X \times_{\alpha^*} E_1^*$$

endowed with the α^*-twisted action. This is a closed compact space which is G-invariant under the α^*-twisted action. Such a space A is called an **affine G-space over** (X, ν).

Definition 2.16. The cocycle α is called **orbital** when $\alpha(g,x) = e$ for all $g \in \mathrm{stab}_G(x)$ for almost every x. The affine G-space A is called an **orbital affine** G**-space** when α is orbital.

Definition 2.17. Let G be a locally compact second countable group and (X,ν) a G-space. $G \curvearrowright (X,\nu)$ is **amenable** when for every affine G-space A over (X,ν) there exists an α^*-invariant function $f : X \to E_1^*$ such that $f(x) \in A_x$ for almost every x: α^*-invariant means $f(x) = \alpha^*(g,x)f(gx)$. $G \curvearrowright (X,\nu)$ is **weakly amenable** when that condition holds for all orbital affine G-spaces over (X,ν).

Proposition 2.3.1 (Stuck-Zimmer [SZ94])**.** *Let G be a locally compact second countable group, (X,ν) a G-space and (B,β) an amenable G-space. Let A be an affine G-space over X. Then there exists G-maps*

$$(B \times X, \beta \times \nu) \to (A,\alpha) \to (X,\nu)$$

such that the composition is the natural projection to X and α is the pushforward of $\beta \times \nu$.

The proof of the previous statement is implicit in [SZ94]; the reader is referred to [CP13] for a concrete proof.

2.4 Amenable Equivalence Relations

The notion of amenability for equivalence relations was introduced by Zimmer in [Zim77]. The reader is also referred to Kechris and Miller [KM04] for more detailed information.

Definition 2.18. Let G be a locally compact second countable group and (X,ν) be a measure-preserving G-space. The **orbit equivalence relation** generated by the action is given by R where xRy when there exists $g \in G$ such that $gx = y$.

Definition 2.19. Let (X,ν) be a measure space. An equivalence relation $R \subseteq X \times X$ is **measurable** when there exists a σ-finite measure ρ on R such that the projection $R \to X$ sends ρ to ν.

When G is a locally compact second countable group and (X,ν) a measure-preserving G-space, the orbit equivalence relation generated by the action is measurable: let $R = \{(x,gx) : x \in X, g \in G\}$ be the equivalence relation generated by the action and let m be a Haar measure on G and consider the map $p : X \times G \to R$ given by $p(x,g) = (x,gx)$; then $\rho = p_*(\nu \times m)$ is σ-finite and makes R measurable.

Definition 2.20. Let (X,ν) be a σ-finite measure space. A **mean** on (X,ν) is a linear functional $m \in L^\infty(X,\nu)^*$ that is positive and has $m(\mathbb{1}) = 1$.

Definition 2.21. Let (X,ν) be a measure space and $R \subseteq X \times X$ be a measurable equivalence relation. A map $m : x \mapsto m_x$ is a **mean** on R when for almost every $x \in X$, m_x is a mean on $[x]$, the equivalence class of x, and the map $x \mapsto m_x$ is measurable in the sense that for any $F \in L^\infty(R)$, writing $F_x : [x] \to \mathbb{R}$ by $F_x(y) = F(x,y)$, it holds that $x \mapsto m_x(F_x)$ is a measurable map.

Definition 2.22. Let (X,ν) be a measure space and $R \subseteq X \times X$ be a measurable equivalence relation. A map $m : x \mapsto m_x$ is an **invariant mean** when it is a mean such that $m_x = m_y$ for almost every $x \in X$ and all $y \in [x]$.

Definition 2.23. The equivalence relation $R_{G\curvearrowright(X,\nu)}$ is **amenable** when there exists an invariant mean for $R_{G\curvearrowright(X,\nu)}$.

Remark 2.24. *Recall that a subgroup $H < G$ is said to be* **coamenable** *in G when there is a G-invariant mean on G/H. If $G \curvearrowright (X,\nu)$ gives rise to an amenable equivalence relation then for almost every $x \in X$, the stabilizer subgroup* $\mathrm{stab}(x)$ *is coamenable in G (since the orbit $[x]$ is isomorphic to $G/\mathrm{stab}(x)$).*

Theorem 2.25 (Zimmer [Zim77])**.** *Let G be a locally compact second countable group and (X,ν) be a measure-preserving G-space. Then the orbit equivalence relation of $G \curvearrowright (X,\nu)$ is amenable if and only if $G \curvearrowright (X,\nu)$ is weakly amenable.*

Theorem 2.26 (Connes-Feldman-Weiss [CFW81])**.** *Let G be a locally compact second countable group and (X,ν) be an ergodic measure-preserving G-space. Then the orbit equivalence relation of $G \curvearrowright (X,\nu)$ is amenable if and only if $G \curvearrowright (X,\nu)$ is orbit equivalent to a free ergodic action of $\mathbb{R}$ or $\mathbb{Z}$ depending on whether G is discrete (two actions $G \curvearrowright (X,\nu)$ and $H \curvearrowright (Y,\eta)$ are orbit equivalent when there exists a measure-space isomorphism $\theta : (X,\nu) \to (Y,\eta)$ such that for all $g \in G$ and almost every $x \in X$ there exists $h \in H$ such that $\theta(gx) = h\theta(x)$).*

Proposition 2.4.1. *Let G be a locally compact second countable group and $\pi : (Y, \eta) \to (X, \nu)$ be a G-map of G-spaces such that $G \curvearrowright (X, \nu)$ is weakly amenable and π is orbital. Then $G \curvearrowright (Y, \eta)$ weakly amenably.*

Proof. Let $R_Y = \{(y, gy) : y \in Y, g \in G\}$ and $R_X = \{(x, gx) : x \in X, g \in G\}$ be the equivalence relations of the actions. Define the set $S = \{(y, g\pi(y)) : y \in Y, g \in G\} \subseteq Y \times X$. For each $y \in Y$, the map $[y] \to [\pi(y)]$ given by $gy \mapsto g\pi(y)$ is one-one since π is orbital. Then the map $\psi : S \to R_Y$ by $\psi(y, g\pi(y)) = (y, gy)$ is a well-defined measurable map. Let m be an invariant mean on R_X. Define a map $M : y \mapsto M_y$ as follows: for $F \in L^\infty(R_Y)$ consider $F \circ \psi : S \to \mathbb{R}$ and write $(F \circ \psi)_y : [\pi(y)] \to \mathbb{R}$ as $(F \circ \psi)_y(g\pi(y)) = F(\psi(y, g\pi(y))) = F(y, gy) = F_y(gy)$. Define $M_y(F_y) = m_{\pi(y)}((F \circ \psi)_y)$. Then $y \mapsto M_y(F_y)$ is measurable since $y \mapsto \pi(y) \mapsto m_\pi(y)$ is measurable and ψ is measurable. Then M is a mean on R_Y. Also, $F_{gy} = F_y$ and $m_{\pi(gy)} = m_{g\pi(y)} = m_{\pi(y)}$ so M is invariant. Hence $G \curvearrowright (Y, \eta)$ is weakly amenable. $\square$

Proposition 2.4.2. *Let G_1 and G_2 be locally compact second countable groups and let (X_1, ν_1) be a weakly amenable G_1-space and (X_2, ν_2) a weakly amenable G_2-space. Then $(X_1 \times X_2, \nu_1 \times \nu_2)$ is a weakly amenable $G_1 \times G_2$-space (with the product action).*

Proof. By Theorem 2.25, there exist measurable maps on X_1 and X_2, written $x_1 \mapsto m_{x_1}$ and $x_2 \mapsto m_{x_2}$, such that m_{x_j} is a mean on $L^\infty([x_j])$ where $[x_j]$ is the G_j-orbit of x_j and such that $m_{y_j} = m_{x_j}$ for all $y_j \in [x_j]$. Define the map $(x_1, x_2) \mapsto m_{x_1,x_2}$ by defining $m_{x_1,x_2}(f_1 \times f_2) = m_{x_1}(f_1)m_{x_2}(f_2)$ and extending by continuity and linearity to all of $L^\infty(X_1 \times X_2, \nu_1 \times \nu_2)$. Then m_{x_1,x_2} are means on $[x_1, x_2]$ and the map is measurable and invariant under the orbit of $G_1 \times G_2$. So by Theorem 2.25 and Proposition 2.4.2, the claim follows. $\square$

2.5 Irreducible Lattices

Definition 2.27. Let G be a locally compact second countable group. A subgroup $\Gamma < G$ is a **lattice** when it is discrete and has finite covolume (there exists an open set $F \subseteq G$ such that $F\Gamma = G$, $F \cap \Gamma = \{e\}$ and $\mathrm{Haar}(F) < \infty$).

Definition 2.28. A lattice Γ in a locally compact second countable group G is **irreducible** when for any noncentral closed normal subgroup $M \lhd G$ that is not cocompact, $\Gamma/(\Gamma \cap M)$ is dense in G/M.

Central and cocompact normal subgroups are excepted in the definition to allow for cases such as $\mathrm{SL}_n(\mathbb{Z}) < \mathrm{SL}_n(\mathbb{R})$ with M being the center and cases such as $G = H \times K$ where K is compact and $\Gamma = \Gamma_0 \times \{e\}$ where Γ_0 is an irreducible lattice in H.

Proposition 2.5.1. *Let $\Gamma < G \times H$ be an irreducible lattice in a product of noncompact nondiscrete locally compact second countable groups. Then $\Gamma \cap (\{e\} \times H)$ is contained in $\{e\} \times Z(H)$ where $Z(H)$ is the center of H.*

Proof. Let $N = \Gamma \cap (\{e\} \times H)$. Then $N \lhd \Gamma$ since $\{e\} \times H \lhd G \times H$. Therefore $\overline{\mathrm{proj}_H N} \lhd \overline{\mathrm{proj}_H \Gamma}$. Since Γ is irreducible and G and H are not compact, $\overline{\mathrm{proj}_H \Gamma} = H$. On the other hand, $N \subseteq \{e\} \times H$ so write $N = \{e\} \times M$ for some $M \lhd H$ and observe that $\mathrm{proj}_H N = M$. Now M is discrete in H since Γ is discrete in $G \times H$. Hence $N \lhd G \times H$ is a discrete (hence closed) normal subgroup.

If N is central then $N < Z(G \times H) \cap (\{e\} \times H) = \{e\} \times Z(H)$. So we may assume N is noncentral. Note that N is not cocompact since G is noncompact. Therefore the projection of Γ to $(G \times H)/N$ is dense: Γ/N is dense in $(G \times H)/N$.

On the other hand, the quotient map $G \times H \to (G \times H)/N$ is an open map so if U is an open neighborhood of e in $G \times H$ such that $U \cap \Gamma = \{e\}$ then the image of U in $(G \times H)/N$ is an open neighborhood of the identity intersecting Γ/N only at the identity. Hence Γ/N is discrete in $(G \times H)/N$.

But then Γ/N is both dense and discrete in $(G \times H)/N$ hence $(G \times H)/N$ is discrete. As N is also discrete, this would mean that $G \times H$ is discrete, contradicting our hypotheses. $\square$

2.6 Induced Actions

Let G be a locally compact second countable group and $\Gamma < G$ a lattice. Given a Γ-space (X, ν), take F to be a fundamental domain for G/Γ with the normalized Haar measure m and define the G-space $G \times_{\Gamma} X$ to be $(F \times X, m \times \nu)$ with the action $g \cdot (f, x) = (gf\alpha(g, f), \alpha(g, f)^{-1}x)$ where $\alpha : G \times F \to \Gamma$ is the cocycle such that $gf\alpha(g, f) \in F$ for all $g \in G$ and $f \in F$. This construction is independent (up to G-isomorphism) of the fundamental domain chosen. The space $G \times_{\Gamma} X$ is the **induced action**. Note that it is measure-preserving when (X, ν) is measure-preserving.

Let $(f, x) \in F \times X$. Observe that $g \cdot (f, x) = (f, x)$ if and only if $gf\alpha(g, f) = f$ and $\alpha(g, f)^{-1}x = x$. Therefore

$$\mathrm{stab}_G(f, x) = f\mathrm{stab}_{\Gamma}(x)f^{-1}$$

for all $(f, x) \in F \times X$.

Proposition 2.6.1. *Let Γ be a lattice in a locally compact second countable group G and let (X, ν) be a Γ-space. Then $\Gamma \curvearrowright (X, \nu)$ weakly amenably if and only if $G \curvearrowright G \times_{\Gamma} X$ weakly amenably.*

Proof. Assume that $\Gamma \curvearrowright (X, \nu)$ weakly amenably. Then by Theorem 2.25, the orbit equivalence relation is amenable so there exists a measurable map $x \mapsto m_x$ such that m_x is a mean on $\ell^{\infty}[x]$ where $[x]$ is the Γ-orbit of a point x and such that $m_y = m_x$ for all $y \in [x]$.

Let F be a fundamental domain for G/Γ with cocycle $\alpha : G \times F \to \Gamma$ such that $gf\alpha(g, f) \in F$ and let ρ be the normalized Haar measure on F. Observe that the G-orbit of a point (f, x) is $G \cdot (f, x) = F \times [x]$. Given $q \in L^{\infty}(F \times [x], \rho \times \text{count})$ (where count is the counting measure on $[x]$), write $q_f(x) = q(f, x)$ to be the fiber of q over $f \in F$. Then $q_f \in \ell^{\infty}[x]$ for almost every $f \in F$. Define a mean $M_{f,x}$ on $L^{\infty}(F \times [x])$ by

$$M_{f,x}(q) = \int_F m_x(q_{f_0})\ d\rho(f_0).$$

One easily checks that $M_{f,x}(\mathbb{1}) = 1$ and that $M_{f,x} \geq 0$ since m_x is a mean. Observe that

$$M_{g\cdot(f,x)}(q) = M_{gf\alpha(g,f),\alpha(g,f)^{-1}x}(q) = \int_F m_{\alpha(g,f)^{-1}x}(q_{f_0})\ d\rho(f_0) = \int_F m_x(q_{f_0})\ d\rho(f_0) = M_{f,x}(q)$$

so $M_{f,x}$ is invariant. The map $(f, x) \mapsto M_{f,x}$ is measurable since $x \mapsto m_x$ is (and $M_{f,x}$ does not depend on f). Therefore the orbit equivalence relation of $G \curvearrowright G \times_{\Gamma} X$ is amenable hence the action is weakly amenable by Theorem 2.25.

Conversely, assume that $G \curvearrowright G \times_{\Gamma} X$ is weakly amenable. Let $(f, x) \mapsto M_{f,x}$ be an invariant mean. Define m_x by, for $q \in \ell^{\infty}[x]$, set $\widetilde{q}(f, x) = q(x) \in L^{\infty}(F \times [x])$ and set

$$m_x(q) = \int_F M_{f,x}(\widetilde{q})\ d\rho(f).$$

Then $x \mapsto m_x$ is measurable since $(f, x) \mapsto M_{f,x}$ is and m_x is a mean. Clearly for $\gamma \in \Gamma$,

$$m_{\gamma x}(q) = \int_F M_{f,\gamma x}(\widetilde{q})\ d\rho(f) = \int_F M_{(f\gamma f^{-1})\cdot(f,x)}(\widetilde{q})\ d\rho(f) = \int_F M_{f,x}(\widetilde{q})\ d\rho(f) = m_x(q)$$

using that $M_{f,x}$ is invariant under the G-action. Therefore $\Gamma \curvearrowright (X, \nu)$ has an amenable orbit equivalence relation hence acts weakly amenably. $\square$

2.7 The Howe-Moore Property

Definition 2.29 (Howe-Moore [HM79]). A locally compact second countable group G has the **Howe-Moore property** when every irreducible unitary representation of $\pi : G \to \mathcal{U}(\mathcal{H})$ without nontrivial invariant vectors has matrix coefficients vanishing at infinity: $\lim_{g\to\infty}\langle \pi(g)x, y\rangle = 0$ as g leaves compact sets for any $x, y \in \mathcal{H}$.

Theorem 2.30 (Schmidt [Sch84]). *Let G be a locally compact second countable group. Then G has the Howe-Moore property if and only if every ergodic G-space is mixing: if (X, ν) is a G-space then $\lim_{g\to\infty} \nu(gE \cap F) = \nu(E)\nu(F)$ as g leaves compact sets for all measurable $E, F \subseteq X$.*

The main result of [CP13] is:

Theorem 2.31 (Creutz-Peterson [CP13]). *Let G be a product of at least two simple nondiscrete noncompact locally compact second countable groups with the Howe-Moore property, at least one of which has property (T), at least one of which is totally disconnected and such that every connected simple factor has propertyy (T). Let $\Gamma < G$ be an irreducible lattice. Then any ergodic measure-preserving action of Γ has finite stabilizers almost surely or finite index stabilizers almost surely.*

2.8 Semisimple Groups

The main class of groups having the Howe-Moore property are the simple real and p-adic Lie groups. Semisimple groups are almost direct products of such groups and as such serve as a main example of our results. We remark that automorphism groups of regular trees also have the Howe-Moore property and so serve as another example.

Definition 2.32. A **semisimple group** is an almost direct product of simple real and p-adic Lie groups.

Theorem 2.33 (Rothman [Rot80]). *Let G be a simple connected locally compact second countable group with the Howe-Moore property. Then G is a simple real Lie group.*

Theorem 2.34 (Howe-Moore [HM79]). *Every simple real and p-adic Lie group has the Howe-Moore property.*

Theorem 2.35 (Zimmer [Zim82]). *Let G be a noncompact nondiscrete simple real Lie group and let (X,ν) be a nontrivial ergodic measure-preserving G-space. Let Λ be any countable subgroup of G. Then the restriction of the action to Λ on (X,ν) is essentially free.*

A direct, easy proof of the previous statement appears in [CP13] though it follows from the work of Zimmer in [Zim82].

2.9 Ergodic Decomposition

Given a G-space (X,ν), consider the G-invariant subalgebra of invariant functions $\mathcal{F} = \{f \in L^\infty(X,\nu) : g \cdot f = f \text{ for all } g \in G\}$ and let $(X/\!\!/G, \overline{\nu})$ be the Mackey point realization (Theorem 2.9) of this algebra. The space $X/\!\!/G$ is referred to as the **ergodic components** of $G \curvearrowright (X,\nu)$. Let $\pi : (X,\nu) \to (X/\!\!/G, \overline{\nu})$ be the quotient map. Then $(\pi^{-1}(y), D_\pi(y))$ is an ergodic G-space for each $y \in X/\!\!/G$ and the disintegration decomposition $\nu = \int_{X/\!\!/G} D_\pi(y)\, d\overline{\nu}(y)$ is the **ergodic decomposition**.

We remark that G acts trivially on $X/\!\!/G$ (an easy consequence of the construction: if it did not act trivially there would be some bounded Borel function on $X/\!\!/G$ that is not G-invariant but the algebra of bounded Borel functions on $X/\!\!/G$ consist only of invariant functions). From this, it is easy to see that each component $(\pi^{-1}(y), D_\pi(y))$ is an ergodic G-space.

We will need the following fact about ergodic decomposition in what follows:

Proposition 2.9.1. *Let (X,ν) be a G-space and (Y,η) be a G-space where G acts trivially. Then $(X \times Y)/\!\!/G = (X/\!\!/G) \times Y$.*

Proof. Write (Z,ζ) for the ergodic components of (X,ν) and $\pi : (X,\nu) \to (Z,\zeta)$ for the decomposition map. Let $\varphi : (X \times Y, \nu \times \eta) \to (Z \times Y, \zeta \times \eta)$ be given by $\varphi(x,y) = (\pi(x), y)$.

Let $E \subseteq X \times Y$ be a positive measure G-invariant set. For each $(z,y) \in Z \times Y$, define the set

$$E_{z,y} = E \cap (\pi^{-1}(z) \times \{y\}).$$

Then $E_{z,y}$ is a $D_\pi(z) \times \delta_y$-measurable set. Since G acts trivially on Y and Z, and $gE = E$ for all $g \in G$, we have that $gE_{z,y} = E_{z,y}$ for all $g \in G$. Now $D_\pi(z)$ is ergodic and δ_y is a point mass, hence $D_\pi(z) \times \delta_y$ is ergodic. Therefore for almost every $(z,y) \in Z \times Y$, either $E_{z,y}$ is null or has full measure.

Let

$$A = \{(z,y) \in Z \times Y : D_\pi(z) \times \delta_y(E_{z,y}) = 1\}$$

and observe that $D_\pi(z) \times \delta_y(E) = D_\pi(z) \times \delta_y(E_{z,y})$ since $D_\pi(z) \times \delta_y$ is supported on $\pi^{-1}(z) \times \{y\}$. Therefore, for $(z,y) \in A$ we have that $D_\pi(z) \times \delta_y(E \triangle \varphi^{-1}(A)) = 0$ since both E and $\varphi^{-1}(A)$ have full $D_\pi(z) \times \delta_y$-measure. On the other hand, for $(z,y) \notin A$ we also have that $D_\pi(z) \times \delta_y(E \triangle \varphi^{-1}(A)) = 0$ since both sets are null.

Therefore

$$\nu \times \eta(E \triangle \varphi^{-1}(A)) = \int_{Z \times Y} D_\pi(z) \times \delta_y(E \triangle \varphi^{-1}(A)) \; d\zeta \times \eta(z,y) = 0$$

meaning that any G-invariant positive measure set in $(X \times Y, \nu \times \eta)$ belongs to the algebra of measurable sets of $(Z \times Y, \eta \times \eta)$ as claimed. □

2.10 The Poisson Boundary

The Poisson boundary of a group will play a relatively minor role in our work compared to its presence in the work of Bader and Shalom [BS06] and in the work of Nevo and Stuck and Zimmer [SZ94],[NZ99]. The main interest we will have in the Poisson boundary is that it is a contractive action and therefore gives rise to relatively contractive maps. The reader is referred to [BS06] and [Cre11] for a detailed account of Poisson boundaries in the abstract setting and to [Fur63], [Fur67], [Fur71], [Kai88] and [Kai92] for information on Poisson boundaries of semisimple groups and lattices in semisimple groups.

Definition 2.36 (Furstenberg [Fur63]). Let G be a locally compact second countable group and $\mu \in P(G)$ a probability measure on G. Consider the map $T : G^{\mathbb{N}} \to G^{\mathbb{N}}$ given by $T(w_1, w_2, w_3, \cdots) = (w_1 w_2, w_3, \cdots)$. The space of T-ergodic components of $(G^{\mathbb{N}}, \mu^{\mathbb{N}})$ is the Poisson boundary of (G, μ).

Theorem 2.37 (Furstenberg [Fur63]). *Let G be a locally compact second countable group and $\mu \in P(G)$. The action of G on the Poisson boundary is a μ-stationary contractive action.*

Theorem 2.38 (Zimmer [Zim84]). *Let G be a locally compact second countable group and $\mu \in P(G)$. The action of G on the Poisson boundary is amenable.*

2.11 Ergodic Decomposition and Poisson Boundaries

We will need a basic fact about the ergodic decomposition of the product of the Poisson boundary and a measure-preserving space due to Bader and Shalom [BS06].

Proposition 2.11.1 (Bader-Shalom, [BS06] Corollary 2.18). *Let (B, β) be the Poisson boundary for (G, μ) where μ is an admissible measure on G and let (X, ν) be an ergodic measure-preserving G-space. Then $(B \times X, \beta \times \nu)$ is an ergodic μ-stationary G-space.*

The only difficulty in the proof of the above statement is the ergodicity. We will need the following extension of their result:

Proposition 2.11.2. *Let (B, β) be the Poisson boundary for (G, μ) where μ is an admissible measure on G, let (C, η) be any G-quotient of (B, β) and let (X, ν) be a measure-preserving G-space. Then $(C \times X) /\!\!/ G = X /\!\!/ G$.*

Proof. Write (Z, ζ) for the ergodic components of (X, ν) and let $\pi : (X, \nu) \to (Z, \zeta)$ be the decomposition map. Let $\varphi : (B \times X, \beta \times \nu) \to (Z, \zeta)$ be given by $\varphi(b, x) = \pi(x)$. Let $\tau : B \times X \to C \times X$ be given by $\tau(b, x) = (\tau_0(b), x)$ where $\tau_0 : B \to C$ is the G-map making (C, η) a G-quotient of (B, β).

Let $E \subseteq C \times X$ be a positive measure G-invariant set. For each $z \in Z$, let $E_z = \tau^{-1}(E) \cap (B \times \pi^{-1}(z))$. Since G acts trivially on Z, E_z is a G-invariant set. Consider $D_\varphi(z) = \beta \times D_\pi(z)$. Since ν is measure-preserving, so is $D_\pi(z)$ for each z. By the above Proposition, $D_\varphi(z)$ is then ergodic for each z.

Therefore, $D_\varphi(z)(E_z)$ is either null or conull for each z. Let $A \subseteq Z$ be the set of z such that $D_\varphi(z)(\tau^{-1}(E)) = D_\varphi(z)(E_z) = 1$. Then $D_\varphi(z)(\tau^{-1}(E) \triangle \varphi^{-1}(A)) = 0$ for almost every z since either both $\tau^{-1}(E)$ and $\varphi^{-1}(A)$ are full (when $z \in A$) or both null (when $z \notin A$). Hence

$$\beta \times \nu(\tau^{-1}(E) \triangle \varphi^{-1}(A)) = \int_Z D_\varphi(z)(\tau^{-1}(E) \triangle \varphi^{-1}(A)) \; d\zeta(z) = 0.$$

Therefore $\eta \times \nu(E \triangle \tau(\varphi^{-1}(A))) = \beta \times \nu(\tau^{-1}(E) \triangle \varphi^{-1}(A)) = 0$ meaning that every G-invariant measurable set in $C \times X$ belongs to the algebra of measurable sets of $X /\!\!/ G$ as claimed. □

2.12 The Invariants Product Functor

We recall now the invariants product functor of Bader and Shalom [BS06]. Let $G = G_1 \times G_2$ be a product of groups and let (X, ν) be an ergodic G-space. Write $X /\!\!/ G_j$ for the space of G_j-ergodic components of X, for $j = 1, 2$. Then G_j acts trivially on $X /\!\!/ G_j$ and G_{3-j} acts ergodically (since G acts ergodically on X). We will write (X_1, ν_1) to be the space of G_2-ergodic components with the push-forward of ν and likewise write (X_2, ν_2) for the space of G_1-ergodic components.

The **invariants product functor** is the functor F^G that assigns $F^G(X, \nu) = (X_1, \nu_1) \times (X_2, \nu_2)$, which we treat as a G-space with the diagonal G-action. That this is indeed a functor is shown in Bader-Shalom in the sense that given a G-map $\pi : (X, \nu) \to (Y, \eta)$ of ergodic G-spaces, define $F^G(\pi) = \pi_1 \times \pi_2$ where $\pi_j : (X_j, \nu_j) \to (Y_j, \eta_j)$ is the Mackey point realization (Theorem 2.9) of the inclusion at the level of σ-algebras, and the following diagram commutes:

$$\begin{array}{ccc} (X,\nu) & \xrightarrow{\quad\quad\pi\quad\quad} & (Y,\eta) \\ \downarrow F^G & & \downarrow F^G \\ (X_1,\nu_1)\times(X_2,\nu_2) & \xrightarrow{\pi_1\times\pi_2} & (Y_1,\eta_1)\times(Y_2,\eta_2) \end{array}$$

In general, the mapping $(X, \nu) \to F^G(X, \nu)$ need not be a G-map (though of course π_j is a G_j-map so $F^G(\pi)$ is always a product of G_1- and G_2-maps). However, in the case of ergodic stationary G-spaces the map is a G-map:

Proposition 2.12.1 (Bader-Shalom [BS06] Proposition 1.10)**.** *Let $G = G_1 \times G_2$ be a product of two locally compact second countable groups and let $\mu_j \in P(G_j)$ be admissible probability measures for $j = 1, 2$. Set $\mu = \mu_1 \times \mu_2$. If (X, ν) is a μ-stationary ergodic G-space then $(X, \nu) \to (X_1, \nu_1) \times (X_2, \nu_2)$ is a relatively measure-preserving G-map.*

2.13 Relatively Contractive Maps

Relatively contractive maps were introduced in [CP13] as a generalization of both the contractive spaces studied by Jaworski [Jaw94], [Jaw95] (under the name SAT) and the notion of proximal maps for stationary actions (see e.g. [FG10]). In [CP13], strong uniqueness properties of such maps is proved and we generalize a result in [CP13] regarding joinings of contractive spaces. This generalization will be the key ingredient in our Intermediate Contractive Factor Theorem.

Definition 2.39 (Jaworski [Jaw94])**.** Let G be a locally compact second countable group and (X, ν) a G-space. Then (X, ν) is **contractive** when for any measurable set $E \subseteq X$ with $\nu(E) > 0$ there exists a sequence $\{g_n\}$ in G such that $\nu(g_n E) \to 1$.

Definition 2.40 (Creutz-Peterson [CP13] Definition 4.4)**.** Let G be a locally compact second countable group and $\pi : (X, \nu) \to (Y, \eta)$ a G-map of G-spaces. Then π is **relatively contractive** when for any measurable set $E \subseteq X$ and almost every $y \in Y$ such that $D_\pi(y)(E) > 0$there exists a sequence $\{g_n\}$ in G such that $g_n^{-1} D_\pi(g_n y)(E) \to 1$.

Theorem 2.41 (Creutz-Peterson [CP13] Theorem 4.15)**.** *Let G be a locally compact second countable group and let $\pi : (X, \nu) \to (Y, \eta)$ and $\psi : (Y, \eta) \to (Z, \zeta)$ be G-maps of G-spaces. If $\psi \circ \pi$ is relatively contractive then π and ψ are relatively contractive.*

Theorem 2.42 (Creutz-Peterson [CP13] Theorem 4.13)**.** *Let G be a locally compact second countable group, (X, ν) a G-space and (B, β) a contractive G-space. Then the natural projection map $p : (B \times X, \beta \times \nu) \to (X, \nu)$ is relatively contractive.*

Theorem 2.43. *Let G be a locally compact second countable group and $\pi : (X, \nu) \to (Y, \eta)$ a G-map of G-spaces. If π is both relatively measure-preserving and relatively contractive then it is an isomorphism.*

Proof. Let E be a measurable set in X. Since π is relatively contractive, for almost every $y \in Y$ such that $D_\pi(y)(E) > 0$ there is a sequence $g_n \in G$ such that $D_\pi(g_n y)(g_n E) \to 1$. Since π is relatively measure-preserving, $D_\pi(g_n y)(g_n E) = D_\pi(y)(E)$. Therefore $D_\pi(y)(E) = 1$ for almost every y such that $D_\pi(y)(E) > 0$. As this holds for all measurable sets E this means π is an isomorphism. $\square$

Corollary 2.44. *Let G be a locally compact second countable group and (X,ν) a contractive G-space. If $\pi : (X,\nu) \to (Y,\eta)$ is a relatively measure-preserving G-map of G-spaces then it is an isomorphism.*

Proof. This follows from the previous theorem and the observation that any map from a contractive space is relatively contractive (the map from (X,ν) to the trivial one-point system is relatively contractive and so by Theorem 2.41 then so is π). ☐

2.14 Joinings

Joinings will play a key role in both our contractive factor theorem and in the study of random subgroups. The reader is referred to [Gla03] for more information on joinings.

Definition 2.45. Let G be a locally compact second countable group and let (X,ν) and (Y,η) be G-spaces. A **joining** of (X,ν) and (Y,η) is a probability measure $\alpha \in P(X \times Y)$ such that $(p_X)_*\alpha = \nu$ and $(p_Y)_*\alpha = \eta$ where p_X and p_Y are the natural projections from $X \times Y$ to X and Y. The space $(X \times Y, \alpha)$ is then a G-space with the diagonal action.

Definition 2.46. Let (X,ν) and (Y,η) be G-spaces with a common G-quotient (Z,ζ), that is a diagram of G-maps and G-spaces as follows:

$$\begin{tikzcd} & (X,\nu) \arrow[d, "\pi"'] \\ (Y,\eta) \arrow[r, "\varphi"] & (Z,\zeta) \end{tikzcd}$$

Treat $X \times Y$ as a G-space with the diagonal action. A G-quasi-invariant Borel probability measure $\rho \in P(X \times Y)$ is a **relative joining** of (X,ν) and (Y,η) over (Z,ζ) when the following diagram of G-maps commutes:

$$\begin{tikzcd} (X \times Y,\rho) \arrow[r, "p_X"] \arrow[d, "p_Y"'] & (X,\nu) \arrow[d, "\pi"'] \\ (Y,\eta) \arrow[r, "\varphi"] & (Z,\zeta) \end{tikzcd}$$

where p_X and p_Y are the natural projections from $X \times Y$ to X and Y, respectively.

In general, the product $\nu \times \eta$ is not a relative joining of (X,ν) and (Y,η) over (Z,ζ) unless (Z,ζ) is trivial since we require that $\pi \circ p_X = \varphi \circ p_Y$ almost everywhere. However, there is a notion of independent joining in the relative case:

Definition 2.47. Let (X,ν) and (Y,η) be G-spaces with common G-quotient (Z,ζ). Let $\pi : (X,\nu) \to (Z,\zeta)$ and $\varphi : (Y,\eta) \to (Z,\zeta)$ be the quotient maps. The probability measure $\rho \in P(X \times Y)$ given by

$$\rho = \int_Z D_\pi(z) \times D_\varphi(z)\ d\zeta(z)$$

is the **independent relative joining** of (X,ν) and (Y,η) over (Z,ζ).

Of course, the independent relative joining is a relative joining. We also note that the independent joining $\nu \times \eta$ is the independent relative joining over the trivial system.

Proposition 2.14.1. *Let $\pi : (X,\nu) \to (Y,\eta)$ be a G-map of G-spaces. Then the independent relative joining of (X,ν) and (Y,η) over (Y,η) is G-isomorphic to (X,ν).*

Proof. The independent relative joining is $(X \times Y, \alpha)$ where

$$\alpha = \int_Y D_\pi(y) \times \delta_y\ d\eta(y).$$

Let $p : X \times Y \to X$ be the projection to X. Let $\alpha_x \in P(X \times Y)$ by $\alpha_x = \delta_x \times \delta_{\pi(x)}$. Then

$$\int_X \alpha_x\ d\nu(x) = \int_Y \int_X \delta_x \times \delta_{\pi(x)}\ dD_\pi(y)(x)\ d\eta(y)$$

$$= \int_Y \int_X \delta_x \times \delta_y \; dD_\pi(y)(x) \; d\eta(y)$$
$$= \int_Y D_\pi(y) \times \delta_y \; d\eta(y) = \alpha$$

and α_x is supported on $p^{-1}(x) = \{x\} \times Y$. Therefore $D_p(x) = \alpha_x$ by uniqueness of disintegration. Since α_x is a point mass, then p is an isomorphism so $(X \times Y, \alpha)$ is isomorphic to (X, ν). □

2.15 Resolutions

The notion of resolution, due to de Cornulier [dC05], is intimately connected with notion of relative property (T). We will make use of resolutions in the easy case when considering a product of two groups, one of which has property (T), to show that weakly amenable actions are in fact essentially transitive in many cases. The reader is referred to [dC05] for a systematic description and proofs.

Definition 2.48 (de Cornulier [dC05])**.** Let G and Q be locally compact second countable groups and let $p : G \to Q$ be a homomorphism with dense image. Let $f : G \to X$ be any map to a topological space. Then f **factors through** p when for every net $\{g_i\}$ in G, if $p(g_i)$ converges in Q then $f(g_i)$ converges in X. Given an action $G \curvearrowright X$ on a topological space, the (Q, f)**-points** of X are $X^Q = \{x \in X : g \mapsto gx$ factors through $f\}$.

Proposition 2.15.1 (de Cornulier [dC05])**.** *Let $f : G \to Q$ be a homomorphism of locally compact second countable groups with dense image and let $G \curvearrowright X$ be any action on a topological space. Then the space of (Q, f)-points X^Q is a closed G-invariant set in X.*

Definition 2.49. Let $f : G \to Q$ be a homomorphism of locally compact second countable groups with dense image and let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a (strongly continuous) unitary representation of G on a Hilbert space. Let $\mathcal{H}^Q$ be the space of (Q, f)-points in $\mathcal{H}$ and let $\pi^Q : Q \to \mathcal{H}^Q$ be the restriction of π to Q on $\mathcal{H}^Q$.

Definition 2.50. Let G be a locally compact second countable group and $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of G on a Hilbert space. Then π has **almost invariant vectors** when there exists a sequence $\{v_n\}$ in $\mathcal{H}$ such that $\|v_n\| = 1$ for all n and such that for each fixed $g \in G$ it holds that $\lim_n \|\pi(g)v_n - v_n\| \to 0$.

Definition 2.51. Let $f : G \to Q$ be a homomorphism of locally compact second countable groups with dense image. Then f is a **resolution** when for every unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ of G on a Hilbert space that has almost invariant vectors, the representation $\pi^Q : Q \to \mathcal{U}(\mathcal{H}^Q)$ also has almost invariant vectors.

Proposition 2.15.2 (de Cornulier [dC05])**.** *Let $G = G_1 \times G_2$ be a product of two locally compact second countable groups. If G_2 has property (T) then the projection map $\mathrm{proj}_1 : G \to G_1$ is a resolution.*

Proposition 2.15.3 (de Cornulier [dC05])**.** *Let G and Q be locally compact second countable groups and let $p : G \to Q$ be a resolution. Let $\Gamma < G$ be a lattice. Then $p : \Gamma \to \overline{p(\Gamma)}$ is a resolution.*

Combining the previous two propositions:

Proposition 2.15.4. *Let G and H be locally compact second countable groups such that H has property (T) and let $\Gamma < G \times H$ be an irreducible lattice. Then the projection map $\mathrm{proj}_G : \Gamma \to G$ is a resolution.*

3 Random Subgroups

Invariant random subgroups are an active area of research and are the natural setting for the study of stabilizers of actions of groups. We present here a systematic approach to treating random subgroups as subgroups of one another and how this interacts with the possible stabilizers of actions of the group.

Definition 3.1. Let G be a locally compact second countable group. Denote by $S(G)$ the space of closed subgroups of G endowed with the Chabauty topology. Let G act on $S(G)$ by conjugation. A Borel probability measure $\eta \in P(S(G))$ that is invariant under the conjugation action is an **invariant random subgroup**.

We generalize the notion of invariant random subgroup to quasi-invariant actions:

Definition 3.2. Let G be a locally compact second countable group and denote by $S(G)$ the space of closed subgroups of G endowed with the Chabauty topology and the action of G by conjugation. A Borel probability measure $\eta \in P(S(G))$ is a **random subgroup** (or more precisely, a **quasi-invariant random subgroup**) when it is quasi-invariant under the conjugation action.

The following generalizes the equivalent statement for measure-preserving actions and invariant random subgroups due to Abert-Glasner-Virág [AGV12]:

Theorem 3.3. *Let $G \curvearrowright (X, \nu)$ be a quasi-invariant action of a locally compact second countable group. Then the map* $\mathrm{stab} : X \to S(G)$ *by* $\mathrm{stab}(x) = \{g \in G : gx = x\}$ *gives rise to a random subgroup* $\mathrm{stab}_*\nu$*. This will be an invariant random subgroup precisely when the action is measure-preserving.*

Conversely, given a random subgroup $\eta \in P(S(G))$ there exists a quasi-invariant action $G \curvearrowright (X, \nu)$ such that $\mathrm{stab}_*\nu = \eta$*. Moreover, this action will be a measure-preserving extension of $(S(G), \eta)$.*

The previous theorem is actually a special case of Theorem 3.9 and will be proved as Corollary 3.11 below.

3.1 Subgroups of Random Subgroups

Subgroups of invariant random subgroups were introduced in [CP13]. We generalize this idea to quasi-invariant random subgroups.

Definition 3.4. Let $\rho, \zeta \in P(S(G))$ be random subgroups of a locally compact second countable group G. Then ρ is a **subgroup** of a ζ when there exists a joining $\alpha \in P(S(G) \times S(G))$ of ρ and ζ such that for α-almost every $(H, L) \in S(G) \times S(G)$, it holds that H is a subgroup of L. This will be written $\rho < \zeta$.

Proposition 3.1.1. *The property of being a subgroup is a transitive relation on random subgroups.*

Proof. Let $\alpha, \beta, \rho \in P(S(G))$ be random subgroups of a group G such that $\alpha < \beta$ and $\beta < \rho$. Let $\psi \in P(S(G) \times S(G))$ be a joining of α and β such that $H < L$ for ψ-almost every (H, L) and let $\varphi \in P(S(G) \times S(G))$ be a joining of β and ρ such that $L < K$ for φ-almost every (L, K). Let $p_A : S(G) \times S(G) \to S(G)$ be the projection to the first coordinate and $p_B : S(G) \times S(G) \to S(G)$ the projection to the second.

Observe that $D_{p_A}(L) = \delta_L \times \varphi_L$ for $\varphi_L \in P(S(G))$ such that $\int_{S(G)} \delta_L \times \varphi_L \; d\beta(L) = \varphi$. Likewise, $D_{p_B}(L) = \psi_L \times \delta_L$ for $\psi_L \in P(S(G))$ such that $\int_{S(G)} \psi_L \times \delta_L \; d\beta(L) = \psi$.

Define $\tau \in P(S(G) \times S(G) \times S(G))$ by

$$\tau = \int_{S(G)} \psi_L \times \delta_L \times \varphi_L \; d\beta(L).$$

Then, letting $p_j : S(G) \times S(G) \times S(G) \to S(G)$ be the projections,

$$(p_1)_*\tau = \int_{S(G)} \psi_L \; d\beta(L) = (p_A)_* \int_{S(G)} \psi_L \times \delta_L \; d\beta(L) = (p_A)_*\psi = \alpha$$

and likewise

$$(p_2)_*\tau = \beta \qquad \text{and} \qquad (p_3)_*\tau = \rho.$$

Therefore τ is a joining of α and β and ρ.

Note that $(p_1 \times p_2)_*\tau = \psi$ and that $(p_2 \times p_3)_*\tau = \varphi$. For τ-almost every (H, L, K) we then have that $H < L$ and $L < K$. Hence $H < K$ for $(p_1 \times p_3)_*\tau$-almost every (H, K). As $(p_1 \times p_3)_*\tau$ is a joining of α and ρ, this shows that $\alpha < \rho$. □

Definition 3.5. Let $\rho, \zeta \in P(S(G))$ be random subgroups of a locally compact second countable group G. Then ρ is a **normal subgroup** of a ζ when there exists a joining $\alpha \in P(S(G) \times S(G))$ of ρ and ζ such that for α-almost every $(H, L) \in S(G) \times S(G)$, it holds that H is a normal subgroup of L. This will be written $\rho \triangleleft \zeta$.

Definition 3.6. A random subgroup $\rho \in P(S(G))$ of a locally compact second countable group is **simple** when the only normal subgroups of it are trivial: if $\eta \lhd \rho$ then for any joining α witnessing that $\eta < \rho$, for α-almost every $(H, L) \in S(G) \times S(G)$, either $H = e$ or $H = L$.

Note that if ρ is a simple ergodic random subgroup and $\eta \lhd \rho$ is also ergodic then $\eta = \rho$ or $\eta = \delta_e$.

The main reason for introducing the notion of subgroups of random subgroups is the following relativization of the fact that stabilizers of quasi-invariant actions give rise to random subgroups:

Theorem 3.7. *Let G be a locally compact second countable group and let $\pi : (X, \nu) \to (Y, \eta)$ be a G-map of G-spaces. Then $\mathrm{stab}_*\nu$ is a subgroup of $\mathrm{stab}_*\eta$.*

Proof. Define $\alpha \in P(S(G) \times S(G))$ by

$$\alpha = \int_X \delta_{\mathrm{stab}(x)} \times \delta_{\mathrm{stab}(\pi(x))} \, d\nu(x).$$

Then the projection to the first coordinate $\mathrm{pr}_1 : S(G) \times S(G) \to S(G)$ has the property that

$$(\mathrm{pr}_1)_*\alpha = \int_X \delta_{\mathrm{stab}(x)} \, d\nu(x) = \mathrm{stab}_*\nu$$

and the projection to the second coordinate has the property that

$$(\mathrm{pr}_2)_*\alpha = \int_X \delta_{\mathrm{stab}(\pi(x))} \, d\nu(x) = \mathrm{stab}_*\pi_*\nu = \mathrm{stab}_*\eta.$$

Therefore α is a joining of $\mathrm{stab}_*\nu$ and $\mathrm{stab}_*\eta$. Now $\mathrm{stab}(x) < \mathrm{stab}(\pi(x))$ for all $x \in X$ since π is a G-map and therefore for α-almost every $(H, L) \in S(G) \times S(G)$ it holds that $H < L$. ☐

Theorem 3.8. *Let G be a locally compact second countable group and $\pi : (X, \nu) \to (Y, \eta)$ a G-map of G-spaces such that $\mathrm{stab}(x)$ is constant on each fiber: for η-almost every $y \in Y$, it holds that $\mathrm{stab}(x)$ is constant for $D_\pi(y)$-almost every $x \in \pi^{-1}(y)$. Then $\mathrm{stab}_*\nu$ is a normal random subgroup of $\mathrm{stab}_*\eta$.*

Proof. Since stab is constant on fibers, it descends to a measurable map $s : Y \to S(G)$ such that $\mathrm{stab}(x) = s(\pi(x))$ for almost every $x \in X$. For such an $x \in X$ and for $g \in \mathrm{stab}(\pi(x))$,

$$g\mathrm{stab}(x)g^{-1} = \mathrm{stab}(gx) = s(\pi(gx)) = s(g\pi(x)) = s(\pi(x)) = \mathrm{stab}(x)$$

meaning that $\mathrm{stab}(x) \lhd \mathrm{stab}(\pi(x))$. The joining $\alpha \in P(S(G) \times S(G))$ given by $\alpha = (\mathrm{stab} \circ (\mathrm{id} \times \pi))_*\nu$ where $\mathrm{stab} \circ (\mathrm{id} \times \pi) : X \to X \times Y$ by $(\mathrm{stab} \circ (\mathrm{id} \times \pi))(x) = (\mathrm{stab}(x), \mathrm{stab}(\pi(x)))$ then shows that $\mathrm{stab}_*\nu$ is a normal subgroup of $\mathrm{stab}_*\eta$. ☐

3.2 Subgroups of Random Subgroups Correspond to Quotient Maps

Theorem 3.9. *Let G be a locally compact second countable group and (X, ν) a G-space. Let $\varphi : X \to P(S(G))$ be a G-equivariant map such that for ν-almost every $x \in X$ and $\varphi(x)$-almost every $H \in S(G)$ it holds that $H < \mathrm{stab}(x)$. Then there exists a G-space (Y, η) and a G-map $\pi : (Y, \eta) \to (X, \nu)$ such that $\mathrm{stab}_*\eta$ is the barycenter of $\varphi_*\nu$. Moreover, π is relatively measure-preserving.*

Proof. Observe that for almost every $x \in X$, $\varphi(x)$ is an invariant random subgroup of $\mathrm{stab}(x)$ since $H < \mathrm{stab}(x)$ for $\varphi(x)$-almost every H and since the G-equivariance of φ gives that for $g \in \mathrm{stab}(x)$, $g \cdot \varphi(x) = g\varphi(x)g^{-1} = \varphi(gx) = \varphi(x)$.

Fix a probability measure $\rho \in P(G)$ in the class of the Haar measure. For each $x \in X$ and $H \in S(\mathrm{stab}(x))$, let $(Q_{x,H}, \rho_{x,H})$ be the Gaussian probability space corresponding to an infinite direct sum of $L^2(G/H)$ where $\rho_{x,H}$ is the pushforward of ρ under the quotient map $q_H : G \to G/H$. Let $Q = ((Q_{x,H}, \rho_{x,H}))_{x \in X, H \in S(\mathrm{stab}(x))}$ be the field of probability spaces just constructed (this is a measurable field following the same reasoning as in [CP13] Theorem 3.3).

Define the cocycle $\alpha : G \times X \times S(G) \to Q$ such that $\alpha(g, x, H) \in \mathrm{Aut}(Q_{x,H}, Q_{gx,gHg^{-1}})$ is the induced automorphism from the operator $T_{g,x,H}$ from the infinte direct sum of $L^2(Q_{x,H}, \rho_{x,H})$ to the infinite direct sum of $L^2(Q_{gx,gHg^{-1}},\rho_{gx,gHg^{-1}})$ given by

$$(T_{g,x,H}f)(kgHg^{-1}) = f(kgH)\sqrt{\frac{d(q_H)_*(\rho g^{-1})}{d\rho_{x,H}}(kgH)}.$$

Define the probability space (Q,ρ) by

$$\big(Q = \bigsqcup_x \bigsqcup_H Q_{x,H}, \rho = \int_X \int_{S(G)} \rho_{x,H}\ d\varphi(x)(H)\ d\nu(x)\big)$$

equipped with the G-action coming from the cocycle α. The cocycle identity holds almost everywhere so by Mackey's point realization [Mac62], as G is locally compact and second countable, after removing a null set we may assume the cocycle identity holds everywhere.

Note that $g \cdot \rho_{x,H} = \rho_{gx,gHg^{-1}}$ and therefore, using the equivariance of φ,

$$\begin{aligned} g\cdot\rho &= \int_X\int_{S(G)} \rho_{gx,gHg^{-1}}\ d\varphi(x)(H)\ d\nu(x) = \int_X\int_{S(G)} \rho_{gx,H}\ dg\varphi(x)(H)\ d\nu(x) \\ &= \int_X\int_{S(G)} \rho_{gx,H}\ d\varphi(gx)(H)\ d\nu(x) = \int_X\int_{S(G)} \rho_{x,H}\ d\varphi(x)(H)\ dg\nu(x) \end{aligned}$$

meaning that ρ is quasi-invariant under the G-action since ν is.

For $x \in X$ and $H \in S(\mathrm{stab}(x))$, the map $g \mapsto \alpha(g,x,H)$ defines an action of $N_{\mathrm{stab}(x)}(H)/H$ on $Q_{x,H}$ which is essentially free (Proposition 1.2 in [AEG94]). Now for $q \in Q_{x,H}$ and $g \in G$ we have that $g\cdot(x,H,q) = (gx, gHg^{-1}, \alpha(g,x,H)q)$ meaning that $g\cdot(x,H,q) = (x,H,q)$ if and only if $gx = x$ and $gHg^{-1} = H$ and $\alpha(g,x,H)q = q$ so if and only if $g \in \mathrm{stab}(x)$ and $g \in N_{\mathrm{stab}(x)}(H)$ and $\alpha(g,x,H)q = q$ hence if and only if $g \in H$. Therefore

$$\mathrm{stab}_*\rho = \int_X\int_H \mathrm{stab}_*\rho_{x,H}\ d\varphi(x)(H)\ d\nu(x) = \int_X\int_H \delta_H\ d\varphi(x)(H)\ d\nu(x) = \int_X \varphi(x)\ d\nu(x)$$

as required.

Define $\pi : (Q,\rho) \to (X,\nu)$ by $\pi(x,H,q) = x$. Then

$$\pi(g\cdot(x,H,q)) = \pi(gx, gHg^{-1}, \alpha(g,x,H)q) = gx$$

so π is a G-map.

To see that π is relatively measure-preserving, observe that for $f \in L^\infty(Q,\rho)$,

$$\begin{aligned} &\int_Q f(x,H,q)\frac{dg\nu}{d\nu}(x)\ d\rho(x,H,q) \\ &\qquad = \int_X\int_{S(G)}\int_{Q_{x,H}} f(x,H,q)\frac{dg\nu}{d\nu}(x)\ d\rho_{x,H}(q)\ d\varphi(x)(H)\ d\nu(x) \\ &\qquad = \int_X\int_{S(G)}\int_{Q_{x,H}} f(x,H,q)\ d\rho_{x,H}(q)\ d\varphi(x)(H)\ dg\nu(x) \\ &\qquad = \int_Q f(x,H,q)\ d(g\cdot\rho)(x,H,q) \end{aligned}$$

and therefore $\frac{dg\cdot\rho}{d\rho}(x,H) = \frac{dg\nu}{d\nu}(x)$ meaning that π is relatively measure-preserving (Theorem 2.7). □

Theorem 3.10. *Let G be a locally compact second countable group and (X,ν) an ergodic G-space. Let $\varphi : X \to P(S(G))$ be a G-equivariant map such that for ν-almost every $x \in X$ and $\varphi(x)$-almost every $H \in S(G)$ it holds that $H < \mathrm{stab}(x)$. Then there exists an ergodic G-space (Y,η) and a G-map $\pi : (Y,\eta) \to (X,\nu)$ such that $\mathrm{stab}_*\eta$ is the barycenter of $\varphi_*\nu$.*

Proof. Let (Q,ρ) be the construction from Theorem 3.9 such that there exists a G-map $\tau : (Q,\rho) \to (X,\nu)$ with the barycenter of $\varphi_*\nu$ being $\mathrm{stab}_*\rho$. Consider the ergodic decomposition $\psi : (Q,\rho) \to (R,\kappa)$. Then

G acts trivially on (R, κ) and almost every fiber $(\psi^{-1}(r), D_\psi(r))$ is an ergodic G-space. Observe that

$$\int_R \mathrm{stab}_* D_\psi(r)\ d\kappa(r) = \mathrm{stab}_* \int_R D_\psi(r)\ d\kappa(r) = \mathrm{stab}_* \rho = \mathrm{bar}\ \varphi_* \nu.$$

Since ν is ergodic, so is $\varphi_*\nu$ (treating $(P(S(G)), \varphi_*\nu)$ as G-space). Therefore $(S(G), \mathrm{bar}\ \varphi_*\nu)$ is also an ergodic G-space. Now ergodic random subgroups are extremal, and therefore, since $\int_R \mathrm{stab}_* D_\psi(r)\ d\kappa(r)$ is a convex combination of random subgroups, $D_\psi(r)$ must be constant and equal to bar $\varphi_*\nu$ for almost every $r \in R$. Therefore for almost every fiber, the map $\pi : (\psi^{-1}(r), D_\psi(r)) \to (\pi(\psi^{-1}(x)), \pi_* D_\psi(r))$ has the required properties. Observe that since (X, ν) is ergodic, $\pi_* D_\psi(r) = \nu$ almost everywhere and therefore $\pi : (\psi^{-1}(r), D_\psi(r)) \to (X, \nu)$ has the required properties. □

Corollary 3.11. *Let $\rho \in P(S(G))$ be a random subgroup of a locally compact second countable group G. Then there exists a G-space (X, ν) such that* $\mathrm{stab}_*\nu = \rho$. *Moreover, if ρ is an invariant random subgroup then (X, ν) is measure-preserving.*

Proof. Apply Theorem 3.9 to the trivial one-point space and the map $\varphi : 0 \to P(S(G))$ given by $\varphi(0) = \rho$. □

3.3 Free Extensions

Definition 3.12. Let G be a locally compact second countable group, (X, ν) a G-space and $\varphi : X \to P(S(G))$ a G-equivariant map such that $\varphi(x)$ is an invariant random subgroup of $\mathrm{stab}(x)$ almost everywhere. The construction from Theorem 3.9 is the **free extension of** (X, ν) **by** φ.

Theorem 3.13. *Let G be a locally compact second countable group and let $\pi : (Y, \eta) \to (X, \nu)$ be a G-map of G-spaces. Let $\varphi : X \to P(S(G))$ be a G-equivariant measurable map such that $\varphi(x) \in P(\mathrm{stab}(x))$ almost everywhere. Let (Z, ζ) be the free extension of (X, ν) over φ and let (W, ρ) be the free extension of (Y, η) over $\varphi \circ \pi$. Then there exists a G-map $\tau : (W, \rho) \to (Z, \zeta)$ such that the resulting diagram commutes.*

Proof. For $y \in Y$ and $H \in S(\mathrm{stab}(y))$, let $W_{y,H}$ be the fiber in W over (y, H). For $x \in X$ and $H \in S(\mathrm{stab}(x))$, let $Z_{x,H}$ be the fiber in Z over (x, H). Define the map $\tau_{y,H} : W_{y,H} \to Z_{\pi(y),H}$ by $\tau(y, H)(y, H, q) = (\pi(y), H, q)$ which is well-defined since $H < \mathrm{stab}(y) < \mathrm{stab}(\pi(y))$. Observe that the cocycle α defining the actions on W and Z are identical on each fiber. Define the map $\tau : W \to Z$ by $\tau(y, H, q) = \tau_{y,H}(y, H, q) = (\pi(y), H, q)$. Then $\tau_*\zeta = \rho$ since $\pi_*\eta = \nu$. Also, $\tau(g \cdot (y, H, q)) = \tau(gy, gHg^{-1}, \alpha(g, y, H)q) = (\pi(gy), gHg^{-1}, \alpha(g, \pi(y), H)q) = g \cdot \tau(y, H, q)$. We also see that, letting $p_X : Z \to X$ and $p_Y : W \to Y$ be the free extension quotient maps,

$$p_X(\tau(y, H, q)) = p_X(\pi(y), H, q) = \pi(y)$$

and that

$$\pi(p_Y(y, H, q)) = \pi(y)$$

meaning that the diagram commutes. □

3.4 Invariant Random Subgroups and Quotient Maps

Theorem 3.14. *Let G be a locally compact second countable group and (X, ν) a measure-preserving G-space. Let $\rho \in P(S(G))$ be an invariant random subgroup such that ρ is a subgroup of* $\mathrm{stab}_*\nu$. *Then there exists a measure-preserving G-space (Y, η) and a G-map $\pi : (Y, \eta) \to (X, \nu)$ such that* $\mathrm{stab}_*\eta = \rho$.

Proof. Let $\alpha \in P(S(G) \times S(G))$ be a joining of ρ and $\mathrm{stab}_*\nu$ such that for α-almost every (H, L) it holds that $H < L$. Let $p : (S(G) \times S(G), \alpha) \to (S(G), \mathrm{stab}_*\nu)$ be the projection to the second coordinate. Define the map $\varphi : X \to P(S(G))$ by $\varphi(x) = D_p(\mathrm{stab}(x))$, the disintegration of α over $\mathrm{stab}_*\nu$ at $\mathrm{stab}(x)$. Then $\varphi(gx) = D_p(\mathrm{stab}(gx)) = D_p(g\mathrm{stab}(x)g^{-1}) = gD_p(\mathrm{stab}(x))g^{-1}$ since D_p is G-equivariant because p is relatively measure-preserving (since ρ and $\mathrm{stab}_*\nu$ are invariant random subgroups).

Theorem 3.9 then yields a measure-preserving G-space (Y, η) (a measure-preserving extension of a measure-preserving action is measure-preserving) and a G-map $(Y, \eta) \to (X, \nu)$. □

Corollary 3.15. *Let G be a locally compact second countable group and let $\rho, \zeta \in P(S(G))$ be invariant random subgroups of G such that ρ is a subgroup of ζ. Then there exists a G-map of measure-preserving G-spaces $\pi : (X, \nu) \to (Y, \eta)$ such that* $\mathrm{stab}_*\nu = \zeta$ *and* $\mathrm{stab}_*\eta = \rho$.

Proof. Let (Y, η) be the space corresponding to ζ from Corollary 3.11 and let (X, ν) be the construction from Theorem 3.14. □

3.5 Free Extensions by Random Subgroups

Definition 3.16. Let G be a locally compact second countable group, (X, ν) a measure-preserving G-space and $\rho \in P(S(G))$ an invariant random subgroup of G such that ρ is a subgroup of $\mathrm{stab}_*\nu$. The construction from Theorem 3.14 is the **free extension of** (X, ν) **by** ρ.

Definition 3.17. Let G be a locally compact second countable group and $\rho \in P(S(G))$ be an invariant random subgroup of G. The **ρ-nonfree action** of G is the free extension of the trivial one-point action by ρ. The (an) **ergodic ρ-nonfree action** of G is any of the ergodic components of the ρ-nonfree action.

4 Quotienting Out By Random Subgroups

We now introduce a generalization of the ergodic decomposition by normal subgroups. Recall that given a locally compact second countable group G and a closed normal subgroup $N \lhd G$, for any G-space (X, ν) one defines the ergodic decomposition of (X, ν) by N to be $(X /\!\!/ N, \overline{\nu})$ as the Mackey point realization (Theorem 2.9) of the algebra of N-invariant functions in $L^\infty(X, \nu)$. Our goal here is to define a similar decomposition over random subgroups.

4.1 The Quotient Space By a Random Subgroup

We now explore the class of random subgroups that live below a G-space (X, ν) and introduce the method of quotienting out by such random subgroups.

Proposition 4.1.1. *Let $G \curvearrowright (X, \nu)$ be a quasi-invariant action of a locally compact second countable group and let $\varphi : X \to S(G)$ be a G-equivariant measurable map such that* $\mathrm{stab}(x) \subseteq \varphi(x)$ *for ν-almost every $x \in X$. Then $\varphi_*\nu$ is a random subgroup of G such that* $\mathrm{stab}_*\nu$ *is a subgroup of $\varphi_*\nu$.*

Proof. That $\varphi_*\nu$ is a random subgroup is immediate from the G-equivariance of φ. The joining $\int \delta_{\mathrm{stab}(x)} \times \delta_{\varphi(x)}\, d\nu(x)$ shows that $\mathrm{stab}_*\nu$ is a subgroup of $\varphi_*\nu$. □

Definition 4.1. Let $G \curvearrowright (X, \nu)$ be a quasi-invariant action of a locally compact second countable group. A random subgroup η of G is a **random subgroup of G below** (X, ν) when there exists a measurable G-equivariant map $\varphi : X \to S(G)$ such that $\mathrm{stab}(x) \subseteq \varphi(x)$ almost everywhere and $\varphi_*\nu = \eta$.

Having established the class of random subgroups that live below a G-space, we now generalize the ergodic decomposition to such random subgroups.

Definition 4.2. Let $G \curvearrowright (X, \nu)$ be a quasi-invariant action of a locally compact second countable group and let $\varphi : X \to S(G)$ be a G-equivariant measurable map such that $\mathrm{stab}(x) \subseteq \varphi(x)$ for ν-almost every $x \in X$. Let $f \in L^\infty(X, \nu)$. Then f is **φ-invariant** when for almost every $x \in X$ and all $g \in \varphi(x)$, it holds that $f(gx) = f(x)$. The space of φ-invariant functions will be written $L^\infty(X, \nu)^\varphi$.

Proposition 4.1.2. *Let $G \curvearrowright (X, \nu)$ be a quasi-invariant action of a locally compact second countable group and let $\varphi : X \to S(G)$ be a G-equivariant measurable map such that* $\mathrm{stab}(x) \subseteq \varphi(x)$ *for ν-almost every $x \in X$. The space of φ-invariant functions $L^\infty(X, \nu)^\varphi$ is a closed G-invariant subalgebra of $L^\infty(X, \nu)$.*

Proof. That it is a subalgebra is clear. Let $f_n \in L^\infty(X, \nu)^\varphi$ such that $f_n \to f \in L^\infty(X, \nu)$. For almost every $x \in X$ and any $g \in \varphi(x)$ then $f_n(gx) = f_n(x) \to f(x)$ and $f_n(gx) \to f(gx)$ so f is also φ-invariant. Given $f \in L^\infty(X, \nu)^\varphi$ and $g \in G$ define $q(x) = f(gx)$. For $h \in \varphi(x)$, observe that $ghg^{-1} \in g\varphi(x)g^{-1} = \varphi(gx)$ so

$$q(hx) = f(ghx) = f((ghg^{-1})gx) = f(gx) = q(x)$$

meaning that $L^\infty(X,\nu)^\varphi$ is G-invariant. □

Definition 4.3. Let $G \curvearrowright (X,\nu)$ be a quasi-invariant action of a locally compact second countable group and let $\varphi : X \to S(G)$ be a G-equivariant measurable map such that $\mathrm{stab}(x) \subseteq \varphi(x)$ for ν-almost every $x \in X$. The **quotient space of** (X,ν) **over** φ is the Mackey point realization (Theorem 2.9) of the G-algebra of φ-invariant functions.

Definition 4.4. Let $G \curvearrowright (X,\nu)$ be a quasi-invariant action of a locally compact second countable group and let η be a random subgroup of G below (X,ν). The **quotient space of** (X,ν) **by** η is the quotient space of (X,ν) over the map φ witnessing that η is below (X,ν).

4.2 Examples of Quotienting By Random Subgroups

Proposition 4.2.1. *Let $G \curvearrowright (X,\nu)$ be a quasi-invariant action of a locally compact second countable group and let $N \lhd G$ be a closed normal subgroup. The quotient space of (X,ν) over $\varphi(x) = \overline{N \cdot \mathrm{stab}(x)}$ is the ergodic decomposition of (X,ν) over N.*

Proof. Let $f \in L^\infty(X,\nu)$ be a φ-invariant function. Then for all $g \in N$ and all $x \in X$ we have that $g \in \varphi(x)$ hence $f(gx) = f(x)$ almost everywhere so f is N-invariant. Now let $f \in L^\infty(X,\nu)$ be an N-invariant function. Then for almost every $x \in X$ and all $g \in N$ we have that $f(gx) = f(x)$. For $g \in stab(x)$ of course $f(gx) = f(x)$ so f is in fact φ-invariant. Hence the space of φ-invariant functions agrees with the space of N-invariant functions. □

Proposition 4.2.2. *Let $G \curvearrowright (X,\nu)$ be a quasi-invariant action of a locally compact second countable group and let $H < G$ be a closed subgroup such that $\mathrm{stab}(x) \subseteq H$ for almost every $x \in X$. The quotient space of (X,ν) over $\varphi(x) = H$ is the ergodic decomposition of (X,ν) over the normal closure of H.*

Proof. Let $f \in L^\infty(X,\nu)$ be φ-invariant. Then $f(hx) = f(x)$ for all $h \in H$ and almost every $x \in X$. Therefore H acts trivially on the quotient space. As the kernel of that action must be normal, it contains the normal closure of H. Conversely, any function invariant under the normal closure of H is invariant under H. □

4.3 The Universal Property of the Quotient Space

We now state and prove several results that jointly amount to a universal property for the quotient space by a random subgroup.

Theorem 4.5. *Let G be a locally compact second countable group and let $\pi : (Y,\eta) \to (X,\nu)$ be a G-map of G-spaces. Let $\varphi : X \to S(G)$ be a G-equivariant measurable map such that $\mathrm{stab}(x) \subseteq \varphi(x)$ for ν-almost every $x \in X$. Let (Z,ζ) be the quotient space of (X,ν) over φ and let $\psi : (X,\nu) \to (Z,\zeta)$ be the corresponding map. Then for almost every $z \in Z$, the disintegration measure $D_\psi(z)$ is $\mathrm{stab}(z)$-ergodic. More precisely, if $f \in L^\infty(X,\nu)$ such that for almost every $x \in X$ and all $g \in \mathrm{stab}(\psi(x))$ it holds that $f(gx) = f(x)$ then f descends to $L^\infty(Z,\zeta)$: there exists $F \in L^\infty(Z,\zeta)$ such that $f = F \circ \psi$.*

Proof. Let $f \in L^\infty(X,\nu)$ such that for almost every $x \in X$ and all $g \in \mathrm{stab}(z)$ it holds that $f(gx) = f(x)$. Since $\varphi(x) \subseteq \mathrm{stab}(\psi(x))$, then f is a φ-invariant function. Since (Z,ζ) is the point realization of all φ-invariant functions, then f descends to a function on Z. □

Theorem 4.6. *Let $G \curvearrowright (X,\nu)$ be a quasi-invariant action and let $\varphi : X \to S(G)$ be a G-equivariant measurable map such that $\mathrm{stab}(x) \subseteq \varphi(x)$ for ν-almost every $x \in X$. Let (Z,ζ) be the quotient space of (X,ν) over φ and let $\psi : (X,\nu) \to (Z,\zeta)$ be the map. Then $\varphi(x) \subseteq \mathrm{stab}(\psi(x))$ for ν-almost every $x \in X$.*

Moreover, the quotient space has the following universal property: if $\pi : (X,\nu) \to (Y,\eta)$ is a G-map such that $\varphi(x) \subseteq \mathrm{stab}(\pi(x))$ for ν-almost every $x \in X$ then there exist G-maps $\psi : (X,\nu) \to (Z,\zeta)$ and $\tau : (Z,\zeta) \to (Y,\eta)$ such that $\tau \circ \psi = \pi$.

Proof. Consider the disintegration $D_\psi : Z \to P(X)$ of ν over ζ. For $f \in L^\infty(X,\nu)$ the function $Ef(x) = D_\psi(\psi(x))(f)$ is the conditional expectation to the space of φ-invariant functions. Therefore, for almost every $x \in X$ and all $g \in \varphi(x)$ it holds that $Ef(gx) = Ef(x)$. Then $D_\psi(\psi(gx)) = D_\psi(\psi(x))$ as this holds for all f. Hence the supports agree meaning that $\psi^{-1}(\psi(gx)) = \psi^{-1}(\psi(x))$ and so $g\psi(x) = \psi(gx) = \psi(x)$. Therefore $\varphi(x) \subseteq \mathrm{stab}(\psi(x))$.

Let $f \in L^\infty(Y,\eta)$. Then $f \circ \pi \in L^\infty(X,\nu)$. For almost every $x \in X$ and all $g \in \varphi(x)$ we have that $f \in \mathrm{stab}(\pi(x))$ so $f \circ \pi(gx) = f \circ \pi(x)$ so f is φ-invariant. Hence $\pi^*(L^\infty(Y,\eta))$ is a closed G-invariant subalgebra of the φ-invariant functions so the maps ψ and τ follow from Mackey's point realization (Theorem 2.9). □

Theorem 4.7. *Let G be a locally compact second countable group and let $\pi : (Y,\eta) \to (X,\nu)$ be a G-map of G-spaces. Let $\varphi : X \to S(G)$ be a G-equivariant measurable map such that $\mathrm{stab}(x) \subseteq \varphi(x)$ for ν-almost every $x \in X$. Let (Z,ζ) be the quotient space of (X,ν) over φ and let (W,ρ) be the quotient space of (Y,η) over $\varphi \circ \pi$. Let $\psi : (X,\nu) \to (Z,\zeta)$ and $\xi : (Y,\eta) \to (W,\rho)$ be the corresponding G-maps. Then there exists a G-map $\tau : (W,\rho) \to (Z,\zeta)$ such that the following diagram commutes:*

$$\begin{array}{ccc} (Y,\eta) & \xrightarrow{\pi} & (X,\nu) \\ \xi\downarrow & & \psi\downarrow \\ (W,\rho) & \xrightarrow{\tau} & (Z,\zeta) \end{array}$$

Moreover, if π is an orbital G-map and ψ has the property that $\mathrm{stab}(\psi(x)) = \varphi(x)$ almost everywhere then τ is orbital (and, in particular, ξ has the property that $\mathrm{stab}(\xi(y)) = \varphi(\pi(y))$ almost everywhere).

Proof. Let $f \in L^\infty(X,\nu)$ be φ-invariant. Then $f \circ \pi \in L^\infty(Y,\eta)$ is $\varphi \circ \pi$-invariant. Therefore at the level of algebras, $L^\infty(Z,\zeta)$ is a closed G-invariant subalgebra of $L^\infty(W,\rho)$ and the required map exists by restricting π to the $\varphi \circ \pi$-invariant functions.

Assume now that π is orbital and that $\mathrm{stab}(\psi(x)) = \varphi(x)$ almost everywhere. Since the diagram commutes and $\mathrm{stab}(\psi(x)) = \varphi(x)$ almost everywhere, for almost every $y \in Y$,

$$\mathrm{stab}(\xi(y)) \subseteq \mathrm{stab}(\tau(\xi(y))) = \mathrm{stab}(\psi(\pi(y))) = \varphi(\pi(y)).$$

On the other hand, since (W,ρ) is the quotient of (Y,η) by $\varphi \circ \pi$, for almost every $y \in Y$,

$$\varphi(\pi(y)) \subseteq \mathrm{stab}(\xi(y)).$$

Therefore $\mathrm{stab}(\xi(y)) = \varphi(\pi(y))$ almost everywhere. Hence, for almost every $y \in Y$,

$$\mathrm{stab}(\xi(y)) = \varphi(\pi(y)) = \mathrm{stab}(\psi(\pi(y))) = \mathrm{stab}(\tau(\xi(y)))$$

and so for almost every $w \in W$, then $\mathrm{stab}(w) = \mathrm{stab}(\tau(w))$ so τ is orbital. □

Corollary 4.8. *Let G be a locally compact second countable group and let $\pi : (Y,\eta) \to (X,\nu)$ be a G-map of G-spaces. Let $\varphi : X \to S(G)$ be a G-equivariant measurable map such that $\mathrm{stab}(x) \subseteq \varphi(x)$ for ν-almost every $x \in X$. Let (Z,ζ) be the quotient space of (X,ν) over φ and let $\psi : (X,\nu) \to (Z,\zeta)$ be the corresponding map.*

Define the map $\phi : X \to S(G)$ by $\phi(x) = \mathrm{stab}(\psi(x))$. Then ϕ is a G-equivariant measurable map and the quotient of (X,ν) by ϕ is isomorphic to (Z,ζ).

Let (W,ρ) be the quotient space of (Y,η) over $\varphi \circ \pi$ and let (W',ρ') be the quotient of (Y,η) over $\phi \circ \pi$. Then there exists a commuting diagram of G-maps:

$$\begin{array}{ccc} (Y,\eta) & \xrightarrow{\pi} & (X,\nu) \\ \downarrow & & \downarrow\psi \\ (W,\rho) & \longrightarrow & (Z,\zeta) \\ \downarrow & & \downarrow\simeq \\ (W',\rho') & \longrightarrow & (Z,\zeta) \end{array}$$

Moreover, if ψ is orbital then the map $(W',\rho') \to (Z,\zeta)$ is orbital.

Proof. Let (Z', ζ') be the quotient of (X, ν) by ϕ. Then there is a G-map $(Z, \zeta) \to (Z', \zeta')$ since $\varphi(x) \subseteq \phi(x)$. On the other hand, by Theorem 4.6, since $\phi(x) = \mathrm{stab}(\psi(x))$ there is a G-map $(Z', \zeta') \to (Z, \zeta)$ and therefore they are isomorphic.

When ψ is orbital, that the map $W' \to Z$ is orbital follows from the previous theorem since by construction, $\mathrm{stab}(\psi(x)) = \phi(x)$. □

4.4 The Quotient Space as a Functor

Definition 4.9. Let G be a locally compact second countable group and let $\Phi : S(G) \to S(G)$ be a conjugation-equivariant map such that $H \subseteq \Phi(H)$ for all $H \in S(G)$ and such that for $H, L \in S(G)$, if $H \subseteq L$ then $\Phi(H) \subseteq \Phi(L)$.

For a G-space (X, ν), define $F^{\Phi}(X, \nu)$ to be the quotient space of (X, ν) by the map $\varphi = \Phi \circ \mathrm{stab}$ where $\mathrm{stab}(x) = \{g \in G : gx = x\}$.

Theorem 4.10. *Let G be a locally compact second countable group and let $\Phi : S(G) \to S(G)$ be a conjugation-equivariant map such that $H \subseteq \Phi(H)$ for all $H \in S(G)$. Then F^{Φ} is a functor on G-spaces and G-maps.*

Proof. Let $\pi : (X, \nu) \to (Y, \eta)$ be a G-map of G-spaces. Let $\varphi : Y \to S(G)$ by $\varphi(y) = \Phi(\mathrm{stab}(y))$ and $\psi : X \to S(G)$ by $\psi(x) = \Phi(\mathrm{stab}(x))$. Then $\varphi \circ \pi(x) = \Phi(\mathrm{stab}(\pi(x))) \supseteq \Phi(\mathrm{stab}(x)) = \psi(x)$. Writing $X /\!\!/ \psi$ for the quotient of (X, ν) by ψ (and likewise writing $X /\!\!/ \varphi \circ \pi$ and $Y /\!\!/ \varphi$), by Theorem 4.6 there exists a G-map $\tau : X /\!\!/ \psi \to X /\!\!/ \varphi \circ \pi$ and, combining this map with the diagram obtained from Theorem 4.7, the following diagram of G-maps exists and commutes (omitting measures for clarity):

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d] & X \arrow[r, "\pi"] \arrow[d] & Y \arrow[d] \\ X /\!\!/ \psi \arrow[r, "\tau"] & X /\!\!/ \varphi \circ \pi \arrow[r] & Y /\!\!/ \varphi. \end{tikzcd}$$

Ignoring the middle column, this says precisely that our construction defines a functor. □

4.5 Quotients of Affine Spaces

Let (X, ν) be an ergodic G-space. Let E be a Banach space and $\alpha : G \times X \to \mathrm{Iso}(E)$ be a cocycle. Let $A_x \subseteq E_1^*$ be a closed convex nonempty subset for each $x \in X$ such that $\alpha^*(g, x)A_{gx} = A_x$ for all $g \in G$ and $x \in X$ (where $\alpha^*(g, x) = (\alpha(g, x)^{-1})^*$ is the adjoint cocycle). Let $\pi : (X, \nu) \to (Z, \zeta)$ be a G-map.

Define the closed subspace

$$E_x^{\pi} = \{e \in E : \alpha(gh, x)e = \alpha(g, x)e \text{ for all } g \in G \text{ and all } h \in \mathrm{stab}(\pi(x))\ \}.$$

Observe that in particular $\alpha(h, x)e = e$ for all $h \in \mathrm{stab}(\pi(x))$ for every $x \in X$ and $e \in E_x^{\pi}$. Also observe that for any $e \in E_x^{\pi}$, any $g \in G$, any $h \in \mathrm{stab}(\pi(x))$ and any $k \in \mathrm{stab}(\pi(hx))$, writing $k = h\ell h^{-1}$ for some $\ell \in \mathrm{stab}(\pi(x))$,

$$\begin{aligned} \alpha(gk, hx)e &= \alpha(gh\ell h^{-1}, hx)e = \alpha(gh\ell, x)\alpha(h^{-1}, hx)e \\ &= \alpha(gh\ell, x)(\alpha(h, x)^{-1})e = \alpha(gh\ell, x)e \\ &= \alpha(gh, x)e = \alpha(g, hx)\alpha(h, x)e = \alpha(g, hx)e \end{aligned}$$

meaning that $E_{hx}^{\pi} = E_x^{\pi}$ for all $h \in \mathrm{stab}(\pi(x))$. By Theorem 4.5 then E_x^{π} is constant on fibers over Z. So write $E_z^{\pi} = E_x^{\pi}$ for $D_{\pi}(z)$-almost every $x \in \pi^{-1}(z)$.

Now given $g \in G$ and $e \in E_z^{\pi}$, for any $x \in \pi^{-1}(z)$ and $h \in \mathrm{stab}(\pi(x))$,

$$\alpha(g, hx)e = \alpha(gh, x)\alpha(h, x)^{-1}e = \alpha(g, hx)e = \alpha(g, x)e$$

so $\alpha(g, \cdot)e$ is $\mathrm{stab}(\pi(x))$-invariant meaning that (again by Theorem 4.5) it descends to $\beta(g, z) = \alpha(g, x)$ for almost every $x \in \pi^{-1}(z)$. Note that for all $g \in$G and $z \in Z$, it holds that $\beta(g, z) \in \mathrm{Iso}(E_z^{\pi} \to E_{gz}^{\pi})$ and that β is a cocycle.

Define $E^\pi = \bigcap_z E_z^\pi$ which is a closed subspace of E. Observe now that for any $q, k \in G$, any $z \in Z$ and any $e \in E^\pi$ it holds that $e \in E^\pi_{q^{-1}kqz}$ and so $\beta(k^{-1}, kqz)\beta(q,z)e \in \beta(k^{-1}, kqz)E^\pi_{kqz} = E^\pi_{qz}$. Therefore for any $g \in G$ and $h \in \mathrm{stab}(kqz)$, writing $h = k\ell k^{-1}$ for $\ell \in \mathrm{stab}(qz)$,

$$\begin{aligned}\beta(gh, kqz)\beta(q,z)e &= \beta(ghk, qz)\beta(k, qz)^{-1}\beta(q,z)e \\ &= \beta(gk\ell, qz)\beta(k^{-1}, kqz)\beta(q,z)e \\ &= \beta(gk, qz)\beta(k^{-1}, kqz)\beta(q,z)e = \beta(g, kqz)\beta(q,z)e\end{aligned}$$

using that $\ell \in \mathrm{stab}(qz)$ and that $\beta(k^{-1}, kqz)\beta(q,z)e \in E^\pi_{qz}$ to move from the second line to the third. This means that $\beta(q,z)e \in E^\pi_{kqz}$ for all $k \in G$. By ergodicity (and Theorem 4.5) then $\beta(q,z)e \in E^\pi$. As this holds for all $g \in G$ and $z \in Z$, this means that $\beta : G \times Z \to \mathrm{Iso}(E^\pi)$ is a well-defined cocycle.

Consider now $a \in A_x$ and $g \in \mathrm{stab}(\pi(x))$. For $e \in E^\pi$,

$$(\alpha^*(g,x)a)(e) = a(\alpha(g,x)^{-1}e) = a(e)$$

so the map $r : E_1^* \to (E^\pi)_1^*$ given by restricting to E^π has the property that if $a \in A_x$ and $g \in \mathrm{stab}(\pi(x))$ then $r(\alpha^*(g,x)a) = r(a)$. By Theorem 4.5, since (X,ν) is ergodic, $\mathrm{stab}(z) \curvearrowright (\pi^{-1}(z), D_\pi(z))$ is ergodic almost surely. Therefore, for each $z \in Z$, the set $B_z \subseteq (E^\pi)_1^*$ given by

$$B_z = \{a\big|_{E^\pi} : a \in A_x \text{ for some } x \in X \text{ such that } \pi(x) = z\}$$

is well-defined and $\beta^*(g,z)B_{gz} = B_z$.

The affine space $A \subseteq X \times_{\alpha^*} E_1^*$ then maps to the affine space $B \subseteq Z \times_{\beta^*} (E^\pi)_1^*$ by $(x,q) \mapsto (\pi(x), r(q))$. The space B is the **quotient affine space of** A **by** π. By construction, B is orbital over (Z,ζ) since $\beta(g,z) = e$ for $g \in \mathrm{stab}(z)$.

Note that the quotient affine space requires that π be a G-map to a G-space (and is not well-defined for an arbitrary G-equivariant $\varphi : X \to S(G)$). However, if one first takes the quotient of (X,ν) by φ and then applies the above construction, one still obtains an orbital affine space over the quotient of (X,ν) by φ. Therefore, given a G-equivariant measurable map $\varphi : X \to S(G)$, we define the **quotient affine space of** A **by** φ to be the quotient of A by π where π is the map $(X,\nu) \to (Z,\zeta)$ such that (Z,ζ) is the quotient of (X,ν) by φ.

Consider now a Borel function $f : A \to \mathbb{R}$ such that $f(g \cdot (x,q)) = f(x,q)$ for all $g \in \mathrm{stab}(\pi(x))$. Then f descends to a function on B by the construction of B. Therefore, if $\alpha \in P(A)$ is any probability measure such that $(\mathrm{proj}_X)_*\alpha = \nu$, the $\mathrm{stab}(\pi(\mathrm{proj}_X(a)))$-invariant functions in $L^\infty(A,\alpha)$ are in fact in $L^\infty(B,\beta)$ (where β is the pushforward of α to B). Likewise, any Borel function on B extends to a Borel function on A that is $\mathrm{stab}(\pi(\mathrm{proj}_X(a)))$-invariant. Therefore, the quotient space of (A,α) by $\mathrm{stab}(\pi(\mathrm{proj}_X(a)))$ is isomorphic to (B,β).

4.6 The Product Random Subgroups Functor

One reason for introducing the notion of quotienting by random subgroups is to construct the product random subgroups functor. The product random subgroups functor will play the role in our work that the ergodic decomposition does in the work of Bader and Shalom [BS06] and in this sense is the key to our study of actions.

Definition 4.11. Let G_1 and G_2 be locally compact second countable groups and set $G = G_1 \times G_2$. Let $\Phi : S(G) \to S(G)$ be given by $\Phi(H) = \overline{\mathrm{proj}_{G_1} H} \times \overline{\mathrm{proj}_{G_2} H}$.

The **product random subgroups functor**, denoted by PRG, is the quotient functor F^Φ: for a G-space (X,ν), the quotient space of (X,ν) by $\Phi \circ \mathrm{stab}_G$ is written $PRG(X,\nu)$ and for a G-map $\pi : (X,\nu) \to (Y,\eta)$, the map between quotient spaces is written $PRG(\pi) : PRG(X,\nu) \to PRG(Y,\eta)$.

Proposition 4.6.1. *Let $G = G_1 \times G_2$ be a product of locally compact second countable groups. Let $\pi : (X,\nu) \to (Y,\eta)$ be a G-map of G-spaces. Then there exists a G-map $PRG(\pi) : PRG(X,\nu) \to PRG(Y,\eta)$ such that $PRG \circ \pi = PRG(\pi) \circ PRG$. That is, PRG is a functor on G-spaces.*

Proof. Clearly, for $H \in S(G)$, $H \subseteq \overline{\mathrm{proj}_{G_1} H} \times \overline{\mathrm{proj}_{G_2} H}$ and for $H, L \in S(G)$ with $H \subseteq L$, $\overline{\mathrm{proj}_{G_1} H} \times \overline{\mathrm{proj}_{G_2} H} \subseteq \overline{\mathrm{proj}_{G_1} L} \times \overline{\mathrm{proj}_{G_2} L}$. Then the result is Theorem 4.10. ☐

Proposition 4.6.2. *Let $G = G_1 \times G_2$ be a product of locally compact second countable groups and let (X, ν) be a G-space. Denote by (X_1, ν) and (X_2, ν_2) the spaces of G_1- and G_2-ergodic components (the invariant products functor applied to (X, ν)) and by $PRG(X, \nu)$ the quotient space of (X, ν) over the map $\Phi \circ \mathrm{stab}$ where $\Phi(H) = \overline{\mathrm{proj}_{G_1} H} \times \overline{\mathrm{proj}_{G_2} H}$. Then there exist G-maps*

$$(X, \nu) \to PRG(X, \nu) \to (X_1 \times X_2, \nu_1 \times \nu_2).$$

Proof. Let $\pi_1 : (X, \nu) \to (X_1, \nu_1)$ be the ergodic decomposition map. Let $g \in \mathrm{stab}_G(x)$. Write $g = g_1 g_2$ for $g_1 \in G_1$ and $g_2 \in G_2$. Then $gx = x$ so $\pi_1(x) = \pi_1(gx) = g\pi_1(x) = g_1\pi(x)$ since G_2 acts trivially on X_1. Therefore $\mathrm{proj}_{G_1} \mathrm{stab}_G(x) \subseteq \mathrm{stab}_{G_1}(\pi_1(x))$. Since the stabilizer subgroups are always closed, then $\overline{\mathrm{proj}_{G_1} \mathrm{stab}_G(x)} \subseteq \mathrm{stab}_{G_1}(\pi_1(x))$. Of course the same holds for G_2.

Let $\pi : (X, \nu) \to (X_1, \nu_1) \times (X_2, \nu_2)$ by $\pi(x) = (\pi_1(x), \pi_2(x))$. Then

$$\mathrm{stab}_G(\pi(x)) = \mathrm{stab}_{G_1}(\pi_1(x)) \times G_2 \cap G_1 \times \mathrm{stab}_{G_2}(\pi_2(x)) \supseteq s(x).$$

By the universal property of the quotient space then there exists $\tau : PRG(X, \nu) \to F(X, \nu)$ such that $\tau \circ \psi = \pi$ and the conclusion follows. ☐

Theorem 4.12. *Let $G = G_1 \times G_2$ be a product of locally compact second countable groups. Let $\mu_1 \in P(G_1)$ and $\mu_2 \in P(G_2)$ be admissible probability measures and set $\mu = \mu_1 \times \mu_2$. Let $PRG(X, \nu)$ be the quotient space of (X, ν) by $\varphi(x) = \overline{\mathrm{proj}_{G_1} \mathrm{stab}(x)} \times \overline{\mathrm{proj}_{G_2} \mathrm{stab}(x)}$. If (X, ν) is an ergodic μ-stationary G-space then the G-map $(X, \nu) \to PRG(X, \nu)$ is relatively measure-preserving.*

Proof. By Proposition 2.12.1, the G-map $(X, \nu) \to (X_1 \times X_2, \nu_1 \times \nu_2)$ is relatively measure-preserving. The previous proposition shows that $PRG(X, \nu)$ is an intermediate quotient of these spaces hence the G-maps $(X, \nu) \to PRG(X, \nu)$ and $PRG(X, \nu) \to (X_1 \times X_2, \nu_1 \times \nu_2)$ are relatively measure-preserving. ☐

5 Relative Joinings Over Relatively Contractive Maps

Relatively contractive maps were introduced in [CP13] and can be used to show that any joining between a contractive space and a measure-preserving space such that the projection to the contractive space is relatively measure-preserving is necessarily the independent joining. We generalize this fact to the case of relative joinings and obtain an analogous result.

Theorem 5.1. *Let (X, ν) and (Y, η) be G-spaces with a common G-quotient (Z, ζ) such that $\varphi : (Y, \eta) \to (Z, \zeta)$ is relatively contractive and $\pi : (X, \nu) \to (Z, \zeta)$ is a G-map. Then there exists at most one relative joining of (X, ν) and (Y, η) over (Z, ζ) such that the projection to (Y, η) is relatively measure-preserving.*

Proof. For convenience, write $W = X \times Y$. Let ρ be a relative joining of (X, ν) and (Y, η) over (Z, ζ) such that $\varphi : (Y, \eta) \to (Z, \zeta)$ is relatively contractive, $p_Y : (W, \rho) \to (Y, \eta)$ is relatively measure-preserving and $p_X : (W, \rho) \to (X, \nu)$ and $\pi : (X, \nu) \to (Z, \zeta)$ are G-maps such that $\pi \circ p_X = \varphi \circ p_Y$ almost everywhere. Denote by $\psi : (W, \rho) \to (Z, \zeta)$ the composition: $\psi = \pi \circ p_X = \varphi \circ p_Y$.

Let $z \in Z$ and let $f \in L^\infty(\pi^{-1}(z), D_\pi(z))$ be arbitrary. Then $f \circ p_X \in L^\infty(\psi^{-1}(z), D_\psi(z))$ since $D_\psi(z) = \int D_{p_X}(x)\, dD_\pi(z)(x)$. Define

$$F(y) = D_{p_Y}(y)(f \circ p_X)$$

and observe that $F \in L^\infty(\varphi^{-1}(z), D_\varphi(z))$.

For an arbitrary $g \in G$, using that p_Y is relatively measure-preserving,

$$\begin{aligned} D_\varphi^{(g)}(z)(F) &= \int_{\varphi^{-1}(z)} F(y)\; dg^{-1}D_\varphi(gz) \\ &= \int_{\varphi^{-1}(gz)} F(g^{-1}y)\; dD_\varphi(gz) \\ &= \int_{\varphi^{-1}(gz)} \int_{p_Y^{-1}(g^{-1}y)} f(p_X(w))\; dD_{p_Y}(g^{-1}y)(w)\; dD_\varphi(gz)(y) \\ &= \int_{\varphi^{-1}(gz)} \int_{p_Y^{-1}(g^{-1}y)} f(p_X(w))\; dg^{-1}D_{p_Y}(y)(w)\; dD_\varphi(gz)(y) \end{aligned}$$

$$= \int_{\varphi^{-1}(gz)} \int_{p_Y^{-1}(y)} f(p_X(g^{-1}w)) \; dD_{p_Y}(y)(w) \; dD_\varphi(gz)(y)$$
$$= \int_{\varphi^{-1}(gz)} \int_{p_Y^{-1}(y)} f(g^{-1}p_X(w)) \; dD_{p_Y}(y)(w) \; dD_\varphi(gz)(y)$$

Now $\int_{\varphi^{-1}(gz)} D_{p_Y}(y) \; dD_\varphi(gz)(y) = D_\psi(gz)$ and therefore

$$\begin{aligned} D_\varphi^{(g)}(z)(F) &= \int_{\psi^{-1}(gz)} f(g^{-1}p_X(w)) \; dD_\psi(gz)(w) \\ &= \int_{p_X(\psi^{-1}(gz))} f(g^{-1}x) \; d((p_X)_* D_\psi(gz))(x) \\ &= \int_{\pi^{-1}(gz)} f(g^{-1}x) \; dD_\pi(gz)(x) \\ &= D_\pi^{(g)}(z)(f). \end{aligned}$$

Now let ρ_1 and ρ_2 both be relative joinings over (Z,ζ). Since φ is relatively contractive, there is a measure one set of $z \in Z$ such that for all $F \in L^\infty(\varphi^{-1}(z), D_\varphi(z))$, we have that $\sup_{g\in G} |D_\varphi^{(g)}(F)| = \|F\|$. Fix z in this measure one set.

Let $f \in L^\infty(\pi^{-1}(z), D_\pi(z))$ be arbitrary. Let $D_{p_Y}^j$ and D_ψ^j for $j = 1,2$ denote the disintegrations of ρ_1 and ρ_2 over η and ζ, respectively. Define, for $j = 1,2$,

$$F_j(y) = D_{p_Y}^j(f \circ p_X)$$

and set $F(y) = F_1(y) - F_2(y)$. As above, $F \in L^\infty(\varphi^{-1}(z), D_\varphi(z))$. Now, by the above, for any $g \in G$,

$$D_\varphi^{(g)}(z)(F_1) = D_\pi^{(g)}(z)(f) = D_\varphi^{(g)}(z)(F_2)$$

and therefore $D_\varphi^{(g)}(z)(F) = 0$.

Since z is in the measure one set where that map is an isometry, $\|F\| = \sup_g |D_\varphi^{(g)}(z)(F)| = 0$. Therefore $F = 0$ almost everywhere. As this holds for all $f \in L^\infty(\pi^{-1}(z), D_\pi(z))$, we conclude that $D_\varphi^1(y) = D_\varphi^2(y)$ for $D_\varphi(z)$-almost-every $y \in \varphi^{-1}(z)$.

Now let $f \in L^\infty(\psi^{-1}(z), D_\psi(z))$ be arbitrary and observe that

$$\begin{aligned} D_\psi^j(z)(f) &= \int_{\psi^{-1}(z)} f(x,y) \; dD_\psi^j(z)(x,y) \\ &= \int_{\varphi^{-1}(z)} \int_{p_Y^{-1}(y)} f(x,y) \; dD_{p_Y}^j(y)(x) \; dD_\varphi(z)(y). \end{aligned}$$

Since $D_{p_Y}^1(y) = D_{p_Y}^2(y)$ for $D_\varphi(z)$-almost every y,

$$D_\psi^1(z)(f) = D_\psi^2(z)(f).$$

This holds for all $f \in L^\infty(\psi^{-1}(z), D_\psi(z))$ and so $D_\psi^1(z) = D_\psi^2(z)$.

Since the above holds for all z in a measure one set,

$$\rho_1 = \int_Z D_\psi^1(z) \; d\zeta(z) = \int_Z D_\psi^2(z) \; d\zeta(z) = \rho_2.$$

□

Corollary 5.2. *Let (X,ν) and (Y,η) be G-spaces with a common G-quotient (Z,ζ) such that $\varphi : (Y,\eta) \to (Z,\zeta)$ is relatively contractive and $\pi : (X,\nu) \to (Z,\zeta)$ is relatively measure-preserving. Then the only relative joining of (X,ν) and (Y,η) over (Z,ζ) such that the projection to (Y,η) is relatively measure-preserving is the independent relative joining.*

Proof. By the previous theorem, we need only show that the independent relative joining $\rho = \int D_\pi \times D_\varphi \; d\zeta$ is a relative joining such that the projection to (Y,η) is relatively measure-preserving. Let D_{p_Y}

be the disintegration of ρ over η. Observe that $p_Y^{-1}(y) = \pi^{-1}(\varphi(y)) \times \{y\}$ and that the support of $D_\pi(\varphi(y)) \times \delta_y$ is the same. Now

$$\begin{aligned}\int_Y D_\pi(\varphi(y)) \times \delta_y \; d\eta(y) &= \int_Z \int_Y D_\pi(z) \times \delta_y \; dD_\varphi(z)(y) \; d\eta(y) \\ &= \int_Z D_\pi(z) \times D_\varphi(z) \; d\zeta(z) = \rho\end{aligned}$$

so by uniqueness, $D_{p_Y}(y) = D_\pi(\varphi(y)) \times \delta_y$ almost everywhere. Then, using that π is relatively measure-preserving,

$$D_{p_Y}(gy) = D_\pi(\varphi(gy)) \times \delta_{gy} = gD_\pi(\varphi(y)) \times g\delta_y = gD_{p_Y}(y)$$

so p_Y is relatively measure-preserving. By the previous theorem, ρ is then the unique relative joining. □

Corollary 5.3. *Let G be a locally compact second countable group and let $(X,\nu), (Y,\eta), (Z,\zeta)$ and (W,ρ) be G-spaces such that the following diagram of G-maps commutes:*

$$\begin{tikzcd} (W,\rho) \arrow[r, "\psi"] \arrow[d, "\tau"'] & (X,\nu) \arrow[d, "\pi"'] \\ (Y,\eta) \arrow[r, "\varphi"] & (Z,\zeta) \end{tikzcd}$$

If τ and π are relatively measure-preserving and ψ and φ are relatively contractive then (W,ρ) is G-isomorphic to the independent relative joining of (X,ν) and (Y,η) over (Z,ζ).

Proof. Consider the map $p : W \to X \times Y$ by $p(w) = (\psi(w), \tau(w))$. Then $p_*\rho$ is a relative joining of (X,ν) and (Y,η) over (Z,ζ). Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the natural projections and observe that the following diagram commutes:

$$\begin{tikzcd} (W,\rho) \arrow[r, "p"] & (X \times Y, p_*\rho) \arrow[r, "p_X"] \arrow[d, "p_Y"'] & (X,\nu) \arrow[d, "\pi"'] \\ & (Y,\eta) \arrow[r, "\varphi"] & (Z,\zeta) \end{tikzcd}$$

since $p_X \circ p = \psi$ and $p_Y \circ p = \tau$.

Now ψ is relatively contractive so p and p_X are relatively contractive (Theorem 2.41) and likewise τ being relatively measure-preserving implies p and p_Y are relatively measure-preserving. Therefore p is an isomorphism (Theorem 2.43). Since φ is relatively contractive and p_Y is relatively measure-preserving and π is relatively measure-preserving, the previous corollary says that $p_*\rho$ is the independent relative joining. □

6 The Intermediate Contractive Factor Theorem for Products

We are now ready to prove a strengthening of the Bader-Shalom Intermediate Factor Theorem [BS06], our key improvement being the removal of the requirement that the G-space be irreducible (that is, ergodic for each G_j):

Theorem 6.1. *Let $G = G_1 \times G_2$ be a product of two locally compact second countable groups and let $\mu_j \in P(G_j)$ be admissible probability measures for $j = 1, 2$. Set $\mu = \mu_1 \times \mu_2$.*

Let (B,β) be the Poisson boundary for (G,μ) and let (X,ν) be a measure-preserving G-space. Let (W,ρ) be a G-space such that there exist G-maps $\pi : (B \times X, \beta \times \nu) \to (W,\rho)$ and $\varphi : (W,\rho) \to (X,\nu)$ with $\varphi \circ \pi$ being the natural projection to X.

Let (W_1,ρ_1) be the space of G_2-ergodic components of (W,ρ) and let (W_2,ρ_2) be the space of G_1-ergodic components. Likewise, let (X_1,ν_1) and (X_2,ν_2) be the ergodic components of (X,ν) for G_2 and G_1, respectively.

Then (W,ρ) is G-isomorphic to the independent relative joining of $(W_1,\rho_1) \times (W_2,\rho_2)$ and (X,ν) over $(X_1,\nu_1) \times (X_2,\nu_2)$.

Proof. Let (X_1, ν_1) and (X_2, ν_2) be the spaces of G_1- and G_2-ergodic components of (X, ν), respectively. Then $F^G(X, \nu) = (X_1, \nu_1) \times (X_2, \nu_2)$. Also, $F^G(W, \rho) = (W_1, \rho_1) \times (W_2, \rho_2)$. Consider now $F^G(B \times X, \beta \times \nu)$.

First, observe that since (B, β) is a Poisson boundary, it is a contractive G-space (Theorem 2.37). Since $F^G(B, \beta) = (B_1, \beta_1) \times (B_2, \beta_2)$ is a relatively measure-preserving quotient of a contractive space, by Corollary 2.44, (B, β) is G-isomorphic to $(B_1, \beta_1) \times (B_2, \beta_2)$.

Observe that (B_1, β_1) is a G_1-quotient of the (G_1, μ_1) Poisson boundary since for $(\mu_1 \times \mu_2)^{\mathbb{N}}$-almost every sequence $\omega \in (G_1 \times G_2)^{\mathbb{N}}$, it holds that $\lim \omega_1 \cdots \omega_n \beta$ is a point mass, and writing $\omega_j = (u_j, v_j)$, $\omega_1 \cdots \omega_n \beta = u_1 \cdots u_n \beta_1 \times v_1 \cdots v_n \beta_2$ shows that $\lim u_1 \cdots u_n \beta_1$ is a point mass for $\mu_1^{\mathbb{N}}$-almost every sequence $u \in G_1^{\mathbb{N}}$ (in fact, (B_1, β_1) is the Poisson boundary as shown in [BS06] but we will not need that).

Now G_1 acts trivially on (B_2, β_2) so by Proposition 2.9.1,

$$(B \times X) /\!\!/ G_1 = (B_1 \times B_2 \times X) /\!\!/ G_1 = (B_1 \times X) /\!\!/ G_1 \times B_2.$$

Since (B_1, β_1) is a quotient of the Poisson boundary of (G_1, μ_1) and (X, ν) is a measure-preserving G_1-space, by Proposition 2.11.2, $(B_1 \times X) /\!\!/ G_1 = X /\!\!/ G_1$ so

$$(B \times X) /\!\!/ G_1 = X /\!\!/ G_1 \times B_2.$$

Likewise, $(B \times X) /\!\!/ G_2 = X /\!\!/ G_2 \times B_1$. Therefore

$$F^G(B \times X, \beta \times \nu) = (B_1, \beta_1) \times (B_2, \beta_2) \times (X_1, \nu_1) \times (X_2, \nu_2)$$

with the diagonal action.

Therefore, applying the functor F^G to the given maps $B \times X \to W \to X$, we obtain the following commuting diagram of G-maps (the measures are omitted for clarity):

$$\begin{tikzcd}
B \times X \arrow[r, "\pi"] \arrow[d] & W \arrow[r, "\varphi"] \arrow[d] & X \arrow[d] \\
B_1 \times B_2 \times X_1 \times X_2 \arrow[r, "F^G(\pi)"] & W_1 \times W_2 \arrow[r, "F^G(\varphi)"] & X_1 \times X_2
\end{tikzcd}$$

The vertical maps are all relatively measure-preserving by Proposition 2.12.1 (Proposition 1.10 in [BS06]). Since (B, β) is a contractive G-space and (X, ν) is a measure-preserving G-space, the natural projection $B \times X \to X$ is a relatively contractive G-map by Theorem 2.42. Therefore π and φ are relatively contractive by Theorem 2.41. Likewise, $(X_1 \times X_2, \nu_1 \times \nu_2)$ is a measure-preserving G-space and the composition $F^G(\varphi) \circ F^G(\pi) = F^G(\varphi \circ \pi)$ is the natural projection to $X_1 \times X_2$. Therefore, since $B_1 \times B_2$ is contractive, $F^G(\varphi \circ \pi)$ is relatively contractive. Hence $F^G(\pi)$ and $F^G(\varphi)$ are both relatively contractive.

Isolating the right-hand side of the diagram:

$$\begin{tikzcd}
(W, \rho) \arrow[r] \arrow[d] & (X, \nu) \arrow[d] \\
(W_1, \rho_1) \times (W_2, \rho_2) \arrow[r] & (X_1, \nu_1) \times (X_2, \nu_2)
\end{tikzcd}$$

is a commuting diagram of G-maps such that the vertical arrows are relatively measure-preserving, the horizontal arrows are relatively contractive and (X, ν) is measure-preserving. By Corollary 5.3, (W, ρ) is G-isomorphic to the independent relative joining of $(W_1, \rho_1) \times (W_2, \rho_2)$ and (X, ν) over $(X_1, \nu_1) \times (X_2, \nu_2)$ as claimed. □

Corollary 6.2 (Bader-Shalom Intermediate Factor Theorem [BS06])**.** *Let $G = G_1 \times G_2$ be a product of two locally compact second countable groups and let $\mu_j \in P(G_j)$ be admissible probability measures for $j = 1, 2$. Set $\mu = \mu_1 \times \mu_2$.*

Let (B, β) be the Poisson boundary for (G, μ) and let (X, ν) be a measure-preserving G-space that is ergodic for each G_j. Let (W, ρ) be a G-space such that there exist G-maps $\pi : (B \times X, \beta \times \nu) \to (W, \rho)$ and $\varphi : (W, \rho) \to (X, \nu)$ with $\varphi \circ \pi$ being the natural projection to X.

Then (W, ρ) is G-isomorphic to $(W_1, \rho_1) \times (W_2, \rho_2) \times (X, \nu)$ where (W_1, ρ_1) is a (G_1, μ_1)-boundary and (W_2, ρ_2) is a (G_2, μ_2)-boundary.

Proof. Since the action of each G_j is ergodic on (X,ν), the ergodic components spaces (X_1,ν_1) and (X_2,ν_2) are both trivial. The previous theorem then implies that (W,ρ) is G-isomorphic to the independent relative joining of $(W_1,\rho_1)\times(W_2,\rho_2)$ and (X,ν) over the trivial system, that is (W,ρ) is G-isomorphic to $(W_1,\rho_1)\times(W_2,\rho_2)\times(X,\nu)$. Since F^G is a functor and the X_j are trivial, applying F^G to the maps $B\times X\to W\to X$ gives G-maps $B\to W_1\times W_2\to 0$. Therefore (W_1,ρ_1) is a (G,μ)-boundary on which the G_2-action is trivial, hence it is a (G_1,μ_1)-boundary. Likewise for (W_2,ρ_2). □

7 Actions of Products of Groups

We now are ready to consider the stabilizers of actions of products of locally compact second countable groups. For clarity, we present first the results for the products of two groups in this section and then later handle the general case. Many of the results in this section hold for arbitrary groups, but some require the hypothesis that the factors be simple, hence the need to handle the case of products of more than two groups separately.

We begin by showing that the weak amenability of the action of a product of groups is equivalent to the weak amenability of the action on the product random subgroups functor space corresponding to the action. From there, we deduce that if certain conditions hold on the spaces of ergodic components that the action is necessarily weakly amenable. Combining this with property (T) (and the relative version in the form of resolutions), we conclude with a classification of actions of products of such groups. The study of actions of irreducible lattices in products of groups will be the subject of the next section.

7.1 Weak Amenability and the Product Random Subgroups Functor

Theorem 7.1. *Let G_1 and G_2 be locally compact second countable groups. Let $G=G_1\times G_2$ and let (X,ν) be a measure-preserving G-space. Assume that the G-action on $PRG(X,\nu)$ is weakly amenable. Then the G-action on (X,ν) is weakly amenable.*

Proof. Let A be an affine orbital G-space over (X,ν). Let (C,ζ) be the Poisson boundary of $G=G_1\times G_2$ for the measure $\mu=\mu_1\times\mu_2$ where μ_j are admissible probability measures on G_j, $j=1,2$. By Theorem 2.38 and Proposition 2.3.1, there are then G-maps

$$(C\times X,\zeta\times\nu)\to(A,\alpha_0)\to(X,\nu)$$

with composition being the natural projection to X and α_0 being the push-forward of $\zeta\times\nu$ to A. By Theorem 2.37, $G\curvearrowright(C,\zeta)$ is contractive hence by Theorem 2.42, the projection $(C\times X,\zeta\times\nu)\to(X,\nu)$ is relatively contractive. Therefore by Theorem 2.41, the maps $(C\times X,\zeta\times\nu)\to(A,\alpha_0)$ and $(A,\alpha_0)\to(X,\nu)$ are both relatively contractive.

Let (A_1,α_1) and (A_2,α_2) be the ergodic decompositions of (A,α_0) for G_2 and G_1, respectively. Likewise, let (X_j,ν_j) and (C_j,ν_j) be the decompositions of (X,ν) and (C,ζ). By Propositon 2.11.2, $(C\times X)/\!\!/G_j=C_j\times X_j$. Since the ergodic decomposition is a functor (being a special case of quotienting by a random subgroup), there exist G_j-maps

$$(C_j\times X_j,\zeta_j\times\nu_j)\to(A_j,\alpha_j)\to(X_j,\nu_j)$$

and therefore, as (C_j,ζ_j) is a contractive G_j-space, the maps $(A_j,\alpha_j)\to(X_j,\nu_j)$ are relatively contractive.

Consider the diagram of G-maps:

$$\begin{tikzcd}
A_1\times A_2\times X \arrow[r, "p_X"] \arrow[d, "p_A"'] & X \arrow[d, "\pi"'] \\
A_1\times A_2 \arrow[r, "\varphi"] & X_1\times X_2
\end{tikzcd}$$

where p_X is the projection to the X coordinate, $p_A=p_{A_1}\times p_{A_2}$ is the diagonal product of the projections to A_1 and A_2, $\pi=\pi_1\times\pi_2$ is the diagonal product of the ergodic decomposition maps of X and $\varphi=\varphi_1\times\varphi_2$ is the product of the natural maps $A_j\to X_j$ obtained by the inclusion at the level of σ-algebras.

By the Intermediate Contractive Factor Theorem (Theorem 6.1), since p_X and φ are relatively contractive and p_A and π are relatively measure-preserving, (A, α_0) is G-isomorphic to the independent relative joining of $(A_1, \alpha_1) \times (A_2, \alpha_2)$ and (X, ν) over $(X_1, \nu_1) \times (X_2, \nu_2)$. That is, (A, α_0) is G-isomorphic to $(A_1 \times A_2 \times X, \alpha)$ where

$$\alpha = \int_{X_1 \times X_2} D_{\varphi_1}(x_1) \times D_{\varphi_2}(x_2) \times D_\pi(x_1, x_2) \; d\nu_1 \times \nu_2(x_1, x_2)$$

as this is the independent relative joining.

Apply the product random subgroups functor to (X, ν) and obtain G-maps

$$(X, \nu) \xrightarrow{q} PRG(X, \nu) \xrightarrow{r} (X_1 \times X_2, \nu_1 \times \nu_2)$$

such that $r \circ q = \pi$ (by the universal property (Theorem 4.6) such a map r exists).

Let B be the affine orbital space over $PRG(X, \nu)$ that is the quotient of A by $\mathrm{stab} \circ q$ (constructed in Subsection 4.5) and let $z : A_1 \times A_2 \times X \to B$ be the corresponding map. Endow B with the pushforward measure β_0. Then (B, β_0) is the quotient of (A, α) by the map $a \mapsto \mathrm{stab}(\pi(p_X(a)))$. Then, by the universal property of the quotient spaces (Theorem 4.6), the diagram above extends to:

$$\begin{tikzcd}
A_1 \times A_2 \times X \arrow[r, "p_X"] \arrow[d, "q_A"'] & X \arrow[d, "q"'] \\
PRG(A) \arrow[r, "\psi"] \arrow[d] & PRG(X) \arrow[d, "\simeq"'] \\
B \arrow[r] \arrow[d, "r_A"'] & PRG(X) \arrow[d, "r"'] \\
A_1 \times A_2 \arrow[r, "\varphi"] & X_1 \times X_2
\end{tikzcd}$$

More precisely, the existence of say, the map r_A follows from the fact that (B, β_0) is the quotient of (A, α_0) by $\mathrm{stab} \circ q$ and it holds that

$$\begin{aligned}
\overline{\mathrm{proj}_1 \; \mathrm{stab}(p(a))} \times \overline{\mathrm{proj}_2 \; \mathrm{stab}(p(a))} &\subseteq G_1 \times \overline{\mathrm{proj}_2 \; \mathrm{stab}(p(a))} \\
&= \overline{G_1 \times \{e\} \cdot \mathrm{stab}(p(a))} = \overline{G_1 \cdot \mathrm{stab}(a)}
\end{aligned}$$

by the orbitality of A over X and therefore the universal property (treating A_2 as the quotient of A by $a \mapsto \overline{G_1 \cdot \mathrm{stab}(a)}$) there exists a map $B \to A_2$ (and likewise for A_1). Since A is orbital over X, B is orbital over $PRG(X)$ by construction (see section 4.5). Since the G-action on $PRG(X, \nu)$ is weakly amenable there then exists an invariant section $\tau : PRG(X, \nu) \to B$. That is, $\tau(gq(x)) = g \cdot \tau(q(x))$.

Define the map $\psi : X \to A_1 \times A_2 \times X$ by $\psi(x) = (r_A(\tau(q(x))), x)$. Then

$$\begin{aligned}
\psi(gx) &= (r_A(\tau(q(gx))), gx) = (r_A(\tau(gq(x))), gx) \\
&= (r_A(\beta(g, q(x))\tau(q(x))), gx) = (gr_A(\tau(q(x))), gx) = g(r_A(\tau(q(x))), x) = g\psi(x)
\end{aligned}$$

which is then, over the isomorphism $A \to A_1 \times A_2 \times X$, an invariant section $X \to A$. As this holds for all affine orbital spaces of (X, ν), the G-action on (X, ν) is weakly amenable. $\square$

Theorem 7.2. *Let G_1 and G_2 be locally compact second countable groups. Let $G = G_1 \times G_2$ and let (X, ν) be an ergodic measure-preserving G-space. Let $(X, \nu) \to (X_j, \nu_j)$ denote the spaces of G_{3-j}-ergodic components. Assume that $G_j \curvearrowright (X_j, \nu_j)$ weakly amenably for both $j = 1, 2$ and that* $\mathrm{stab}_*\nu_j$ *are simple invariant random subgroups for $j = 1, 2$. Then one of the following holds:*

- *$G \curvearrowright (X, \nu)$ essentially free;*
- *$G \curvearrowright (X, \nu)$ weakly amenably;*
- $\mathrm{stab}_*\nu = \delta_{\{e\}} \times \mathrm{stab}_*\nu_2$*; or*

- $\mathrm{stab}_*\nu = \mathrm{stab}_*\nu_1 \times \delta_{\{e\}}$.

Proof. Let $\pi_1 : (X,\nu) \to (X_1,\nu_1)$ be the decomposition into G_2-ergodic components. For ν_1-almost every x_1, the G_2-action on $(\pi_1^{-1}(x_1), D_{\pi_1}(x_1))$ is G_2-ergodic. Since $\mathrm{proj}_{G_1}\ \mathrm{stab}_G(x)$ is G_2-invariant, by ergodicity it is constant almost everywhere on almost every ergodic component, that is, for ν_1-almost every x_1 the subgroup $\mathrm{proj}_{G_1}\ \mathrm{stab}_G(x)$ is constant $D_{\pi_1}(x_1)$-almost everywhere. Therefore the map $s_1 : X \to S(G_1)$ by $s_1(x) = \overline{\mathrm{proj}_{G_1}\ \mathrm{stab}_G(x)}$ is constant on fibers over X_1. By Theorem 3.8, then $(s_1)_*\nu \triangleleft \mathrm{stab}_*\nu_1$.

Since $\mathrm{stab}_*\nu_1$ is simple, for ν-almost every $x \in X$ it holds that $s_1(x) = \{e\}$ or $s_1(x) = \mathrm{stab}(\pi_1(x))$. Since the set $\{x \in X : s_1(x) = \{e\}\}$ is G-invariant (because $s_1(gx) = (\mathrm{proj}_{G_1} g)s_1(x)(\mathrm{proj}_{G_1} g)^{-1}$), by the ergodicity of $G \curvearrowright (X,\nu)$ it is either measure zero or measure one. Therefore $(s_1)_*\nu = \delta_{\{e\}}$ or $(s_1)_*\nu = \mathrm{stab}_*\nu_1$. Likewise, $(s_2)_*\nu = \delta_{\{e\}}$ or $(s_2)_*\nu = \mathrm{stab}_*\nu_2$.

Consider first the case when $(s_1)_*\nu = \delta_{\{e\}}$. Then $\mathrm{stab}(x) \subseteq \{e\} \times s_2(x)$ almost everywhere. If in addition, $(s_2)_*\nu = \delta_{\{e\}}$ then $\mathrm{stab}(x) = \{e\}\times\{e\}$ almost everywhere so G acts essentially freely. So instead suppose $(s_2)_*\nu = \mathrm{stab}_*\nu_2$. Then $\mathrm{stab}(x) = \{e\} \times H_x$ for some $H_x < G_2$ and $s_2(x) = \overline{\mathrm{proj}_{G_2}\ \mathrm{stab}(x)} = \overline{H_x} = H_x$ since H_x is necessarily closed (as $\mathrm{stab}(x)$ is always closed). Therefore $H_x = \mathrm{stab}(\pi_2(x))$ almost everywhere (since $(s_2)_*\nu = \mathrm{stab}_*\nu_2$) meaning that $\mathrm{stab}(x) = \{e\} \times \mathrm{stab}(\pi_2(x))$ almost everywhere and so $\mathrm{stab}_*\nu = \delta_{\{e\}} \times \mathrm{stab}_*\nu_2$. The symmetric case follows the same way.

Consider now the case when $(s_1)_*\nu = \mathrm{stab}_*\nu_1$ and $(s_2)_*\nu = \mathrm{stab}_*\nu_2$ and consider the G-maps $q : (X,\nu) \to PRG(X,\nu)$ and $r : PRG(X,\nu) \to (X_1 \times X_2, \nu_1 \times \nu_2)$ such that $r \circ q = \pi_1 \times \pi_2$. By construction, for almost every $x \in X$ it holds that $s_1(x) \times s_2(x) \subseteq \mathrm{stab}(q(x))$. Since $(s_1)_*\nu = \mathrm{stab}_*\nu_1$ and $(s_2)_*\nu = \mathrm{stab}_*\nu_2$, then for almost every $x \in X$, we have that $\mathrm{stab}(\pi_1(x)) \times \mathrm{stab}(\pi_2(x)) \subseteq \mathrm{stab}(q(x))$. But as $PRG(X,\nu)$ is an extension of $(X_1 \times X_2, \nu_1 \times \nu_2)$, this means that $PRG(X,\nu)$ is orbital over $(X_1 \times X_2, \nu_1 \times \nu_2)$. Since each $G_j \curvearrowright (X_j,\nu_j)$ weakly amenably, $G_1 \times G_2 \curvearrowright (X_1 \times X_2, \nu_1 \times \nu_2)$ weakly amenably by Proposition 2.4.2. Then by Proposition 2.4.1, $G \curvearrowright PRG(X,\nu)$ weakly amenably so by Theorem 7.1, $G \curvearrowright (X,\nu)$ weakly amenably. □

Theorem 7.3. *Let G_1 and G_2 be locally compact second countable groups. Let $G = G_1 \times G_2$ and let (X,ν) be an ergodic measure-preserving G-space. If $\mathrm{proj}_{G_2}\ \mathrm{stab}(x)$ is dense in G_j almost everywhere for both $j = 1, 2$ then $G \curvearrowright (X,\nu)$ weakly amenably.*

Proof. When both projections are dense almost everywhere, $PRG(X,\nu)$ is the quotient by $G_1 \times G_2$ hence G acts trivially on $PRG(X,\nu)$. As (X,ν) is ergodic, so is $PRG(X,\nu)$ and therefore $PRG(X,\nu)$ is the trivial (one-point) space. Clearly every group acts weakly amenably on the trivial space, so the conclusion follows by Theorem 7.1. □

7.2 Irreducible Actions

Theorem 7.4. *Let G_1 and G_2 be locally compact second countable groups. Set $G = G_1 \times G_2$ and let (X,ν) be a measure-preserving G-space such that each $G_j \curvearrowright (X,\nu)$ ergodically for both $j = 1, 2$. Then there exist normal subgroups $N_1 \triangleleft G_1$ and $N_2 \triangleleft G_2$ such that, setting $N = N_1 \times N_2$, it holds that G/N acts essentially freely on the space of N-ergodic components $(X,\nu)/\!\!/N$ and N acts weakly amenably on almost every N-ergodic component.*

Proof. Consider the functions $s_j(x) = \overline{\mathrm{proj}_{G_j}\ \mathrm{stab}(x)}$ for each $j = 1, 2$. Each s_j is G_{3-j}-invariant so by ergodicity is constant almost surely. Set $N_j = s_j(x)$. Since $N_j = s_j(g_j x) = g_j s_j(x) g_j^{-1} = g_j N_j g_j^{-1}$ for any $g_j \in G_j$, we have that $N_j \triangleleft G_j$.

Let (Y,η) be the space of $N_1 \times N_2$-ergodic components and let $\pi : (X,\nu) \to (Y,\eta)$ be the G-map. Then $\mathrm{stab}(\pi(x)) = \mathrm{stab}(x) \cdot N_1 \times N_2 = N_1 \times N_2$ almost everywhere so $G/(N_1 \times N_2)$ acts essentially freely on (Y,η).

Now for almost every $x \in X$, we have that $\mathrm{stab}(x) < N_1 \times N_2$ and that $\mathrm{proj}_{N_j}\ \mathrm{stab}(x)$ is dense in N_j for both $j = 1, 2$. Since $N_1 \times N_2$ acts ergodically on almost every ergodic component $y \in Y$, by Theorem 7.3, we have that $N_1 \times N_2$ acts weakly amenably on $(\pi^{-1}(y), D_\pi(y))$. □

Corollary 7.5. *Let G_1 and G_2 be simple locally compact second countable groups. Set $G = G_1 \times G_2$ and let (X,ν) be a measure-preserving G-space such that each $G_j \curvearrowright (X,\nu)$ ergodically for both $j = 1, 2$. Then either $G \curvearrowright (X,\nu)$ is essentially free or $G \curvearrowright (X,\nu)$ weakly amenably.*

Proof. Assume the action is not essentially free. By Theorem 7.4, as G_1 and G_2 are simple, there exists $N \triangleleft G$ of the form $N = \{e\}$, $N = G_1 \times \{e\}$, $N = \{e\} \times G_2$ or $N = G$ such that G/N acts essentially freely on the space of N-ergodic components and N acts weakly amenably on almost every ergodic component. Since the action is not essentially free, N is not the trivial group. Since both G_j act ergodically, G/N acts essentially freely on the trivial space meaning that $N = G$. Therefore G acts weakly amenably on the only N-ergodic component which is (X, ν) itself. □

7.3 Weakly Amenable Actions and Property (T)

We now show how the presence of property (T) in only one of the two groups in the product is enough to rule out weakly amenable actions that are not essentially transitive. We begin with some basic facts about such actions and then employ resolutions to rule them out.

Proposition 7.3.1. *Let G be a locally compact second countable group and (X, ν) an ergodic measure-preserving G-space such that $G \curvearrowright (X, \nu)$ weakly amenably and not essentially transitively. Then there exists a sequence of almost invariant (but not invariant) vectors in $L_0^2(X, \nu)$, the subspace of $L^2(X, \nu)$ orthogonal to the constants.*

Proof. Since $G \curvearrowright (X, \nu)$ is weakly amenable, by the Connes-Feldman-Weiss theorem (Theorem 2.26), it is orbit equivalent to an action of $\mathbb{Z}$ or $\mathbb{R}$ on a probability space (Y, η). Krasa [Kra85] has shown that if a group H is amenable as a discrete group (which both $\mathbb{Z}$ and $\mathbb{R}$ are) and there is a unique invariant mean on $L^\infty(Y, \eta)$ then there exists a positive measure orbit (when H is countable, this is due to del Junco and Rosenblatt [dJR79]). Clearly the uniqueness of an invariant mean and the existence of a positive measure orbit are characteristics of the equivalence relation, so we conclude that if $G \curvearrowright (X, \nu)$ has a unique invariant mean then the action is essentially transitive (using ergodicity, the positive measure orbit is of full measure).

Rosenblatt [Ros81] (Theorem 1.4) showed that if $G \curvearrowright (X, \nu)$ admits more than one invariant mean then there exists a positive measure set $E \subseteq X$ and an approximately invariant net (A_α) of measurable sets such that $A_\alpha \subseteq X \setminus E$ for all α. Approximately invariant means that for all $g \in G$ it holds that $\lim_\alpha (\nu(A_\alpha))^{-1} \nu(g^{-1} A_\alpha \triangle A_\alpha) = 0$.

Define the functions $f_\alpha = \mathbb{1}_{A_\alpha} - \nu(A_\alpha) \in L_0^2(X, \nu)$. Then $\|f_\alpha\|_2^2 = \nu(A_\alpha)(1 - \nu(A_\alpha))$. For $g \in G$, $\int \left| g \cdot f_\alpha - f_\alpha \right|^2 d\nu = \nu(g^{-1} A_\alpha \triangle A_\alpha)$. Let $q_\alpha = \|f_\alpha\|_2^{-1} f_\alpha$. Then for $g \in G$,

$$\|g \cdot q_\alpha - q_\alpha\|_2^2 = \frac{\nu(g^{-1} A_\alpha \triangle A_\alpha)}{\nu(A_\alpha)(1 - \nu(A_\alpha))} \leq \frac{\nu(g^{-1} A_\alpha \triangle A_\alpha)}{\nu(A_\alpha)} \frac{1}{\nu(E)} \to 0$$

since $\nu(E) > 0$ and $A_\alpha \subseteq X \setminus E$. So the $\{q_\alpha\}$ are almost invariant (norm one) vectors.

The reader is referred to Hjorth and Kechris [HK05] Appendix A for a detailed account of the theory of nonuniqueness of invariant means for equivalence relations arising from group actions. □

Proposition 7.3.2. *Let $G = G_1 \times G_2$ be a product of two locally compact second countable groups such that G_2 has property (T). Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of G on a Hilbert space that has almost invariant vectors that are not invariant. Then π restricted to the space of G_2-invariant, but not G_1-invariant, vectors has almost invariant vectors (as G-, hence as a G_1-) representation.*

Proof. Without loss of generality, we may assume that $\mathcal{H}$ has no G-invariant vectors by simply restricting π to the complement of the invariant vectors. By Proposition 2.15.2, the projection map $\mathrm{proj}_1 : G \to G_1$ is a resolution since G_2 has property (T). The space of (G_1, proj_1)-points in $\mathcal{H}$, denoted $\mathcal{H}^{G_1}$, is a closed G-invariant subspace (Proposition 2.15.1). Since π has almost invariant vectors and proj_1 is a resolution, $\pi^{G_1} : G_1 \to \mathcal{H}^{G_1}$ also has almost invariant vectors.

Observe that if $v \in \mathcal{H}$ is G_2-invariant and if $\{g_n\}$ is any sequence in G such that $\mathrm{proj}_1\, g_n$ converges to some $g_\infty \in G_1$ then $\pi(g_n)v = \pi(\mathrm{proj}_1\, g_n)v \to \pi(g_\infty)v$ since π is a continuous. Therefore the space of G_2-invariant vectors is contained in $\mathcal{H}^{G_1}$. Suppose now that for some $v \in \mathcal{H}$ there exists $h \in G_2$ such that $\pi(h)v \neq v$. Consider the sequence $\{g_n\}$ in G given by $g_n = e$ for n even and $g_n = h$ for n odd. Then $\mathrm{proj}_1\, g_n = e$ for all n which converges in G_1 but $\pi(g_n)v = v$ for n even and $\pi(g_n)v = \pi(h)v \neq v$ for n odd. Therefore $v \notin \mathcal{H}^{G_1}$. So we conclude that the space of G_1-points is precisely the space of G_2-invariant vectors. □

Theorem 7.6. *Let $G = G_1 \times G_2$ be a product of two locally compact second countable groups such that G_2 has property (T). Let (X, ν) be an ergodic measure-preserving G-space such that $G \curvearrowright (X, \nu)$ weakly amenably and not essentially transitively. Let $\mathcal{H}$ be the subspace of $L^2(X, \nu)$ consisting of the G_2-invariant functions that are not G-invariant. Then there exists a sequence of almost invariant vectors in $\mathcal{H}$.*

Proof. By Proposition 7.3.1 there is a sequence of almost invariant vectors in $L^2(X, \nu)$ that are not invariant. Since G_2 has property (T), by Proposition 7.3.2, there is a sequence of almost invariant vectors in the space of G_2-invariant but not G_1-invariant functions. $\square$

7.4 Actions of Products of Groups, at least one with Property (T)

Corollary 7.7. *Let G_1 be a simple locally compact second countable group with property (T) and let G_2 be any locally compact second countable group. Set $G = G_1 \times G_2$ and let (X, ν) be a measure-preserving G-space such that each $G_j \curvearrowright (X, \nu)$ ergodically for both $j = 1, 2$. Then either the kernel of the G-action is of the form $N = \{e\} \times N_2$ for some $N_2 \lhd G_2$ and the G/N-action on (X, ν) is essentially free or else $G \curvearrowright (X, \nu)$ is essentially transitive.*

Proof. By Theorem 7.4, there exists $N_1 \lhd G_1$ and $N_2 \lhd G_2$ such that, setting $N = N_1 \times N_2$, we have that G/N acts essentially freely on the space of N-ergodic components and N acts weakly amenably on almost every ergodic component. Since G_1 is simple, either N_1 is trivial or $N_1 = G_1$.

Consider the case when N_1 is trivial. Then $G/N = G_1 \times (G_2/N_2)$ acts essentially freely on the space of N-ergodic components. Let $\pi : (X, \nu) \to (Y, \eta)$ be the G-map to the space of N-ergodic components. Since $G/N \curvearrowright (X, \nu)$ essentially freely, $\mathrm{stab}(x) < N = \{e\} \times N_2$ almost surely. Therefore $\{e\} \times \overline{\mathrm{proj}_{G_2} \mathrm{stab}(x)} = \mathrm{stab}(x)$. But $\{e\} \times \overline{\mathrm{proj}_{G_2} \mathrm{stab}(x)}$ is G_1-invariant hence constant by ergodicity. Therefore $\mathrm{stab}(x)$ is constant almost surely meaning that $\mathrm{stab}(x) = \ker(G \curvearrowright (X, \nu))$ which is of the form $\{e\} \times N_2'$ for some $N_2' \lhd G_2$.

Now consider the case when $N_1 = G_1$. Since G_1 acts ergodically, the space of N-ergodic components is trivial and (X, ν) is the only N-ergodic component. As G/N acts essentially freely on the space of N-ergodic components, which is trivial, then $N = G$ so $N_2 = G_2$. So $N = G$ acts weakly amenably on (X, ν). Suppose that G does not act essentially transitively on (X, ν). Then, by Theorem 7.6, the space of G_1-invariant but not G_2-invariant functions in $L^2(X, \nu)$ has almost invariant vectors. But G_1 acts ergodically on (X, ν) so the only G_1-invariant functions are the constants which are themselves G_2-invariant. This contradiction means that G acts essentially transitively. $\square$

Corollary 7.8. *Let G be a product of at least two simple noncompact locally compact second countable groups, at least one with property (T) and let (X, ν) be a measure-preserving G-space that is ergodic for each simple factor of G. Then $G \curvearrowright (X, \nu)$ is either essentially free or essentially transitive.*

Proof. Write $G = G_1 \times H$ where G_1 is simple and has property (T) and $H = \prod_j H_j$ is a product of simple groups. By Corollary 7.7, either the kernel of the G-action is of the form $\{e\} \times N$ for some $N \lhd H$ and the G/N-action on (X, ν) is essentially free or else $G \curvearrowright (X, \nu)$ is essentially transitive. Therefore, the only thing to check is that if the kernel is of the form $\{e\} \times N$ then N is necessarily trivial (making the G-action on (X, ν) essentially free).

Suppose then that the kernel is $\{e\} \times N$ for N nontrivial. First note that if H is a single simple group then $N = H$ in which case H acts ergodically and trivially on (X, ν) making it trivial. So we may assume that H has at least two factors. Since N is nontrivial, it has nontrivial projection to some simple factor of H. Without loss of generality, we assume that N projects nontrivially to H_1.

Let $n \in N$ such that $\mathrm{proj}_{H_1} n \neq e$ and let $h_1 \in H_1$. Write $n = (n_1, n_2)$ where $n_1 \in H_1$ and $n_2 \in \prod_{j \neq 1} H_j$. Then $h_1 n h_1^{-1} n^{-1} = (h_1 n_1 h_1^{-1} n_1^{-1}, e) \in H_1$ and also, as N is normal, $h_1 n h_1^{-1} n^{-1} \in N$. Now, if $n_1 \notin Z(H_1)$ then there exists $h_1 \in H_1$ such that $h_1 n_1 h_1^{-1} n_1^{-1}$ is nontrivial. Since H_1 is simple and noncompact, $Z(H_1)$ is trivial. Therefore, there exists $h_1 n h_1^{-1} n^{-1} \in N \cap H_1$ that is nontrivial. Since $N \cap H_1 \lhd H_1$, then $N \cap H_1 = H_1$. Hence H_1 acts trivially on (X, ν) and ergodically meaning that (X, ν) is trivial. $\square$

7.5 Actions of Products of Property (T) Groups

Corollary 7.9. *Let G_1 and G_2 be locally compact second countable groups with property (T). Set $G = G_1 \times G_2$ and let (X, ν) be an ergodic measure-preserving G-space. Assume that there exists simple closed subgroups $H_j < G_j$ such that the space of G_{3-j}-ergodic components is isomorphic to $(G_j/H_j, \mathrm{Haar})$ for $j = 1, 2$ and such that any nontrivial normal subgroup of H_j has finite index in H_j. Then either at least one $G_j \curvearrowright (X, \nu)$ essentially free or $G \curvearrowright (X, \nu)$ is essentially transitive.*

Proof. First observe that since (X, ν) is measure-preserving, so is $(G_j/H_j, \mathrm{Haar})$ and therefore H_j has finite covolume in G_j. Recall that the map $s_1(x) = \overline{\mathrm{proj}_{G_1} \mathrm{stab}(x)}$ has the property that $s_1(x) \triangleleft \mathrm{stab}(\pi_1(x)) = g_1 H_1 g_1^{-1}$ where π_1 is the ergodic decomposition map. Since H_1 has the property that every normal subgroup is either trivial or of finite index, $s_1(x)$ is either trivial or has finite index in some conjugate of H_1 almost everywhere. Since G_1 acts ergodically on $(G_1/H_1, \mathrm{Haar})$, either $s_1(x) = \{e\}$ almost everywhere or $s_1(x)$ has finite index in a conjugate of H_1 almost everywhere. The case when $s_1(x) = \{e\}$ corresponds to G_1 acting essentially freely on (X, ν). As the same reasoning holds for $s_2(x) = \overline{\mathrm{proj}_{G_2} \mathrm{stab}(x)}$, we may assume from here on that $\varphi(x) = s_1(x) \times s_2(x)$ is of the form $\varphi(x) = gKg^{-1}$ where $K = K_1 \times K_2$ with K_j of finite index in H_j.

Since K_j has finite index in H_j, K_j also has finite covolume in G_j. Let $\psi : (X, \nu) \to PRG(X, \nu)$ be the product random subgroups functor map. Then $\mathrm{stab}(\psi(x))$ has finite covolume in G since $\varphi(x) \subseteq \mathrm{stab}(\psi(x))$. We may therefore define an invariant mean m_x on $PRG(X, \nu)$ using the Haar measures on $G/\mathrm{stab}(\psi(x))$. We conclude that $PRG(X, \nu)$ is then weakly amenable. Hence (X, ν) is weakly amenable by Theorem 7.1. Since G has property (T) then the action is essentially transitive. □

8 Actions of Lattices in Product Groups

Having completed our study of the actions of products of two groups, we now turn to the study of actions of irreducible lattices in such products. As with the previous section, we first state and prove the results for the product of two simple groups and in the following section generalize to the case of arbitrary products. Unlike in the case of actions of the products of groups, a full classification of the stabilizers of actions of lattices is only possible under the additional assumption that the groups have the Howe-Moore property.

8.1 The Projected Action

In addition to the product random subgroups functor (which we apply to the induced action), we need a similar object that can be obtained directly from the action of a lattice. The projected action, which we define presently, is in essence the same as the product random subgroups functor, with the caveat that it is not, strictly speaking, a quotient of the action of the lattice. The idea in the proofs in this section is to make use of information gained from studying the projected action to conclude facts about the product random subgroups functor applied to the induced action.

Theorem 8.1. *Let G_1 and G_2 be noncompact locally compact second countable groups and set $G = G_1 \times G_2$. Let $\Gamma < G$ be an irreducible lattice and let (X, ν) be an ergodic measure-preserving Γ-space. Consider the maps $s_j : X \to S(G_j)$ given by $s_j(x) = \overline{\mathrm{proj}_{G_j} \mathrm{stab}_\Gamma(x)}$ for $j = 1, 2$. Then each $(s_j)_*\nu$ is an invariant random subgroup of G_j.*

Proof. Clearly $(s_j)_*\nu \in P(S(G_j))$ so the only thing to check is that it is conjugation-invariant. For $\gamma \in \Gamma$,

$$\begin{aligned}(\mathrm{proj}_{G_j} \gamma) s_j(x) (\mathrm{proj}_{G_j} \gamma)^{-1} &= (\mathrm{proj}_{G_j} \gamma)\overline{\mathrm{proj}_{G_j} \mathrm{stab}(x)}(\mathrm{proj}_{G_j} \gamma)^{-1} \\ &= \overline{\mathrm{proj}_{G_j} \gamma \mathrm{stab}(x) \gamma^{-1}} = \overline{\mathrm{proj}_{G_j} \mathrm{stab}(\gamma x)}\end{aligned}$$

so $(s_j)_*\nu$ is invariant under conjugation by $\mathrm{proj}_{G_j} \Gamma$ since ν is Γ-invariant. Since Γ is irreducible, $\mathrm{proj}_{G_j} \Gamma$ is dense in G_j and as the action by conjugation is continuous this means that $(s_j)_*\nu$ is conjugation invariant. □

Definition 8.2. Let G_1 and G_2 be locally compact second countable groups and set $G = G_1 \times G_2$. Let $\Gamma < G$ be an irreducible lattice and let (X, ν) be a measure-preserving Γ-space. The **projected action** of $\Gamma \curvearrowright (X, \nu)$ is the spaces $G_j \curvearrowright (Y_j, \eta_j)$ for $j = 1, 2$ that the ergodic $(s_j)_*$-nonfree actions of G_j.

8.2 Actions of Lattices in Products of Howe-Moore Groups

Theorem 8.3. *Let G_1 and G_2 be simple nondiscrete noncompact locally compact second countable groups with the Howe-Moore property and set $G = G_1 \times G_2$. Let $\Gamma < G$ be an irreducible lattice and let (X, ν) be an ergodic measure-preserving Γ-space. Then one of the following holds:*

- $\Gamma \curvearrowright (X, \nu)$ *is essentially free;*
- $\Gamma \curvearrowright (X, \nu)$ *is weakly amenable;*
- $\mathrm{stab}_*\nu$ *is supported on the finite index subgroups of* Γ*;*
- $\mathrm{stab}_*\nu$ *is supported on the torsion elements of* Γ*; or*
- *one G_j is totally disconnected and acts ergodically and essentially freely on the induced space $G \times_\Gamma X$ and the other G_{3-j} does not act ergodically on the induced space.*

Proof. Assume that $\Gamma \curvearrowright (X, \nu)$ is not essentially free. For $\gamma \in \Gamma$, let $E_\gamma = \{x \in X : \gamma x = x\}$. Let (Y_j, η_j) be the projected action to G_1 and G_2. Since the Γ-action is not essentially free, there exists $\gamma \in \Gamma \setminus \{e\}$ such that $\nu(E_\gamma) > 0$. Let $L = \{\gamma \in \Gamma \setminus \{e\} : \nu(E_\gamma) > 0\}$. So $L \neq \emptyset$.

Consider first the case when there exists $\gamma \in L$ such that $\langle \mathrm{proj}_{G_1} \gamma\rangle$ is unbounded in G_1 (that is, $\overline{\langle \mathrm{proj}_{G_1} \gamma\rangle}$ is noncompact). Since $\gamma \in \mathrm{stab}(x)$ on the positive measure set E_γ, there exists $F_\gamma \subseteq Y_1$ with $\eta_1(F_1) > 0$ such that $\mathrm{proj}_{G_1} \gamma \in \mathrm{stab}(y)$ for all $y \in F_\gamma$ (because $\mathrm{stab}_*\eta_1 = (s_1)_*\nu$). Then $\overline{\langle \mathrm{proj}_{G_1} \gamma\rangle} \subseteq \mathrm{stab}(y)$ for all $y \in F_\gamma$. Since G_1 has the Howe-Moore property and (Y_1, η_1) is an ergodic measure-preserving G_1-space, by Theorem 2.30, $G_1 \curvearrowright (Y_1, \eta_1)$ is mixing. As $\langle \mathrm{proj}_{G_1} \gamma\rangle$ is unbounded, $\lim_n \eta_1((\mathrm{proj}_{G_1} \gamma)^n F_\gamma \cap F_\gamma) = (\eta_1(F_\gamma))^2$. But $\mathrm{proj}_{G_1} \gamma$ fixes F_γ so $\eta_1((\mathrm{proj}_{G_1} \gamma)^n F_\gamma \cap F_\gamma) = \eta_1(F_\gamma)$. Therefore $\eta_1(F_\gamma) = (\eta_1(F_\gamma))^2$ meaning that $\eta_1(F_\gamma) = 1$. Then $\mathrm{proj}_{G_1} \gamma \in \ker(G_1 \curvearrowright Y_1)$ and as G_1 is simple then (Y_1, η_1) is the trivial space since it is G_1-ergodic. So $\overline{\mathrm{proj}_{G_1} \mathrm{stab}_\Gamma(x)} = G_1$ almost everywhere.

So we conclude that if there exists $\gamma_j \in \Gamma$ for both $j = 1, 2$ such that $\nu(E_{\gamma_j}) > 0$ and $\langle \mathrm{proj}_{G_j} \gamma_j\rangle$ is unbounded in G_j then $\overline{\mathrm{proj}_{G_j} \mathrm{stab}_\Gamma(x)} = G_j$ almost everywhere. Let $G \times_\Gamma X = (F \times X, m \times \nu)$ be the induced action to G from $\Gamma \curvearrowright (X, \nu)$ (here F is a fundamental domain for Γ with cocycle $\alpha : G \times F \to \Gamma$ such that $gf\alpha(g, f) \in F$ and the action is given by $g \cdot (f, x) = (gf\alpha(g, f), \alpha(g, f)^{-1}x)$). Let $(Z, \zeta) = PRG(G \times_\Gamma X)$ be the product random subgroups space for the induced action and let $\psi : (F \times X, m \times \nu) \to (Z, \zeta)$ be the defining map. Then

$$\mathrm{stab}_G(f, x) = f \mathrm{stab}_\Gamma(x) f^{-1}$$

for almost every $(f, x) \in F \times X$. So

$$\mathrm{proj}_{G_j} \mathrm{stab}_G(f, x) = (\mathrm{proj}_{G_j} f) \mathrm{proj}_{G_j} \mathrm{stab}_\Gamma(x) (\mathrm{proj}_{G_j} f)^{-1}$$

is dense almost everywhere for both $j = 1, 2$. Then by Theorem 7.3, $G \curvearrowright G \times_\Gamma X$ weakly amenably so by Proposition 2.6.1, $\Gamma \curvearrowright (X, \nu)$ weakly amenably.

Consider now the case when for every $\gamma \in \Gamma$ such that $\nu(E_\gamma) > 0$, it holds that $\langle \mathrm{proj}_{G_j} \gamma\rangle$ is bounded in G_j for both $j = 1, 2$. Then $\langle \gamma\rangle \subseteq \overline{\langle \mathrm{proj}_{G_1} \gamma\rangle} \times \overline{\langle \mathrm{proj}_{G_2} \gamma\rangle}$ which is a compact subgroup. But $\langle \gamma\rangle$ is discrete and therefore finite meaning that γ is torsion. Therefore, in this case, $\mathrm{stab}_*\nu$ is supported on the torsion elements since every non-torsion $\gamma \in \Gamma$ has $\nu(E_\gamma) = 0$.

So we are left with the case when there exists $\gamma_1 \in \Gamma$ with $\nu(E_{\gamma_1}) > 0$ and $\langle \mathrm{proj}_{G_1} \gamma\rangle$ unbounded in G_1 but that for every $\gamma \in \Gamma$ with $\nu(E_\gamma) > 0$, it holds that $\langle \mathrm{proj}_{G_2} \gamma\rangle$ is bounded in G_2 (or the reverse situation, which is the same by symmetry). Then, as above, $\mathrm{proj}_{G_1} \mathrm{stab}_\Gamma(x)$ is dense almost everywhere. Let $\psi : G \times_\Gamma X \to PRG(G \times_\Gamma X)$ be the map defining the product random subgroups space of the induced action. Let (Z_j, ζ_j) be the space of G_{3-j}-ergodic components of $G \times_\Gamma X$ for $j = 1, 2$. Let

$\tau : PRG(G \times_\Gamma X) \to (Z_1 \times Z_2, \zeta_1 \times \zeta_2)$ be the map from Proposition 4.6.2 such that $\tau \circ \psi$ is the ergodic decomposition. Then for $m \times \nu$-almost every $(f, x) \in F \times X$,

$$\mathrm{stab}(\tau \circ \psi(f, x)) \supseteq \mathrm{stab}(\psi(f, x)) \supseteq G_1 \times (\mathrm{proj}_{G_2} f)\overline{\mathrm{proj}_{G_2} \mathrm{stab}_\Gamma(x)}(\mathrm{proj}_{G_2} f)^{-1}.$$

Therefore $\mathrm{stab}_{G_1}(z_1) = G_1$ for ζ_1-almost every $z_1 \in Z_1$. Since G_1 acts ergodically on (Z_1, ζ_1), the space is trivial. As (Z_1, ζ_1) is the space of G_2-ergodic components this means that G_2 acts ergodically on $G \times_\Gamma X$. Furthermore, $\mathrm{stab}_{G_2}(f, x) = (\{e\} \times G_2) \cap f\mathrm{stab}_\Gamma(x)f^{-1}$. Since Γ is irreducible, $\Gamma \cap (\{e\} \times G_2) = \{e\}$ by Proposition 2.5.1. So if $g \in \mathrm{stab}_{G_2}(f, x)$ then $f^{-1}gf \in \{e\} \times G_2 \cap \mathrm{stab}_\Gamma(x) = \{e\}$ so $g = e$. Hence $G_2 \curvearrowright G \times_\Gamma X$ ergodically and essentially freely.

We now show that we are in the fifth case. Suppose that G_1 also acts ergodically on the induced space $G \times_\Gamma X$. Then both G_j act ergodically on $G \times_\Gamma X$ so by Corollary 7.5, $G \curvearrowright G \times_\Gamma X$ is either essentially free or weakly amenable. Therefore $\Gamma \curvearrowright (X, \nu)$ is either essentially free or weakly amenable by Proposition 2.6.1.

Suppose now that G_2 is connected (since it is simple, if it is not connected then it is totally disconnected). Then G_2 is a simple real Lie group since it has the Howe-Moore property (Theorem 2.33). By Theorem 2.35, since $\mathrm{proj}_{G_2} \Gamma$ is a countable subgroup of G_2, either (Y_2, η_2) is the trivial space or else $\mathrm{proj}_{G_2} \Gamma$ acts essentially freely on (Y_2, η_2).

If (Y_2, η_2) is trivial then $\mathrm{stab}_\Gamma(x)$ projects densely to G_2 almost everywhere. As we already have that it projects densely almost everywhere to G_1 this means that $G \curvearrowright G \times_\Gamma X$ weakly amenably by Theorem 7.3 and so by Proposition 2.6.1, $\Gamma \curvearrowright (X, \nu)$ weakly amenably. So we are left with the case when $\mathrm{proj}_{G_2} \Gamma$ acts essentially freely on (Y_2, η_2). But this means exactly that $\mathrm{proj}_{G_2} \mathrm{stab}_\Gamma(x) = \{e\}$ almost everywhere. So if $\nu(E_\gamma) > 0$ then $\gamma \in (G_1 \times \{e\}) \cap \Gamma = \{e\}$ since Γ is irreducible (Proposition 2.5.1) which means that $\Gamma \curvearrowright (X, \nu)$ essentially freely. So if the Γ-action is not essentially free then G_2 must be totally disconnected. □

Remark 8.4. *The fifth possibility in the previous theorem, that one of the groups be totally disconnected with certain other properties, is exactly the case that the results of [CP13] handle. In this sense, our work here complements perfectly that of [CP13].*

8.3 Actions of Lattices in Products of Howe-Moore Groups, at least one with property (T)

Corollary 8.5. *Let G_1 and G_2 be simple nondiscrete noncompact locally compact second countable groups with the Howe-Moore property such that at least one G_j has property (T). Set $G = G_1 \times G_2$ and let $\Gamma < G$ be an irreducible lattice. Let (X, ν) be an ergodic measure-preserving Γ-space. Then one of the following holds:*

- $\Gamma \curvearrowright (X, \nu)$ *is essentially free;*
- $\mathrm{stab}_*\nu$ *is supported on the finite index subgroups of* Γ*;*
- $\mathrm{stab}_*\nu$ *is supported on the torsion elements of* Γ*;*
- *one G_j is totally disconnected and acts ergodically and essentially freely on the induced space $G \times_\Gamma X$ and the other G_{3-j} does not act ergodically on the induced space; or*
- *one G_j does not have property (T), is totally disconnected, and there is a nontrivial ergodic G_j-space (Y, η) that is a Γ-quotient of (X, ν) and such that $G_j \curvearrowright (Y, \eta)$ is not essentially transitive and $\Gamma \curvearrowright (Y, \eta)$ is weakly amenable.*

Proof. Assume none of the first four possibilities hold. By the previous theorem, $\Gamma \curvearrowright (X, \nu)$ weakly amenably. Note that if $\Gamma \curvearrowright (X, \nu)$ essentially transitively then (X, ν) is necessarily a finite atomic space by ergodicity, in which case the stabilizers are finite index subgroups of Γ. As this possibility is assumed not to hold, we have that Γ does not act essentially transitively on (X, ν).

Now if both G_1 and G_2 have property (T) then Γ also has property (T) (being a lattice) and therefore $\Gamma \curvearrowright (X, \nu)$ is essentially transitive since being weakly amenable, it is orbit equivalent to an action of $\mathbb{Z}$ by Theorem 2.26 and the corresponding cocycle into $\mathbb{Z}$ must take values in a compact (finite) subgroup.

Without loss of generality, we may therefore assume that G_2 has property (T) and that G_1 does not. Then $\mathrm{proj}_1 : G \to G_1$ is a resolution by Propositon 2.15.4. Since Γ is a lattice in G, by Proposition 2.15.4, the map $\mathrm{proj}_1 : \Gamma \to \overline{\mathrm{proj}_1 \Gamma} = G_1$ is also a resolution.

Let $\pi : \Gamma \to \mathcal{U}(L^2(X,\nu))$ be the Koopman representation. By Proposition 7.3.1, since $\Gamma \curvearrowright (X,\nu)$ is weakly amenable but not essentially transitive, there exists a sequence $f_n \in L^2(X,\nu)$ that are not Γ-invariant but are almost invariant.

Let $\{\gamma_n\}$ and $\{\gamma_n'\}$ be sequences in Γ such that $\mathrm{proj}_1 \gamma_n \to g$ in G_1 and that $\mathrm{proj}_1 \gamma_n' \to g$ in G_1. Then the sequence $\{a_n\}$ given by $a_{2n} = \gamma_n$ and $a_{2n+1} = \gamma_n'$ also has the property that $\mathrm{proj}_1 a_n \to g$ in G_1. Consider the space of G_1-points in $L^2(X,\nu)$:

$$\mathcal{F} = \{f \in L^2(X,\nu) : g \mapsto \pi(g)f \text{ factors through } \mathrm{proj}_1\}.$$

By Proposition 2.15.1, this is a closed Γ-invariant space. For $f \in \mathcal{F}$, then $\pi(\gamma_n)f \to q \in \mathcal{F}$ and $\pi(\gamma_n') \to q' \in \mathcal{F}$ and $\pi(a_n)f \to q'' \in \mathcal{F}$. By the construction of $\{a_n\}$ then $q'' = q = q'$ so we can define an action of G_1 on $\mathcal{F}$ by

$$g_1 \cdot f = \lim \pi(\gamma_n)f \text{ for any } \{\gamma_n\} \text{ such that } \mathrm{proj}_1 \gamma_n \to g_1 \text{ in } G_1.$$

Since $\mathcal{F}$ is a closed Γ-invariant subalgebra, the point realization (Y,η) corresponding to it is a G_1-space that is a Γ-quotient of (X,ν). Now $\Gamma \curvearrowright (X,\nu)$ ergodically hence $G_1 \curvearrowright (Y,\eta)$ ergodically and there exists a Γ-map $\psi : (X,\nu) \to (Y,\eta)$.

Since proj_1 is a resolution, there exists a sequence $\{q_n\}$ of G_1-almost invariant, but not invariant, functions in $L^2(Y,\eta)$ and in particular, (Y,η) is nontrivial. Since $\Gamma \curvearrowright (X,\nu)$ is weakly amenable, the same holds for $\Gamma \curvearrowright (Y,\eta)$.

Note that if G_1 is connected then $\mathrm{proj}_1 \Gamma$ acts essentially freely on (Y,η) by Theorem 2.35. Therefore $\mathrm{proj}_1 \mathrm{stab}_\Gamma(x) = \{e\}$ almost everywhere. So $\mathrm{stab}_\Gamma(x) \subseteq \{e\} \times G_2$ almost surely. But Γ is irreducible so $\Gamma \cap \{e\} \times G_2 = \{e\}$ by Proposition 2.5.1. Then $\mathrm{stab}_\Gamma(x) = \{e\}$ almost surely so $\Gamma \curvearrowright (X,\nu)$ is essentially free which we have assumed is not the case.

Clearly $\mathrm{proj}_1 \mathrm{stab}_\Gamma(x) \subseteq \mathrm{stab}_{G_1}(\psi(x))$ and therefore $\overline{\mathrm{proj}_1 \mathrm{stab}_\Gamma(x)} \subseteq \mathrm{stab}_{G_1}(\psi(x))$ for all $x \in X$. Let (Z,ζ) be the quotient space of (X,ν) by the map $\varphi(x) = \overline{\mathrm{proj}_1 \mathrm{stab}_\Gamma(x)}$. Then by the universal property of quotient spaces there exist Γ-maps

$$(X,\nu) \to PRG(X,\nu) \to (Z,\zeta) \to (Y,\eta).$$

Suppose that $G_1 \curvearrowright (Y,\eta)$ is essentially transitive. Then, taking a continuous compact model for the G-action on Y, there exists $y_0 \in Y$ such that $G_1 \cdot y_0$ is homeomorphic to $G_1/\mathrm{stab}_{G_1}(y_0)$ and $G_1 \cdot y_0$ has a finite G_1-invariant measure (since $\eta(G_1 \cdot y_0) = 1$) meaning that $\mathrm{stab}_{G_1}(y_0) = \Lambda$ is a lattice in G_1 or else that (Y,η) is purely atomic (since the action is ergodic).

Consider first the case when (Y,η) is purely atomic. Since G_1 acts continuously, then (Y,η) is trivial (as G_1 is nondiscrete and acts continuously) by ergodicity. But then the sequence of G_1-almost invariant vectors are in fact invariant, a contradiction. So (Y,η) is not atomic.

Therefore we are left with the case when $\overline{\mathrm{proj}_1 \mathrm{stab}_\Gamma(x)}$ is contained in a G_1-conjugate of a fixed lattice $\Lambda < G_1$ almost surely. Then $\mathrm{proj}_1 \mathrm{stab}_\Gamma(x)$ is discrete almost surely so $\overline{\mathrm{proj}_1 \mathrm{stab}_\Gamma(x)} = \mathrm{proj}_1 \mathrm{stab}_\Gamma(x)$.

Let $\pi : (X,\nu) \to (Y,\eta)$ be the Γ-map. For $\gamma \in \Gamma$, let $E_\gamma = \{x\ inX : \gamma x = x\}$. Then $\nu(E_\gamma) > 0$ for infinitely many γ (since we have assumed the stabilizers are infinite almost surely). Then for all $x \in E_\gamma$ it holds that $\mathrm{proj}_1 \gamma \in \mathrm{stab}_{G_1}(\pi(x))$. Let $F = \pi(E_\gamma)$. Then $\eta(F) > 0$. So there exists a positive Haar measure set $Q \subseteq G$ such that for $g \in Q$, it holds that $\mathrm{proj}_1 \gamma \in \mathrm{stab}(gy_0) = g\Lambda g^{-1}$. Then $g^{-1}\mathrm{proj}_1 \gamma g \in \Lambda$ for all $g \in Q$. But Λ is discrete and Q has positive Haar measure meaning that G_1 is then discrete, a contradiction. ☐

Combining our work with the results in [CP13], we obtain:

Corollary 8.6. *Let G_1 and G_2 be simple nondiscrete noncompact locally compact second countable groups with the Howe-Moore property such that at least one G_j has property (T). Set $G = G_1 \times G_2$ and let $\Gamma < G$ be an irreducible lattice. Let (X,ν) be an ergodic measure-preserving Γ-space. Then one of the following holds:*

- *$\Gamma \curvearrowright (X,\nu)$ is essentially free;*

- $\mathrm{stab}_*\nu$ *is supported on the finite index subgroups of* Γ*; or*
- $\mathrm{stab}_*\nu$ *is supported on the torsion elements of* Γ*; or*
- *one* G_j *is totally disconnected, has property* (T) *and acts ergodically and essentially freely on the induced space* $G \times_\Gamma X$ *and the other* G_{3-j} *is connected, does not have property* (T) *and does not act ergodically on the induced space.*

Proof. By the previous corollary, if none of the first three possibilities occur then one of:

- one G_j is totally disconnected and acts ergodically and essentially freely on the induced space $G \times_\Gamma X$ and the other G_{3-j} does not act ergodically on the induced space; or
- one G_j does not have property (T), is totally disconnected, and there is a nontrivial ergodic G_j-space (Y, η) that is a Γ-quotient of (X, ν) and such that $G_j \curvearrowright (Y, \eta)$ is weakly amenable but not essentially transitive.

The result in [CP13] (Theorem 2.31) rules out the second case since in that case either both groups are totally disconnected or the connected group has property (T). Likewise, in the first case, the only possibility not covered by [CP13] is that there is a connected group in the product that does not have property (T). □

9 Higher-Order Product Groups

We now generalize the results of the previous two sections to products of arbitrarily (finitely) many groups and irreducible lattices in such products.

9.1 The Higher-Order Product Random Subgroups Functor

Definition 9.1. Let G_j be locally compact second countable groups for $j = 1, \ldots, k$. Set $G = G_1 \times \cdots \times G_k$. Given a G-space (X, ν), define $PRG^{(k)}_{G_1, G_2, \ldots, G_k}(X, \nu)$ to be the quotient space of (X, ν) by the map

$$\varphi(x) = \overline{\mathrm{proj}_{G_1}\ \mathrm{stab}(x)} \times \cdots \times \overline{\mathrm{proj}_{G_k}\ \mathrm{stab}(x)}.$$

Note that $PRG^{(1)}_{G_1}(X, \nu) = (X, \nu)$ in the case when there is a single group.

Proposition 9.1.1. *Let* G_j *be locally compact second countable groups for* $j = 1, \ldots, k$ *where* $k \geq 2$. *Set* $G = G_1 \times \cdots \times G_k$ *and let* (X, ν) *be a* G*-space. Then*

$$PRG^{(2)}_{G_1, G_2 \times \cdots \times G_k}(PRG^{(k-1)}_{G_1 \times G_2, G_3, \ldots, G_k}(X, \nu)) = PRG^{(k)}_{G_1, \ldots, G_k}(X, \nu).$$

Proof. This will follow from the universal property of the quotient space by a random subgroup (Theorem 4.6). Let

$$(X, \nu) \xrightarrow{\pi} PRG^{(k-1)}_{G_1 \times G_2, G_3, \ldots, G_k}(X, \nu) \xrightarrow{\psi} PRG^{(2)}_{G_1, G_2 \times \cdots \times G_k}(PRG^{(k-1)}_{G_1 \times G_2, G_3, \ldots, G_k}(X, \nu))$$

be the G-maps defining the quotient spaces. Observe that since

$$\overline{\mathrm{proj}_{G_1 \times G_2}\ \mathrm{stab}(x)} \times \overline{\mathrm{proj}_{G_3}\ \mathrm{stab}(x)} \times \cdots \times \overline{\mathrm{proj}_{G_k}\ \mathrm{stab}(x)} \subseteq \mathrm{stab}(\pi(x))$$

it holds that

$$\overline{\mathrm{proj}_{G_1}\ \mathrm{stab}(\pi(x))} \supseteq \overline{\mathrm{proj}_{G_1}\ \overline{\mathrm{proj}_{G_1 \times G_2}\ \mathrm{stab}(x)}} \supseteq \overline{\mathrm{proj}_{G_1}\ \mathrm{stab}(x)}$$

and likewise that

$$\overline{\mathrm{proj}_{G_2 \times \cdots \times G_k}\ \mathrm{stab}(\pi(x))} \supseteq \overline{\mathrm{proj}_{G_2}\ \mathrm{stab}(x)} \times \cdots \times \overline{\mathrm{proj}_{G_k}\ \mathrm{stab}(x)}.$$

Hence by the universal property of $PRG^{(k)}$ there exist G-maps

$$(X, \nu) \to PRG^{(k)}_{G_1, \ldots, G_k}(X, \nu) \to PRG^{(2)}_{G_1, G_2 \times \cdots \times G_k}(PRG^{(k-1)}_{G_1 \times G_2, G_3, \ldots, G_k}(X, \nu)).$$

On the other hand, since

$$\begin{aligned}\overline{\mathrm{proj}_{G_1}\,\mathrm{stab}(x)}\times\cdots\times\overline{\mathrm{proj}_{G_k}\,\mathrm{stab}(x)}\\ \supseteq\ \overline{\mathrm{proj}_{G_1\times G_2}\,\mathrm{stab}(x)}\times\overline{\mathrm{proj}_{G_3}\,\mathrm{stab}(x)}\times\cdots\times\overline{\mathrm{proj}_{G_k}\,\mathrm{stab}(x)}\end{aligned}$$

by the universal property of $PRG^{(k-1)}$ there are G-maps

$$(X,\nu)\to PRG^{(k-1)}_{G_1\times G_2,G_3,\ldots,G_k}(X,\nu)\to PRG^{(k)}_{G_1,\ldots,G_k}(X,\nu).$$

Then by the universal property of $PRG^{(2)}$ and the obvious inclusion of the stabilizers there exist G-maps

$$\begin{aligned}PRG^{(k-1)}_{G_1\times G_2,G_3,\ldots,G_k}(X,\nu)&\to PRG^{(2)}_{G_1,G_2\times\cdots\times G_k}(PRG^{(k-1)}_{G_1\times G_2,G_3,\ldots,G_k}(X,\nu))\\ &\to PRG^{(k)}_{G_1,\ldots,G_k}(X,\nu)\end{aligned}$$

and we therefore conclude that

$$PRG^{(k)}_{G_1,\ldots,G_k}(X,\nu)=PRG^{(2)}_{G_1,G_2\times\cdots\times G_k}(PRG^{(k-1)}_{G_1\times G_2,G_3,\ldots,G_k}(X,\nu)).$$

□

9.2 Actions of Higher-Order Product Groups

Theorem 9.2. *Let G_j be locally compact second countable groups for $j=1,\ldots,k$. Set $G=G_1\times\cdots\times G_k$ and let (X,ν) be a measure-preserving G-space. If $G\curvearrowright PRG^{(k)}_{G_1,\ldots,G_k}(X,\nu)$ weakly amenably then $G\curvearrowright(X,\nu)$ weakly amenably.*

Proof. By Proposition 9.1.1 since $G\curvearrowright PRG^{(k)}_{G_1,\ldots,G_k}(X,\nu)$ weakly amenably, it then holds that $G\curvearrowright PRG^{(2)}_{G_1,G_2\times\cdots\times G_k}(PRG^{(k-1)}_{G_1\times G_2,G_3,\ldots,G_k}(X,\nu))$ weakly amenably. Then, by Theorem 7.1, we also have $G\curvearrowright PRG^{(k-1)}_{G_1\times G_2,G_3,\ldots,G_k}(X,\nu)$ weakly amenably. Proceeding inductively, we then have that $G\curvearrowright PRG^{(k-j+1)}_{G_1\times\cdots\times G_j,G_{j+1},\ldots,G_k}(X,\nu)$ weakly amenably for all $1\leq j\leq k$. Hence in particular, it holds that $G\curvearrowright PRG^{(1)}_{G_1\times\cdots\times G_k}(X,\nu)=(X,\nu)$ weakly amenably. □

Theorem 9.3. *Let G_j be locally compact second countable groups for $j=1,\ldots,k$ with $k\geq 2$. Set $G=G_1\times\cdots\times G_k$ and let (X,ν) be an ergodic measure-preserving G-space. Let $(X,\nu)\to(X_j,\nu_j)$ be the ergodic decomposition into $\widetilde{G}_j=G_1\times\cdots\times G_{j-1}\times\{e\}\times G_{j+1}\times\cdots\times G_k$-ergodic components. Assume that $G_j\curvearrowright(X_j,\nu_j)$ weakly amenably and that $\mathrm{stab}_*\nu_j$ is a simple invariant random subgroup for all $j=1,\ldots,k$. Then either there exists at least one G_j such that $\mathrm{proj}_{G_j}\,\mathrm{stab}(x)=\{e\}$ almost everywhere or else $G\curvearrowright(X,\nu)$ weakly amenably.*

Proof. Consider the maps $s_j(x)=\overline{\mathrm{proj}_{G_j}\,\mathrm{stab}(x)}$. Since $s_j(x)$ is a $\widetilde{G}_j$-invariant function, it descends to a function on (X_j,ν_j). Therefore by Theorem 3.8, $(s_j)_*\nu\lhd\mathrm{stab}_*\nu_j$. Since the set $\{x\in X:s_j(x)=\{e\}\}$ is G-invariant, by ergodicity either $(s_j)_*\nu=\delta_{\{e\}}$ or else $(s_j)_*\nu=\mathrm{stab}_*\nu_j$ for each j. If $(s_j)_*\nu=\delta_{\{e\}}$ then the conclusion follows. So we may assume that $(s_j)_*\nu=\mathrm{stab}_*\nu_j$ for all $j=1,\ldots,k$. This says precisely that $PRG^{(k)}_{G_1,\ldots,G_k}(X,\nu)$ is orbital over $(X_1\times\cdots\times X_k,\nu_1\times\cdots\times\nu_k)$ (which is a quotient of the product random subgroups functor by the universal property since the $\widetilde{G}_j$-ergodic components are a quotient by an invariant random subgroup with larger stabilizers). Then the fact that each G_j acts weakly amenably on (X_j,ν_j) says that $G\curvearrowright(X_1\times\cdots\times X_k,\nu_1\times\cdots\times\nu_k)$ weakly amenably which in turn means, by Proposition 2.4.1, that $G\curvearrowright PRG^{(k)}_{G_1,\ldots,G_k}(X,\nu)$ weakly amenably. Then, by Theorem 9.2, $G\curvearrowright(X,\nu)$ weakly amenably. □

Theorem 9.4. *Let G_j be locally compact second countable groups for $j=1,\ldots,k$ with $k\geq 2$. Set $G=G_1\times\cdots\times G_k$ and let (X,ν) be an ergodic measure-preserving G-space. If $\mathrm{proj}_{G_j}\,\mathrm{stab}(x)$ is dense in G_j almost everywhere for each $j=1,\ldots,k$ then $G\curvearrowright(X,\nu)$ weakly amenably.*

Proof. When the projections are all dense, $G \curvearrowright PRG^{(k)}_{G_1,\ldots,G_k}(X,\nu)$ trivially. By ergodicity, it is then the trivial one-point space. As every group acts weakly amenably on a point, the conclusion follows from Theorem 9.2. □

Corollary 9.5. *Let G_j be locally compact second countable groups for $j = 1,\ldots,k$ with $k \geq 2$ each with property (T). Set $G = G_1 \times \cdots \times G_k$ and let (X,ν) be an ergodic measure-preserving G-space. Assume that there exist simple closed subgroups $H_j < G_j$ such that the spaces of $\prod_{\ell \neq j} G_\ell$-ergodic components is isomorphic to $(G_j/H_j, \mathrm{Haar})$ for each j and such that any nontrivial normal subgroup of H_j has finite index in H_j. Then either at least one $G_j \curvearrowright (X,\nu)$ essentially free or $G \curvearrowright (X,\nu)$ is essentially transitive.*

Proof. The same reasoning as in Corollary 7.9 gives that if none of the G_j act essentially freely then the G-action on $PRG^{(k)}_{G_1,\ldots,G_k}(X,\nu)$ is weakly amenable, hence $G \curvearrowright (X,\nu)$ is weakly amenable. Since G has property (T), the action is then essentially transitive. □

9.3 Actions of Lattices in Higher-Order Product Groups

Theorem 9.6. *Let $G = G_1, \times \cdots \times G_k$ be a product of at least two simple nondiscrete noncompact locally compact second countable groups with the Howe-Moore property. Let $\Gamma < G$ be an irreducible lattice and let (X,ν) be an ergodic measure-preserving Γ-space. Then one of the following holds:*

- $\Gamma \curvearrowright (X,\nu)$ *is essentially free;*
- $\mathrm{stab}_*\nu$ *is supported on the torsion elements of* Γ*;*
- $\Gamma \curvearrowright (X,\nu)$ *is weakly amenable; or*
- *at least one G_j is totally disconnected.*

Proof. Assume the action is not essentially free. For $\gamma \in \Gamma$, let $E_\gamma = \{x \in X : \gamma x = x\}$. Let $L = \{\gamma \in \Gamma \setminus \{e\} : \nu(E_\gamma) > 0\}$. Then L is nonempty since the action is not essentially free. For each j, let $s_j(x) = \overline{\mathrm{proj}_{G_j}\ \mathrm{stab}(x)}$. Then $(s_j)_*\nu$ is an invariant random subgroup of G_j by Theorem 8.1. Let $(Y_j, \eta_j$ be the ergodic $(s_j)_*\nu$-nonfree action of G_j. If there exists $\gamma \in L$ such that $\overline{\langle \mathrm{proj}_{G_j}\ \gamma\rangle}$ is noncompact then, as in the proof of Theorem 8.3, since G_j has the Howe-Moore property, $s_j(x) = G_j$ almost everywhere. Define the set

$$S = \{j \in \{1,\ldots,k\} : \exists\ \gamma \in L \text{ such that } \overline{\langle \mathrm{proj}_{G_j}\ \gamma\rangle} \text{ is noncompact}\}.$$

Then for every $j \in S$, it holds that $s_j(x) = G_j$ almost everywhere.

Consider now the set

$$T = \{j \in \{1,\ldots,k\} : G_j \text{ is connected}\}.$$

Let $j \in T$. Then G_j is a simple real Lie group since it has the Howe-Moore property (Theorem 2.33). Since $\mathrm{proj}_{G_j}\ \Gamma$ is a countable subgroup of G_j, by Theorem 2.35, either $\mathrm{proj}_{G_j}\ \Gamma$ acts essentially freely on (Y_j,η_j) or else (Y_j,η_j) is trivial. For $\gamma \in L$, since $\nu(E_\gamma) > 0$, on a $(s_j)_*\nu$-positive measure set $\mathrm{proj}_{G_j}\ \gamma$ is in the stabilizer of $y \in Y_j$. But $\mathrm{proj}_{G_j}\ \Gamma$ acts essentially freely so this is a contradiction. Therefore (Y_j,η_j) is trivial. This means that $s_j(x) = G_j$ almost everywhere.

If S is empty then every $\gamma \in L$ has the property that $\langle\gamma\rangle$ is contained in a compact group. Since Γ is discrete, this means that γ has a finite orbit hence is a torsion element. So if S is empty then $\mathrm{stab}_*\nu$ is supported on the torsion elements of Γ.

If $|S \cup T| = k$ then $PRG^{(k)}_{G_1,\ldots,G_k}(G \times_\Gamma X)$ has the property that almost every stabilizer projects densely into each of the G_j. Then, by Theorem 9.4, $G \curvearrowright G \times_\Gamma X$ is weakly amenable and so, by Proposition 2.6.1, $\Gamma \curvearrowright (X,\nu)$ is weakly amenable.

Therefore we are left with the case when there exists some $j \notin S \cup T$ and S is nonempty. Then G_j is totally disconnected. □

Corollary 9.7. *Let $G = G_1, \times \cdots \times G_k$ be a product of at least two simple nondiscrete noncompact locally compact second countable groups with the Howe-Moore property, at least one with property (T). Let $\Gamma < G$ be an irreducible lattice and let (X,ν) be an ergodic measure-preserving Γ-space. Then one of the following holds:*

- $\Gamma \curvearrowright (X, \nu)$ *is essentially free;*
- $\mathrm{stab}_*\nu$ *is supported on the torsion elements of* Γ*;*
- $\mathrm{stab}_*\nu$ *is supported on the finite index subgroups of* Γ*; or*
- *at least one* G_j *is totally disconnected and at least one* G_j *is connected and none of the connected* G_j *have property* (T).

Proof. By Theorem 9.6, if neither of the first two possibilities occur then either $\Gamma \curvearrowright (X, \nu)$ is weakly amenable or at least one G_j is totally disconnected. Consider first when $\Gamma \curvearrowright (X, \nu)$ is weakly amenable. Suppose that $\Gamma \curvearrowright (X, \nu)$ is not essentially transitive (that is, $\mathrm{stab}_*\nu$ is not supported on the finite index subgroups of Γ). Then, as in the proof of Corollary 8.5, there exists a totally disconnected G_j and a Γ-quotient of (X, ν) that is a G_j-space on which Γ acts weakly amenably. So we are left with the case when there is a totally disconnected G_j.

Then, by [CP13] (Theorem 2.31), the only case left is when there is a connected simple factor that does not have property (T). Moreover, in [CP13] Corollary 5.2, the requirement that all the connected factors have property (T) is used in the following way: one gets an irreducible lattice Γ_0 in the product of the connected factors acting on a space (X_0, ν_0) weakly amenably and then uses property (T) to conclude the action has either finite orbits or finite stabilizers. Applying our work to Γ_0 in the product of the connected factors, the proof of Corollary 8.5 then gives that one connected factor having property (T) is enough to conclude Γ_0 acts with either finite stabilizers or finite index stabilizers. Then the proof of [CP13] Corollary 5.2 goes through when only one connected factor has property (T) and so $\Gamma \curvearrowright (X, \nu)$ weakly amenably necessarily implies the action is essentially transitive. □

9.4 Actions of Semisimple Groups and Lattices

We conclude with a strengthening of the results on actions of semisimple real Lie groups and irreducible lattices in them due to Nevo-Stuck-Zimmer [SZ94],[NZ99]. We remark that our methods give a more general statement than theirs, except in the case of a lattice in a simple higher-rank Lie group, in which case the only known proof is the algebraic proof they give (as opposed to the more geometric methods we employ):

Corollary 9.8. *Let* G *be a semisimple group with trivial center and no compact factors, at least one simple factor having property* (T)*. Let* (X, ν) *be an ergodic* G*-space such that each simple factor of* G *acts ergodically on* (X, ν)*. Then* $G \curvearrowright (X, \nu)$ *is essentially free or essentially transitive.*

Proof. The case when G is simple is covered by Nevo-Stuck-Zimmer [SZ94],[NZ99]. The case when G has at least two simple factors follows from Corollary 7.8. □

Corollary 9.9. *Let* G *be a semisimple group with trivial center and no compact factors with at least one simple factor being a connected (real) Lie group with property* (T)*. Let* $\Gamma < G$ *be an irreducible lattice and* (X, ν) *be a nonatomic ergodic measure-preserving* Γ*-space. Then* $\Gamma \curvearrowright (X, \nu)$ *is essentially free.*

Proof. When G has a single simple factor, by hypothesis then G is a simple real Lie group with property (T) hence the results of Nevo-Stuck-Zimmer [SZ94],[NZ99] give the conclusion. When G has at least two simple factors, Corollary 9.7 states that if the action is not essentially free then either $\mathrm{stab}_*\nu$ is supported on the torsion elements of Γ or $\mathrm{stab}_*\nu$ is supported on the finite index subgroups of Γ (the final possibility in that Corollary is ruled out by our hypothesis that there is a simple connected factor with property (T)). If $\mathrm{stab}_*\nu$ is supported on the finite index subgroups of Γ then by ergodicity, (X, ν) is finite and atomic and we have assumed (X, ν) is nonatomic. Suppose $\gamma \in \Gamma$ is torsion, so $\gamma^m = e$ for some $m \in \mathbb{N}$. Then $\mathrm{proj}_{G_j} \gamma$ is torsion in G_j but G_j is simple and connected hence torsion-free. Therefore the action is essentially free. □

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