Measure-Theoretically Mixing Subshifts of Minimal Word Complexity

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Abstract We resolve a long-standing open question on the relationship between measure-theoretic dynamical complexity and symbolic complexity by establishing the exact word complexity at which measure-theoretic strong mixing manifests:

For every superlinear $f : \mathbb{N} \to \mathbb{N}$, i.e. $f(q)/q \to \infty$, there exists a subshift admitting a (strongly) mixing of all orders probability measure with word complexity p such that $p(q)/f(q) \to 0$.

For a subshift with word complexity p which is non-superlinear, i.e. $\liminf p(q)/q < \infty$, every ergodic probability measure is partially rigid.

Introduction

Among measure-theoretic dynamical properties of measure-preserving transformations, strong mixing of all orders is the 'most complex': every finite collection of measurable sets tends asymptotically toward independence, necessarily implying a significant amount of randomness. Despite this, 'low complexity' mixing transformations exist–there are mixing transformation with zero entropy–raising the question of how deterministic a mixing transformation can be.

Word complexity, the number $p(q)$ of distinct words of length q appearing in the language of the subshift, provides a more fine-grained means of quantifying complexity in the zero entropy setting, leading to the question of how low the word complexity of a mixing transformation can be.

Ferenczi [Fer95] initially conjectured that mixing transformations' word complexity should be superpolynomial but quickly refuted this himself [Fer96] showing that the staircase transformation, proven mixing by Adams [Ada98], has quadratic word complexity. Recent joint work of the author and R. Pavlov and S. Rodock [CPR22] exhibited subshifts admitting mixing measures with word complexity functions which are subquadratic but superlinear by more than a logarithm. We exhibit subshifts admitting mixing measures with complexity arbitrarily close to linear:

Theorem A. For every $f : \mathbb{N} \to \mathbb{N}$ which is superlinear, $f(q)/q \to \infty$, there exists a subshift, admitting a strongly mixing probability measure, with word complexity p such that $p(q)/f(q) \to 0$.

Our examples, which we call quasi-staircase transformations, are mixing rank-one transformations hence mixing of all orders [Kal84], [Ryz93]. We establish their word complexity is optimal:

Theorem B. Every subshift of non-superlinear word complexity, $\liminf p(q)/q < \infty$, equipped with an ergodic probability measure is partially rigid hence not strongly mixing,

Non-superlinear complexity subshifts are conjugate to S-adic shifts (Donoso, Durand, Maass and Petite [DDMP21]). Named by Vershik and the subject of a well-known conjecture of Host, S-adic subshifts are quite structured (see e.g. [Ler12] for more information on S-adicity).

Our work may be viewed as saying there is a sharp divide in 'measure-theoretic complexity', precisely at superlinear word complexity, between highly structured and highly complicated: as soon as the word complexity is 'large enough' to escape the S-adic structure and partial rigidity, there is already 'enough room' for (strong) mixing of all orders.

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Cyr and Kra established that superlinear complexity is the dividing line for a subshift admitting countably many ergodic measures: there exists subshifts with complexity arbitrarily close to linear which admit uncountably many ergodic measures [CK20b] and non-superlinear complexity implies at most countably many [CK19], [Bos85]. Our work implies that in the non-superlinear case, the at most countably measures are all partially rigid (with a uniform rigidity constant). Their result, like ours, indicates that superlinear word complexity is the line at which complicated measure-theoretic phenomena can manifest.

Beyond the structure imposed by S-adicity, linear complexity subshifts are known to be structured in various ways (e.g. [CFPZ19], [CK20a], [DDMP16], [DOP21], [PS22a], [PS22b]). Our work indicates there is no hope for similar phenomena in any superlinear setting.

1 Definitions and preliminaries

1.1 Symbolic dynamics

Definition 1.1. A **subshift** on the finite set $\mathcal{A}$ is any subset $X \subset \mathcal{A}^{\mathbb{Z}}$ which is closed in the product topology and shift-invariant: for all $x = (x_n)_{n\in\mathbb{Z}} \in X$ and $k \in \mathbb{Z}$, the translate $(x_{n+k})_{n\in\mathbb{Z}}$ of x by k is also in X.

Definition 1.2. A **word** is any element of $\mathcal{A}^{\ell}$ for some ℓ, the **length** of w, written $\|w\|$. A word w is a **subword** of a word or biinfinite sequence x if there exists k so that $w_i = x_{i+k}$ for all $1 \leq i \leq \|w\|$. A word u is a **prefix** of w when $u_i = w_i$ for $1 \leq i \leq \|u\|$ and a word v is a suffix of w when $v_i = w_{i+\|w\|-\|v\|}$ for $1 \leq i \leq \|v\|$.

For words v, w, we denote by vw their concatenation–the word obtained by following v immediately by w. We write such concatenations with product or exponential notation, e.g. $\prod_i w_i$ or 0^n.

Definition 1.3. The **language** of a subshift X is $\mathcal{L}(X) = \{w : w \text{ is a subword of some } x \in X\}$.

Definition 1.4. The **word complexity function** of a subshift X over $\mathcal{A}$ is the function $p_X : \mathbb{N} \to \mathbb{N}$ defined by $p_X(q) = |\mathcal{L}(X) \cap \mathcal{A}^q|$, the number of words of length q in the language of X.

When X is clear from context, we suppress the subscript and just write $p(n)$.

For subshifts on the alphabet $\{0, 1\}$, we consider:

Definition 1.5. The set of **right-special** words is $\mathcal{L}^{RS}(X) = \{w \in X : w0, w1 \in \mathcal{L}(X)\}$.

Cassaigne [Cas97] showed the well-known: $p(q) = p(m) + \sum_{\ell=m}^{q-1} |\{w \in \mathcal{L}^{RS} : \|w\| = \ell\}|$ for $m < q$.

1.2 Ergodic theory

Definition 1.6. A **transformation** T is a measurable map on a standard Borel or Lebesgue measure space $(Y, \mathcal{B}, \mu)$ that is measure-preserving: $\mu(T^{-1}B) = \mu(B)$ for all $B \in \mathcal{B}$.

Definition 1.7. Two transformations T on $(Y, \mathcal{B}, \mu)$ and T' on $(Y', \mathcal{B}', \mu')$ are **measure-theoretically isomorphic** when there exists a bijective map ϕ between full measure subsets $Y_0 \subset Y$ and $Y_0' \subset Y'$ where $\mu(\phi^{-1}A) = \mu'(A)$ for all measurable $A \subset Y_0'$ and $(\phi \circ T)(y) = (T' \circ \phi)(y)$ for all $y \in Y_0$.

Definition 1.8. A transformation T is **ergodic** when $A = T^{-1}A$ implies that $\mu(A) = 0$ or $\mu(A^c) = 0$.

Theorem 1.9 (Mean Ergodic Theorem)**.** *If T is ergodic and on a finite measure space and $f \in L^2(Y)$,*

$$\lim_{n\to\infty} \int \Big| \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^{-i} - \int f \, d\mu \Big| \, d\mu = 0$$

Definition 1.10. A transformation T is **mixing** when for all $A, B \in \mathcal{B}$, $\mu(T^n A \cap B) \to \mu(A)\mu(B)$.

1.3 Rank-one transformations

A **rank-one transformation** is a transformation T constructed by "cutting and stacking". Here Y represents a (possibly infinite) interval, $\mathcal{B}$ is the induced σ-algebra from $\mathbb{R}$, and μ is Lebesgue measure. We give a brief description, referring the reader to [FGH$^+$21] or [Sil08] for more details.

The transformation is defined inductively on larger and larger portions of the space through Rohlin towers or **columns**, denoted C_n. Each column C_n consists of **levels** $I_{n,j}$ where $0 \leq j < h_n$ is the height of the level within the column. All levels $I_{n,j}$ in C_n are intervals with the same length, $\mu(I_n)$, and the total number of levels in a column is the **height** of the column, denoted by h_n. The transformation T is defined on all levels $I_{n,j}$ except the top one I_{n,h_n-1} by sending each $I_{n,j}$ to $I_{n,j+1}$ using the unique order-preserving affine map.

Start with $C_1 = [0,1)$ with height $h_1 = 1$. To obtain C_{n+1} from C_n, we require a **cut sequence**, $\{r_n\}$ such that $r_n \geq 1$ for all n. Make r_n vertical cuts of C_n to create $r_n + 1$ **subcolumns** of equal width. Denote a **sublevel** of C_n by $I_{n,j}^{[i]}$ where $0 \leq a < h_n$ is the height of the level within that column, and i represents the position of the subcolumn, where $i = 0$ represents the leftmost subcolumn and $i = r_n$ is the rightmost subcolumn. After cutting C_n into subcolumns, add extra intervals called **spacers** on top of each subcolumn to function as levels of the next column. The **spacer sequence**, $\{s_{n,i}\}$ such that $0 \leq i \leq r_n$ and $s_{n,i} \geq 0$, specifies how many sublevels to add above each subcolumn. Spacers are the same width as the sublevels, act as new levels in the column C_{n+1}, and are taken to be the leftmost intervals in $[1,\infty)$ not in C_n. After the spacers are added, stack the subcolumns with their spacers right on top of left, i.e. so that $I_{n,0}^{[i+1]}$ is directly above $I_{n,h_n-1}^{[i]}$. This gives the next column, C_{n+1}.

Each column C_n defines T on $\bigcup_{j=0}^{h_n-2} I_{n,j}$ and the partially defined map T on C_{n+1} agrees with that of C_n, extending the definition of T to a portion of the top level of C_n where it was previously undefined. Continuing this process gives the sequence of columns $\{C_1, \ldots, C_n, C_{n+1}, \ldots\}$ and T is then the limit of the partially defined maps.

Though this construction could result in Y being an infinite interval with infinite Lebesgue measure, Y has finite measure if and only if $\sum_n \frac{1}{r_n h_n} \sum_{i=0}^{r_n} s_{n,i} < \infty$, see [CS10]. All rank-one transformations we define satisfy this condition, and for convenience we renormalize so that $Y = [0,1)$. Every rank-one transformation is ergodic and invertible.

The reader should be aware that we are making r_n cuts and obtaining $r_n + 1$ subcolumns (following Ferenczi [Fer96]), while other papers (e.g. [Cre21]) use r_n as the number of subcolumns.

1.4 Symbolic models of rank-one transformations

For a rank-one transformation defined as above, we define a subshift $X(T)$ on the alphabet $\{0,1\}$ which is measure-theoretically isomorphic to T:

Definition 1.11. The **symbolic model** $X(T)$ of a rank-one transformation T is given by the sequence of words: $B_1 = 0$ and

$$B_{n+1} = B_n 1^{s_{n,0}} B_n 1^{s_{n,1}} \cdots B_n 1^{s_{n,r_n}} = \prod_{i=0}^{r_n} B_n 1^{s_{n,i}}$$

and $X(T)$ is the set of all biinfinite sequences such that every subword is a subword of some B_n.

The words B_n are a symbolic coding of the column C_n: 0 represents C_1 and 1 represents the spacers. There is a natural measure associated to $X(T)$:

Definition 1.12. The **empirical measure** for a symbolic model $X(T)$ of a rank-one transformation T is the measure ν defined by, for each word w,

$$\nu([w]) = \lim_{n\to\infty} \frac{|\{1 \leq j \leq \|B_n\| - \|w\| \ : \ B_{n[j,j+\|w\|)} = w\}|}{\|B_n\| - \|w\|}$$

Danilenko [Dan16] (combined with [dJ77] and [Kal84]) proved that the symbolic model $X(T)$ of a rank-one subshift, equipped with its empirical measure, is measure-theoretically isomorphic to the cut-and-stack construction (see [AFP17]; see [FGH+21] for the full generality including odometers).

Due to this isomorphism, we move back and forth between rank-one and symbolic model terminology as needed and write $\mathcal{L}(T)$ for the language of $X(T)$.

2 Quasi-staircase transformations

Definition 2.1. Given nondecreasing sequences of integers $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ tending to infinity such that $c_1 \geq 1$ and $c_{n+1} \geq c_n + b_n$, a **quasi-staircase transformation** is a rank-one transformation with cut sequence $r_n = a_n b_n$ and spacer sequence $s_{n,t} = c_n + \left\lfloor \frac{t}{a_n} \right\rfloor$ for $0 \leq t < r_n$ and $s_{n,r_n} = 0$.

The symbolic representation of a quasi-staircase is $B_1 = 0$ and

$$B_{n+1} = \Big(\prod_{i=0}^{b_n - 1} \left(B_n 1^{c_n+i}\right)^{a_n} \Big) B_n$$

The height sequence of a quasi-staircase is $h_1 = 1$ and $h_{n+1} = (a_n b_n + 1)h_n + a_n b_n c_n + \frac{1}{2} a_n b_n (b_n - 1)$.

2.1 Quasi-staircase right-special words

Lemma 2.2. *Let $01^z0 \in \mathcal{L}(T)$. Then there are unique n and i with $0 \leq i < b_n$ such that $z = c_n + i$. $01^{c_n+i}0$ is not a subword of B_m for $m \leq n$ and every occurrence of $01^{c_n+i}0$ is as a suffix of $1^{c_{n+1}}(\prod_{j=0}^{i-1}(B_n 1^{c_n+j})^{a_n})(B_n 1^{c_n+i})^q 0$ for some $1 \leq q \leq a_n$ (adopting the convention that $\prod_0^{-1}$ is the empty word).*

Proof. As every B_n begins and ends with 0, the only such words are of the form $01^{c_n+i}0$. Since $c_{n+1} \geq c_n + b_n$, such n and i are unique. This also gives that 1^{c_n} is not a subword of B_n.

The word $01^{c_n+i}0$ only occurs inside B_{n+1} due to $c_{n+1} \geq c_n + b_n$, and only as part of the $(B_n 1^{c_n+i})^{a_n}$ in its construction, and B_{n+1} is always preceded by $1^{c_{n+1}}$ □

Proposition 2.3. *If $w \in \mathcal{L}^{RS}(T)$ then at least one of the following holds:*

(i) $w = 1^{\|w\|}$

(ii) *w is a suffix of $1^{c_n+i-1}\left(B_n 1^{c_n+i}\right)^{a_n}$ for some n and $0 \leq i < b_n$*

(iii) *w is a suffix of $1^{c_n+b_n-1} B_n 1^{c_n}$ for some n*

(iv) $w = 1^{c_n}(B_n 1^{c_n})^{a_n}$

Proof. Let $w \in \mathcal{L}^{RS}(T)$. Since $c_1 \geq 1$, the word $00 \notin \mathcal{L}(T)$ so w does not end in 0. If $w = 1^{\|w\|}$ then w is of form (i) so from here on, assume that w contains at least one 0.

Let $z \geq 1$ such that w has 01^z as a suffix. Then $w0$ has 01^z0 as a suffix so $z = c_n + i$ for some unique $n \geq 1$ and $0 \leq i < b_n$ by Lemma 2.2. As $w0$ has $01^{c_n+i}0$ as a suffix, $w0$ shares a suffix with the word $1^{c_{n+1}}(\prod_{j=0}^{i-1}(B_n 1^{c_n+j})^{a_n})(B_n 1^{c_n+i})^q 0$ for some $1 \leq q \leq a_n$.

First consider the case when $i > 0$. If w is a suffix of $1^{c_n+i-1}(B_n 1^{c_n+i})^{a_n}$ then it is of form (ii) so we need only consider w that have $01^{c_n+i-1}(B_n 1^{c_n+i})^q$ as a suffix. For such w, the word $w1$ has the suffix $01^{c_n+i-1}(B_n 1^{c_n+i})^{q-1} B_n 1^{c_n+i+1}$ but that word is only in $\mathcal{L}(T)$ if $q - 1 = a_n$ which is impossible.

Now consider the case when $i = 0$, i.e. $z = c_n$. If w is a suffix of $1^{c_n-1}(B_n 1^{c_n})^{a_n}$ then it is of form (ii) so we may assume that w has $1^{c_n-1}(B_n 1^{c_n})^q$ as a strict suffix for some $1 \leq q \leq a_n$. Since $B_n 1^{c_n}$ is always preceded by 1^{c_n} (possibly as part of some $1^{c_{n+1}+i}$ or 1^{c_n+i}), w cannot have $01^{c_n-1} B_n 1^{c_n}$ as a subword so w has $1^{c_n}(B_n 1^{c_n})^q$ as a suffix for some $1 \leq q \leq a_n$.

Take q maximal so that w has $1^{c_n}(B_n 1^{c_n})^q$ as a suffix.

Consider first when w has $1^{c_n}(B_n1^{c_n})^{a_n}$ as a suffix, i.e. when $q = a_n$. If $w = 1^{c_n}(B_n1^{c_n})^{a_n}$ then it is of form (iv). If w has $01^{c_n}(B_n1^{c_n})^{a_n}$ as a suffix then $w0 \notin \mathcal{L}(T)$ as $0(1^{c_n}B_n)^{a_n}1^{c_n}0 \notin \mathcal{L}(T)$. If w has $11^{c_n}(B_n1^{c_n})^{a_n}$ as a suffix then $w1$ has $1^{c_n+1}(B_n1^{c_n})^{a_n-1}B_n1^{c_n+1}$ as a suffix but that is not in $\mathcal{L}(T)$.

So we may assume $q < a_n$. Since $1^{c_n}(B_n1^{c_n})^q$ is then of form (ii), we may assume $1^{c_n}(B_n1^{c_n})^q$ is a strict suffix of w.

Consider when w has $01^{c_n}(B_n1^{c_n})^q$ as a suffix. As $01^{c_n}(B_n1^{c_n})^q$ only appears as a suffix of $B_n1^{c_n}(B_n1^{c_n})^q$ and that word is always preceded by 1^{c_n} (possibly as part of some $1^{c_{n+1}+i}$), w then shares a suffix with $1^{c_n}(B_n1^{c_n})^{q+1}$. As q is maximal, then w is a suffix of $1^{c_n-1}(B_n1^{c_n})^{q+1}$ and, as $q < a_n$, this means w is of form (ii).

We are left with the case when w has $1^{c_n+1}(B_n1^{c_n})^q$ as a suffix for some $1 \leq q < a_n$. If $q \geq 2$ then $w1$ has $1^{c_n+1}(B_n1^{c_n})^{q-1}B_n1^{c_n+1}$ as a suffix but that is not in $\mathcal{L}(T)$ for $q-1 \geq 1$. So we are left with the situation when w shares a suffix with $1^{c_n+1}B_n1^{c_n}$. So $w0$ shares a suffix with $1^{c_n+1}B_n1^{c_n}0$ which must share a suffix with $1^{c_{n+1}}B_n1^{c_n}0$, meaning that w shares a suffix with $1^{c_{n+1}}B_n1^{c_n}$. If w is a suffix of $1^{c_n+b_n-1}B_n1^{c_n}$ then it is of form (iii). If not then w has the suffix $1^{c_n+b_n}B_n1^{c_n}$ so $w1$ has suffix $1^{c_n+b_n}B_n1^{c_n+1}$ which is not in $\mathcal{L}(T)$ since $B_n1^{c_n+1}$ is always preceded by $B_n1^{c_n}$ or $B_n1^{c_n+1}$. □

Lemma 2.4. *$1^\ell \in \mathcal{L}^{RS}(T)$ for all ℓ.*

Proof. For n such that $\ell < c_n$, as the word $1^{c_n}B_n$ is a subword of B_{n+1}, so are $1^{\ell+1}$ and $1^\ell 0$ since $\ell < c_n$ and B_n starts with 0. □

Lemma 2.5. *If w is a suffix of $1^{c_n}(B_n1^{c_n})^{a_n}$ then $w \in \mathcal{L}^{RS}(T)$.*

Proof. B_{n+2} has $1^{c_{n+1}}B_{n+1} = 1^{c_{n+1}-c_n}1^{c_n}B_{n+1}$ as a subword which has $1^{c_n}(B_n1^{c_n})^{a_n}B_n$ as a subword which gives $1^{c_n}(B_n1^{c_n})^{a_n}0 \in \mathcal{L}(T)$. B_{n+1} has $(B_n1^{c_n})^{a_n}B_n1^{c_n+1}$ as a prefix which has suffix $1^{c_n}(B_n1^{c_n})^{a_n-1}B_n1^{c_n+1}$ and that word is $1^{c_n}(B_n1^{c_n})^{a_n}1$ giving $1^{c_n-1}(B_n1^{c_n})^{a_n}1 \in \mathcal{L}(T)$. □

Lemma 2.6. *If w is a suffix of $1^{c_n+i-1}(B_n1^{c_n+i})^{a_n}$ for $0 < i < b_n$ then $w \in \mathcal{L}^{RS}(T)$.*

Proof. B_{n+1} has $1^{c_n+i-1}(B_n1^{c_n+i})^{a_n}B_n$ as a subword which gives $1^{c_n+i-1}(B_n1^{c_n+i})^{a_n}0 \in \mathcal{L}(T)$. When $i < b_n - 1$, B_{n+1} has $(1^{c_n+i}B_n)^{a_n}1^{c_n+i+1}$ as a subword which gives $1^{c_n+i-1}(B_n1^{c_n+i})^{a_n}1 \in \mathcal{L}(T)$; when $i = b_n - 1$, B_{n+2} has the subword $(1^{c_n+b_n-1}B_n)^{a_n}1^{c_{n+1}}$ so $1^{c_n+b_n-2}(B_n1^{c_n+b_n-1})^{a_n}1^{c_{n+1}-c_n-b_n+1} \in \mathcal{L}(T)$ so $1^{c_n+b_n-2}(B_n1^{c_n+b_n-1})^{a_n}1 \in \mathcal{L}(T)$ as $c_{n+1} \geq c_n + b_n$. □

Lemma 2.7. *If w is a suffix of $1^{c_n+b_n-1}B_n1^{c_n}$ then $w \in \mathcal{L}^{RS}(T)$.*

Proof. B_{n+2} has $B_{n+1}1^{c_{n+1}}B_{n+1}$ as a subword which has $B_{n+1}1^{c_{n+1}}B_n1^{c_n}B_n$ as a prefix, and that word has $1^{c_n+b_n-1}B_n1^{c_n}0$ as a subword since $c_n + b_n - 1 < c_{n+1}$. Also B_{n+2} has $B_{n+1}1^{c_{n+1}}$ as a subword which has $1^{c_n+b_n-1}B_n1^{c_{n+1}}$ as a suffix which then has $1^{c_n+b_n-1}B_n1^{c_n}1$ as a subword. □

2.2 The level-n complexity functions

Definition 2.8. For a word w, define the **tail length** $z(w)$ such that $w = u01^{z(w)}$ for some (possibly empty) word u with the conventions that $z(1^{\|w\|}) = \infty$ and $z(u0) = 0$.

Definition 2.9. For $1 \leq n < \infty$, the set of **level-n generating words** is

$$W_n = \{w \in \mathcal{L}^{RS}(T) : c_n \leq z(w) < c_{n+1}\}$$

Proposition 2.10. $\mathcal{L}^{RS}(T) = \{1^\ell : \ell \in \mathbb{N}\} \sqcup \bigsqcup_{n=1}^{\infty} W_n$.

Proof. $\{c_n\}$ is strictly increasing so the W_n are disjoint. Lemma 2.4 says $1^\ell \in \mathcal{L}^{RS}(T)$ for all ℓ and as every word in W_n has 0 as a subword, these are disjoint from the W_n. If $z(w) < c_1$ then $w0 \notin \mathcal{L}(T)$ by Lemma 2.2 so all right-special words with 0 as a subword are in some W_n. □

Definition 2.11. The **level-n complexity** is $p_n(q) = |\{w \in W_n : \|w\| < q\}|$.

By definition, $p_n(\ell+1) - p_n(\ell) = |\{w \in W_n : \|w\| = \ell\}|$.

Proposition 2.12. *The complexity function p satisfies $p(q) = 1 + q + \sum_{n=1}^{\infty} p_n(q)$.*

Proof. Using Proposition 2.10 and that $p(\ell+1) - p(\ell) = |\{w \in \mathcal{L}^{RS} : \|w\| = \ell\}|$,

$$\begin{aligned} p(q) - p(1) &= \sum_{\ell=1}^{q-1} (p(\ell+1) - p(\ell)) = \sum_{\ell=1}^{q-1} |\{w \in \mathcal{L}^{RS}(T) : \|w\| = \ell\}| \\ &= \sum_{\ell=1}^{q-1} \Big(\sum_{n=1}^{\infty} |\{w \in W_n : \|w\| = \ell\}| + |\{1^{\ell}\}| \Big) = \sum_{\ell=1}^{q-1} \Big(\sum_{n=1}^{\infty} (p_n(\ell+1) - p_n(\ell)) + 1 \Big) \\ &= \sum_{n=1}^{\infty} \Big(\sum_{\ell=1}^{q-1} (p_n(\ell+1) - p_n(\ell)) \Big) + q - 1 = \sum_{n=1}^{\infty} (p_n(q) - p_n(1)) + q - 1 \end{aligned}$$

All words in W_n have length at least $1 + c_n > 1$ so $p_n(1) = 0$. The claim follows as $p(1) = 2$. $\square$

2.3 Counting quasi-staircase words

Lemma 2.13. *If $w \in W_n$ then exactly one of the following holds:*

(i) w *is a suffix of* $1^{c_n+i-1}\big(B_n 1^{c_n+i}\big)^{a_n}$ *and* $\|w\| > c_n + i$ *for some* $0 \le i < b_n$*;*

(ii) w *is a suffix of* $1^{c_n+b_n-1} B_n 1^{c_n}$ *and* $\|w\| > h_n + 2c_n$*; or*

(iii) $w = 1^{c_n}(B_n 1^{c_n})^{a_n}$

Proof. The only words in Proposition 2.3 which have $c_n \le z(w) < c_{n+1}$ are of the stated forms; Lemmas 2.5, 2.6 and 2.7 state that these words are in $\mathcal{L}^{RS}(T)$. The forms do not overlap due to the restriction on $\|w\|$ in form (ii). $\square$

Lemma 2.14. *Fix $0 \le i < b_n$. For $c_n + i < \ell < a_n h_n + (a_n+1)(c_n+i)$ there is exactly one word in W_n of form (i) for that value of i; for ℓ not in that range, there are no words of form (i) for that i in W_n.*

Proof. For $w \in W_n$ of form (i), $w = u1^{c_n+i}$ where u is a nonempty suffix of $1^{c_n+i-1}(B_n 1^{c_n+i})^{a_n-1} B_n$. The word u is unique if it exists which is exactly when $c_n + i = \|1^{c_n+i}\| < \|w\| \le \|1^{c_n+i-1}(B_n 1^{c_n+i})^{a_n}\| = a_n h_n + (a_n+1)(c_n+i) - 1$. $\square$

Lemma 2.15. *For $h_n + 2c_n < \ell < h_n + 2c_n + b_n$ there is exactly one word in W_n of form (ii); for ℓ not in that range, there are no words of form (ii) in W_n.*

Proof. To be of that form, $w = u1^{c_n}$ where u is a nonempty suffix of $1^{c_n+b_n-1}B_n$ that has 1^{c_n+1} as a prefix. The word u is unique if it exists and it exists exactly when $h_n + 2c_n + 1 = \|1^{c_n+1}B_n 1^{c_n}\| \le \|w\| \le \|1^{c_n+b_n-1}B_n 1^{c_n}\| = h_n + 2c_n + b_n - 1$. $\square$

Lemma 2.16. *If $\ell \le c_n$ then $p_n(\ell+1) - p_n(\ell) = 0$.*

Proof. Every $w \in W_n$ has subwords 1^{c_n} and 0 so $\|w\| \ge c_n + 1$. $\square$

Lemma 2.17. *If $c_n < \ell < c_n + b_n$ then $p_n(\ell+1) - p_n(\ell) = \ell - c_n$.*

Proof. Lemma 2.14 applies for $0 \le i < \ell - c_n$ but not for $\ell - c_n \le i < b_n$. Lemma 2.15 does not apply. $\square$

Lemma 2.18. *If $c_n + b_n \le \ell \le h_n + 2c_n$ then $p_n(\ell+1) - p_n(\ell) = b_n$.*

Proof. Lemma 2.14 applies for all $0 \leq i < b_n$ and Lemma 2.15 does not apply. □

Lemma 2.19. *If $h_n + 2c_n < \ell < h_n + 2c_n + b_n$ then $p_n(\ell + 1) - p_n(\ell) = b_n + 1$.*

Proof. Lemma 2.14 applies for all $0 \leq i < b_n$ and Lemma 2.15 applies. □

Lemma 2.20. *If $h_n + 2c_n + b_n \leq \ell < a_n h_n + (a_n + 1)c_n$ then $p_n(\ell + 1) - p_n(\ell) = b_n$.*

Proof. Lemma 2.14 applies for all $0 \leq i < b_n$ and Lemma 2.15 does not apply. □

Lemma 2.21. $p_n(a_n h_n + (a_n + 1)c_n + 1) - p_n(a_n h_n + (a_n + 1)c_n) = b_n + 1$.

Proof. Lemma 2.14 applies for all $0 \leq i < b_n$ and Lemma 2.15 does not apply. Lemma 2.13 form (*iii*) gives one additional word in W_n. □

Lemma 2.22. *If $a_n h_n + (a_n + 1)c_n + 1 < \ell < a_n h_n + (a_n + 1)(c_n + b_n - 1)$ then $p_n(\ell + 1) - p_n(\ell) \leq b_n$.*

Proof. Lemma 2.14 applies for some subset of $0 \leq i < b_n$ and Lemma 2.15 does not apply. □

Lemma 2.23. $p(a_n h_n + (a_n + 1)c_n + (a_n + 1)(b_n - 1)) - p(a_n h_n + (a_n + 1)c_n) = \frac{1}{2}(a_n + 1)b_n(b_n - 1) + 1$.

Proof. For each $0 \leq i < b_n$, Lemma 2.14 applies for $\ell = a_n h_n + (a_n + 1)c_n + y$ exactly when $0 \leq y < (a_n + 1)i$, therefore there are a total of $(a_n + 1)\frac{1}{2}b_n(b_n - 1)$ words in W_n of the enclosed lengths from Lemma 2.14. Lemma 2.15 does not apply and Lemma 2.13 form (*iii*) gives one additional word. □

Lemma 2.24. *If $a_n h_n + (a_n + 1)(c_n + b_n - 1) \leq \ell$ then $p_n(\ell + 1) - p_n(\ell) = 0$.*

Proof. Neither Lemma 2.14 nor 2.15 apply. □

2.4 Bounding the complexity of quasi-staircases

Since $p_n(\ell + 1) - p_n(\ell) = 0$ for $\ell \geq a_n h_n + (a_n + 1)(c_n + b_n - 1)$, we define:

Definition 2.25. The **post-productive sequence** is

$$m_n = a_n h_n + (a_n + 1)(c_n + b_n - 1)$$

Lemma 2.26. $p_n(m_n) = h_{n+1} - h_n$

Proof. By Lemma 2.16, $p_n(c_n) = \sum_{\ell=0}^{c_n - 1}(p_n(\ell + 1) - p_n(\ell)) = 0$.
By Lemma 2.17, $p_n(c_n + b_n) - p_n(c_n) = \sum_{\ell=c_n}^{c_n + b_n - 1}(\ell - c_n) = \frac{1}{2}b_n(b_n - 1)$.
By Lemma 2.18, $p_n(h_n + 2c_n + 1) - p_n(c_n + b_n) = (h_n + c_n + 1 - b_n)b_n$.
By Lemma 2.19, $p_n(h_n + 2c_n + b_n) - p_n(h_n + 2c_n + 1) = (b_n + 1)(b_n - 1)$.
By Lemma 2.20, $p_n(a_n h_n + (a_n + 1)c_n) - p_n(h_n + 2c_n + b_n) = ((a_n - 1)h_n + (a_n - 1)c_n - b_n)b_n$.
By Lemma 2.23, $p_n(m_n) - p(a_n h_n + (a_n + 1)c_n) = \frac{1}{2}(a_n + 1)b_n(b_n - 1) + 1$. Therefore

$$\begin{aligned} p_n(m_n) &= \frac{1}{2}b_n(b_n - 1) + (h_n + c_n + 1 - b_n)b_n + (b_n + 1)(b_n - 1) \\ &\quad + ((a_n - 1)h_n + (a_n - 1)c_n - b_n)b_n + \frac{1}{2}(a_n + 1)b_n(b_n - 1) + 1 \\ &= a_n b_n h_n + a_n b_n c_n + \frac{1}{2}a_n b_n(b_n - 1) + b_n(b_n - 1) + b_n - b_n^2 + b_n^2 - 1 - b_n^2 + 1 = h_{n+1} - h_n \end{aligned}$$ □

Definition 2.27. For $q \in \mathbb{N}$ define

$$\alpha(q) = \max\{n : m_n \leq q\} \qquad \text{and} \qquad \beta(q) = \min\{n : q < c_{n+1}\}$$

Lemma 2.28. $\alpha(q) \leq \beta(q)$

Proof. If $\beta(q) \leq \alpha(q) - 1$ then $m_{\alpha(q)} \leq q < c_{\beta(q)+1} \leq c_{\alpha(q)-1+1} = c_{\alpha(q)} < m_{\alpha(q)}$ is impossible. □

Lemma 2.29. *If $q < c_n$ then $p_n(q) = 0$. If $c_n \leq q < m_n$ then $p_n(q) \leq (q - c_n + 1)b_n$. If $m_n \leq q$ then $p_n(q) = h_{n+1} - h_n$.*

Proof. Lemma 2.16 gives $p_n(\ell+1) - p_n(\ell) = 0$ for $0 \leq \ell < c_n$. Lemmas 2.17, 2.18, 2.19, 2.20 and 2.22 all give $p_n(\ell+1) - p_n(\ell) \leq b_n$ for $c_n \leq \ell < m_n$ except for Lemma 2.19 which gives $p_n(\ell+1) - p_n(\ell) = b_n + 1$ for exactly $b_n - 1$ values of ℓ and Lemma 2.21 which gives one additional word. Then, for $c_n \leq q < m_n$,

$$p_n(q) = \sum_{\ell=0}^{q-1} (p_n(\ell+1) - p_n(\ell)) = \sum_{\ell=0}^{c_n-1} 0 + \sum_{\ell=c_n}^{q-1} (p_n(\ell+1) - p_n(\ell)) \leq (q - c_n)b_n + b_n$$

Lemma 2.24 says $p_n(\ell+1) - p_n(\ell) = 0$ for $\ell \geq m_n$ so when $q \geq m_n$, $p_n(q) = p_n(m_n)$ and Lemma 2.26 gives the final statement. □

Proposition 2.30. $p(q) \leq q\Big(2 + \sum_{n=\alpha(q)}^{\beta(q)} b_n\Big)$ *for all q.*

Proof. For n such that $\beta(q) < n$, by Lemma 2.16, $p_n(q) = 0$. Proposition 2.12 and Lemma 2.29 give, using that $h_1 = 1$ so $1 + \sum_{n=1}^{\alpha(q)} (h_{n+1} - h_n) = h_{\alpha(q)+1}$,

$$\begin{aligned} p(q) &= q + 1 + \sum_{n=1}^{\alpha(q)} p_n(q) + \sum_{n=\alpha(q)+1}^{\beta(q)} p_n(q) + \sum_{n=\beta(q)+1}^{\infty} p_n(q) \\ &\leq q + 1 + \sum_{n=1}^{\alpha(q)} (h_{n+1} - h_n) + \sum_{n=\alpha(q)+1}^{\beta(q)} (q - c_n + 1)b_n + 0 \leq q + h_{\alpha(q)+1} + \sum_{n=\alpha(q)+1}^{\beta(q)} q b_n \end{aligned}$$

and

$$\begin{aligned} h_{\alpha(q)+1} &= h_{\alpha(q)} + b_{\alpha(q)}(a_{\alpha(q)}h_{\alpha(q)} + a_{\alpha(q)}c_{\alpha(q)} + \frac{1}{2}a_{\alpha(q)}(b_{\alpha(q)} - 1)) \\ &\leq h_{\alpha(q)} + b_{\alpha(q)}m_{\alpha(q)} \leq m_{\alpha(q)}(1 + b_{\alpha(q)}) \leq q(1 + b_{\alpha(q)}) \end{aligned}$$

□

3 Quasi-staircase complexity arbitrarily close to linear

Proposition 3.1. *Let $\{d_n\}$ be a nondecreasing sequence of integers such that $d_n \to \infty$ and $d_1 = d_2 = 1$ and $d_{n+1} - d_n \in \{0, 1\}$ and $d_{n+1} - d_n$ does not take the value 1 for consecutive n.*

Let $\{b_n\}$ be a nondecreasing sequence of integers such that $b_n \to \infty$ and $b_1 = 3$ and $b_n \leq n + 2$.

Set $a_n = 2n + 2$. Set $c_1 = 1$ and for $n > 1$,

$$c_n = \begin{cases} m_{n-d_n} & \text{when } d_n = d_{n-1} \\ c_{n-1} + b_{n-1} & \text{when } d_n = d_{n-1} + 1 \end{cases}$$

Then $\{a_n\}, \{b_n\}, \{c_n\}$ define a quasi-staircase such that $\sum \frac{a_n b_n^2 + a_{n+1}b_{n+1} + c_{n+1}}{h_n} < \infty$.

Proof. Since $r_n = a_n b_n$, we have $6n + 6 \leq r_n \leq (2n+2)(n+2)$. Then $\prod_{j=1}^{n-1}(r_j + 1) \geq n!$ so $h_n \geq \prod_{j=1}^{n-1}(r_j + 1) \geq n!$ so $\sum \frac{a_n b_n^2 + a_{n+1}b_{n+1}}{h_n} \leq \sum \frac{(2n+2)(n+2)^2 + (2n+4)(n+3)}{n!} < \infty$.

For n such that $d_n = d_{n-1} + 1$, we have $c_n = c_{n-1} + b_{n-1}$ and $d_{n-1} - d_{n-2} = 0$ since $\{d_n\}$ never increases for two consecutive values, so $c_{n-1} = m_{n-1-d_{n-1}}$. As $n - d_n = n - 1 - d_{n-1}$, then $c_n = m_{n-1-d_{n-1}} + b_{n-1} = m_{n-d_n} + b_{n-1}$. So $m_{n-d_n} \leq c_n \leq m_{n-d_n} + b_{n-1}$ for all n.

Since $b_n \geq 3$, we have $r_n \leq \frac{1}{2}r_n(b_n - 1)$. As $b_n \leq a_n + 1$ and $a_n + 1 \leq a_n b_n$,

$$m_n = a_n h_n + (a_n + 1)c_n + r_n + b_n - a_n - 1 \leq (a_n b_n + 1)h_n + a_n b_n c_n + \frac{1}{2}r_n(b_n - 1) = h_{n+1}$$

and therefore, since $d_n \geq 1$ so $n - d_n + 1 \leq n$,

$$c_n \leq m_{n-d_n} + b_{n-1} \leq h_{n-d_n+1} + b_{n-1} \leq h_n + b_{n-1} \leq 2h_n$$

meaning that, as $r_n + b_n \leq h_n$,

$$m_n = (a_n + 1)c_n + a_n h_n + r_n + b_n - a_n - 1 \leq 2(a_n + 1)h_n + a_n h_n + h_n \leq 3(a_n + 1)h_n$$

We now claim that $c_{n+1} \geq c_n + b_n$ for all n. The case when $c_n = m_{n-d_n}$, which occurs when $d_n = d_{n-1}$, is all we need to check. Since $d_2 = d_1 = 1$, we have $c_2 = m_1 \geq c_1 + b_1$. Since $d_n \leq \frac{n}{2}$, we have $a_{n-d_n-1} = 2(n - d_n - 1) + 2 \geq 2(\frac{n}{2} - 1) + 2 = n$. As $b_n \leq n + 2$ and $b_n \geq 3$,

$$\begin{aligned}
c_n - c_{n-1} - b_{n-1} &\geq m_{n-d_n} - (m_{n-d_{n-1}-1} + b_{n-2}) - b_{n-1} = m_{n-d_n} - m_{n-d_n-1} - b_{n-2} - b_{n-1} \\
&\geq a_{n-d_n}h_{n-d_n} - 3(a_{n-d_n-1} + 1)h_{n-d_n-1} - n - (n + 1) \\
&\geq a_{n-d_n}a_{n-d_n-1}b_{n-d_n-1}h_{n-d_n-1} - 3(a_{n-d_n-1} + 1)h_{n-d_n-1} - 2n - 1 \\
&= (a_{n-d_n}a_{n-d_n-1}b_{n-d_n-1} - 3(a_{n-d_n-1} + 1))h_{n-d_n-1} - 2n - 1 \\
&\geq (3a_{n-d_n}a_{n-d_n-1} - 3a_{n-d_n-1} - 3)h_{n-d_n-1} - 2n - 1 \\
&\geq 3a_{n-d_n-1}(a_{n-d_n} - 2)h_{n-d_n-1} - 2n - 1 \geq 3n(n - 2) - 2n - 1 > 0
\end{aligned}$$

for $n \geq 3$. Then $c_{n+1} \geq c_n + b_n$ for all n so $\{c_n\}$, $\{a_n\}$, $\{b_n\}$ define a quasi-staircase transformation.

Now observe that

$$\sum_n \frac{c_n}{h_n} \leq \sum_n \frac{m_{n-d_n} + b_{n-1}}{h_n} \leq 3\sum_n \frac{(a_{n-d_n} + 1)h_{n-d_n}}{h_n} + \sum_n \frac{b_n}{h_n}$$

and the second sum converges as shown at the start of the proof.

As $h_n \geq h_{n-d_n} \prod_{j=n-d_n}^{n-1}(r_j + 1)$,

$$\begin{aligned}
\sum_n \frac{(a_{n-d_n} + 1)h_{n-d_n}}{h_n} &\leq \sum_n \frac{a_{n-d_n} + 1}{\prod_{j=n-d_n}^{n-1}(r_j + 1)} = \sum_n \frac{a_{n-d_n} + 1}{r_{a_{n-d_n}} + 1} \frac{1}{\prod_{j=n-d_n+1}^{n-1}(r_j + 1)} \\
&\leq \sum_n \frac{1}{(r_{n-d_n+1} + 1)^{d_n-1}} = \sum_n (r_{n-d_n+1} + 1)^{1-d_n}
\end{aligned}$$

Since $r_{n-d_n+1} \geq 2(n - d_n + 1) \geq 2(n - n/2 + 1) \geq n$, we have $(r_{n-d_n+1} + 1)^{1-d_n} \leq n^{1-d_n}$. Then, as $d_n \geq 3$ eventually, $\sum(r_{n-d_n+1} + 1)^{1-d_n} \leq \sum n^{1-d_n} < \infty$. Therefore $\sum \frac{c_n}{h_n} < \infty$.

Now observe that

$$\sum_{n:c_{n+1}=c_n+b_n} \frac{c_{n+1}}{h_n} \leq \sum_n \frac{c_n + b_n}{h_n} < \infty$$

and, since $d_n \geq 3$ implies $m_{n-d_n} \leq m_{n-3} \leq 2a_{n-3}h_{n-3}$,

$$\sum_{n:c_{n+1}=m_{n-d_n},d_n\geq 3} \frac{c_{n+1}}{h_n} \leq \sum_n \frac{2a_{n-3}h_{n-3}}{h_n} < \sum_n \frac{2a_{n-3}h_{n-3}}{a_{n-1}a_{n-2}a_{n-3}h_{n-3}} = \sum_n \frac{2}{2n(2n-2)} < \infty$$

and $d_n \geq 3$ eventually so $\sum \frac{c_{n+1}}{h_n} < \infty$. □

Lemma 3.2. *If $f : \mathbb{N} \to \mathbb{N}$ is any function such that $f(q) \to \infty$ then there exists $g : \mathbb{N} \to \mathbb{N}$ which is nondecreasing such that $g(1) = 1$ and $g(q) \leq f(q)$ and $g(q + 2) - g(q) \leq 1$ for all q and $g(q) \to \infty$.*

Proof. Set $f^*(q) = \inf_{q' \geq q} f(q')$. Then $f^*(q) \to \infty$ and $f^*(q)$ is nondecreasing and $f^*(q) \leq f(q)$ for all q. Set $g(1) = 1 \leq f^*(1)$. For $n \geq 0$, set $g(2n+2) = g(2n+1)$ and for $n \geq 1$ set

$$g(2n+1) = g(2n) + \begin{cases} 1 & \text{when } f^*(2n+1) > f^*(2n-1) \\ 0 & \text{otherwise} \end{cases}$$

Then g is nondecreasing and $g(q+2) - g(q) \leq 1$ for all q. Since f^* is integer-valued, if $f^*(2n+1) - f^*(2n-1) \neq 0$ then $f^*(2n+1) - f^*(2n-1) \geq 1$. Then $g(2n+1) - g(2n-1) \leq f^*(2n+1) - f^*(2n-1)$ so for all n we have

$$g(2n+1) = g(1) + \sum_{m=1}^{n} (g(2m+1) - g(2m-1)) \leq f^*(1) + \sum_{m=1}^{n} (f^*(2m+1) - f^*(2m-1)) = f^*(2n+1)$$

so, as $g(2n+2) = g(2n+1) \leq f^*(2n+1) \leq f^*(2n+2)$, we have $g(q) \leq f^*(q) \leq f(q)$ for all q. If $g(q) \leq C$ for all q then $f^*(2n+1) = f^*(2n-1)$ eventually, contradicting that $f^*(q) \to \infty$. Therefore $g(q) \to \infty$. $\square$

Theorem 3.3. *Let $f : \mathbb{N} \to \mathbb{N}$ be any function such that $f(q) \to \infty$. There exists a quasi-staircase transformation with $\sum \frac{a_n b_n^2 + a_{n+1} b_{n+1} + c_{n+1}}{h_n} < \infty$, $\frac{b_n}{a_n} \to 0$ and complexity satisfying $\frac{p(q)}{qf(q)} \to 0$.*

Proof. By Lemma 3.2, we may assume f is nondecreasing and that $f(n+2) - f(n) \leq 1$ for all n. Then $f(n+1) - f(n) \in \{0, 1\}$ and is never 1 for two consecutive values. We may also assume $f(1) = 1$.

Set $d_1 = d_2 = 1$ and $d_n = \left\lfloor \sqrt[3]{f(n)} \right\rfloor$ for $n > 2$. Then $d_n \to \infty$ is nondecreasing. Also $d_{n+1} - d_n \in \{0, 1\}$ and is never 1 for two consecutive values.

Set $b_n = 3$ for all n such that $\sqrt[3]{f(n)} < 3$ and $b_n = \left\lfloor \sqrt[3]{f(n)} \right\rfloor$ for n such that $\sqrt[3]{f(n)} \geq 3$. Then $b_n \to \infty$ is nondecreasing and $b_n \leq f(n) + 2 \leq n + 2$ as $f(n) \leq n$ since $f(1) = 1$ and $f(n+2) - f(n) \leq 1$ imply $f(n) \leq 1 + \frac{n}{2}$.

Take the quasi-staircase transformation from Proposition 3.1 with defining sequences $\{a_n\}$ and $\{c_n\}$.

As $a_n = 2n + 2$ and $b_n = \max(3, \sqrt[3]{f(n)}) \leq \sqrt[3]{n}$, we have $\frac{b_n}{a_n} \to 0$.

Since $0 \leq d_{n+1} - d_n \leq 1$, the sequence $n - d_n$ is nondecreasing and attains every value in $\mathbb{N}$. For each q, let n_q be the largest n such that $m_{n-d_n} \leq q$. Then $q < m_{n_q+1-d_{n_q+1}}$ so $n_q + 1 - d_{n_q+1} > n_q - d_{n_q}$ and so $1 > d_{n_q+1} - d_{n_q}$ meaning that $d_{n_q+1} = d_{n_q}$. Therefore $c_{n_q+1} = m_{n_q+1-d_{n_q+1}} = m_{n_q-d_{n_q}+1}$.

So $\alpha(q) = n_q - d_{n_q}$ as $m_{n_q-d_{n_q}} \leq q < m_{n_q+1-d_{n_q+1}} = m_{n_q-d_{n_q}+1}$ and $\beta(q) \leq n_q$ since $q < m_{n_q-d_{n_q}+1} = c_{n_q+1}$. By Proposition 2.30, since $q \geq n_q$ and f is nondecreasing to infinity and $n_q \to \infty$,

$$\begin{aligned} \frac{p(q)}{qf(q)} &\leq \frac{2 + \sum_{n=\alpha(q)}^{\beta(q)} b_n}{f(q)} \leq \frac{2 + \sum_{n=n_q-d_{n_q}}^{n_q} b_n}{f(q)} \leq \frac{2 + (d_{n_q}+1) b_{n_q}}{f(n_q)} \\ &\leq \frac{2 + (\sqrt[3]{f(n_q)} + 1) \sqrt[3]{f(n_q)}}{f(n_q)} = \frac{2}{f(n_q)} + \frac{1}{\sqrt[3]{f(n_q)}} + \frac{1}{(\sqrt[3]{f(n_q)})^2} \to 0 \end{aligned}$$

$\square$

4 Mixing for quasi-staircase transformations

Proposition 4.1. *Let T be a quasi-staircase transformation given by $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ with height sequence $\{h_n\}$ such that $\sum \frac{c_n + b_n}{h_n} < \infty$. Then T is on a finite measure space.*

Proof. Writing S_n for the spacers added above the n^{th} column,

$$\mu(S_n) = (c_n r_n + \frac{1}{2} r_n (b_n - 1)) \mu(I_{n+1}) = \left(c_n \frac{r_n}{r_n + 1} + \frac{1}{2} \frac{r_n (b_n - 1)}{r_n + 1} \right) \mu(I_n) \leq \frac{c_n + b_n}{h_n} \mu(C_n)$$

and therefore $\mu(C_{n+1}) = \mu(C_n) + \mu(S_n) \leq \left(1 + \frac{c_n + b_n}{h_n}\right) \mu(C_n)$. Then $\mu(C_{n+1}) \leq \prod_{j=1}^{n} \left(1 + \frac{c_j + b_j}{h_j}\right) \mu(C_1)$, meaning that $\log(\mu(C_{n+1})) \leq \log(\mu(C_1)) + \sum_{j=1}^{n} \log(1 + \frac{c_j+b_j}{h_j})$. As $\frac{c_n+b_n}{h_n} \to 0$, since $\log(1+x) \approx x$ for $x \approx 0$, $\lim_n \log(\mu(C_{n+1})) \lesssim \log(\mu(C_1)) + \sum_{j=1}^{\infty} \frac{c_j+b_j}{h_j} < \infty$. □

For the remainder of this section, all transformations are on probability spaces.

Recall that a sequence $\{t_n\}$ is **mixing** when for all measurable sets A and B, $\mu(T^{t_n}A \cap B) \to \mu(A)\mu(B)$.

Notation 4.2. *For measurable sets A and B, write*

$$\lambda_B(A) = \mu(A \cap B) - \mu(A)\mu(B)$$

So $\{t_n\}$ is mixing when $\lambda_B(T^{t_n}A) \to 0$ for all measurable A and B. The following is left to the reader:

Lemma 4.3. *If A and A' are disjoint then*

$$\lambda_B(A \sqcup A') = \lambda_B(A) + \lambda_B(A') \qquad \text{and} \qquad |\lambda_B(A)| \leq \mu(A)$$

and, writing $\chi_B(x) = \mathbb{1}_B(x) - \mu(B)$, for $n \in \mathbb{Z}$, $\lambda_B(T^nA) = \int_A \chi_B \circ T^n d\mu$.

For a rank-one transformation T, a sequence $\{t_n\}$ is **rank-one uniform mixing** when for every union of levels B, $\sum_{j=0}^{h_n-1} |\lambda_B(T^{t_n}I_{n,j})| \to 0$. Rank-one uniform mixing for a sequence implies mixing for that sequence [CS04] Proposition 5.6.

Notation 4.4. *For $h_n \leq j < h_n + c_n$, let $I_{n,j} = T^{j-h_n+1}I_{n,h_n-1}$ be the union of the $(j-h_n)^{th}$ stage of the c_n spacer levels added above every subcolumn. Write*

$$\tilde{h}_n = h_n + c_n$$

Lemma 4.5. *Let T be a rank-one transformation, B a union of levels in some column C_N and $n \geq N$. Then for any $0 \leq j < \tilde{h}_n$ and $0 \leq i \leq r_n$,*

$$\lambda_B(I_{n,j}^{[i]}) = \frac{1}{r_n+1}\lambda_B(I_{n,j})$$

Proof. Since B is a union of levels in C_N, either $I_{n,j} \subseteq B$ or $I_{n,j} \cap B = \emptyset$. If $I_{n,j} \subseteq B$ then $\mu(I_{n,j}^{[i]} \cap B) = \mu(I_{n,j}^{[i]}) = \frac{1}{r_n+1}\mu(I_{n,j}) = \frac{1}{r_n+1}\mu(I_{n,j} \cap B)$ and if $I_{n,j} \cap B = \emptyset$ then $\mu(I_{n,j}^{[i]} \cap B) = 0 = \frac{1}{r_n+1}\mu(I_{n,j} \cap B)$. □

Lemma 4.6. *Let T be a quasi-staircase transformation. Then for any n and $0 \leq \ell < b_n$ and $k, i \geq 0$ such that $i + k \leq a_n$ and any $j \geq k\ell$,*

$$T^{k\tilde{h}_n} I_{n,j}^{[\ell a_n + i]} = I_{n,j-k\ell}^{[\ell a_n + i + k]}$$

Proof. There are $c_n + \left\lfloor \frac{\ell a_n + i}{a_n} \right\rfloor = c_n + \ell$ spacers above $I_{n,j}^{[\ell a_n+i]}$ so $T^{\tilde{h}_n} I_{n,j}^{[\ell a_n+i]} = I_{n,j-\ell}^{[\ell a_n+i+1]}$. Since $i+k \leq a_n$, there are also $c_n + \ell$ spacers above each $I_{n,j-v\ell}^{[\ell a_n+i+v]}$ for $1 \leq v < k$ so applying $T^{h_n+c_n}$ repeated k times, the claim follows. □

Lemma 4.7. *Let T be a quasi-staircase transformation, $k \in \mathbb{N}$, B a union of levels in some C_N and $n \geq N$. If $k < a_n$ and $kb_n < h_n$ then*

$$\sum_{j=0}^{h_n-1} |\lambda_B(T^{k\tilde{h}_n}I_{n,j})| \leq \int \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n-1} \chi_B \circ T^{-k\ell} \right| d\mu + \frac{k+1}{a_n} + \frac{kb_n}{h_n}$$

Proof. By Lemma 4.6 and then Lemma 4.5, for $kb_n \leq j < h_n$,

$$|\lambda_B(T^{k\tilde{h}_n}I_{n,j})| = \left| \sum_{\ell=0}^{b_n-1} \sum_{i=0}^{a_n-1} \lambda_B(T^{k\tilde{h}_n} I_{n,j}^{[\ell a_n+i]}) + \lambda_B(T^{k\tilde{h}_n} I_{n,j}^{[r_n]}) \right|$$

$$
\begin{aligned}
&\leq \left| \sum_{\ell=0}^{b_n-1} \sum_{i=0}^{a_n-k-1} \lambda_B(T^{k\tilde{h}_n} I_{n,j}^{[\ell a_n+i]}) \right| + (b_n k+1)\mu(I_{n+1}) \\
&= \left| \sum_{\ell=0}^{b_n-1} \sum_{i=0}^{a_n-k-1} \lambda_B(I_{n,j-k\ell}^{[\ell a_n+i+k]}) \right| + (b_n k+1)\mu(I_{n+1}) \\
&= \left| \sum_{\ell=0}^{b_n-1} \sum_{i=0}^{a_n-k-1} \frac{1}{r_n+1}\lambda_B(I_{n,j-k\ell}) \right| + \frac{b_n k+1}{r_n+1}\mu(I_n) \\
&= \left| \frac{1}{r_n+1} \sum_{\ell=0}^{b_n-1} \sum_{i=0}^{a_n-k-1} \lambda_B(T^{-k\ell} I_{n,j}) \right| + \frac{b_n k+1}{r_n+1}\mu(I_n) \\
&= \frac{a_n-k}{r_n+1} \left| \sum_{\ell=0}^{b_n-1} \lambda_B(T^{-k\ell} I_{n,j}) \right| + \frac{b_n k+1}{r_n+1}\mu(I_n) \leq \frac{1}{b_n} \left| \sum_{\ell=0}^{b_n-1} \lambda_B(T^{-k\ell} I_{n,j}) \right| + \frac{k+1}{a_n}\mu(I_n) \\
&= \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n-1} \int_{I_{n,j}} \chi_B \circ T^{-k\ell} \, d\mu \right| + \frac{k+1}{a_n}\mu(I_n) \leq \int_{I_{n,j}} \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n-1} \chi_B \circ T^{-k\ell} \right| \, d\mu + \frac{k+1}{a_n}\mu(I_n)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sum_{j=0}^{h_n-1} |\lambda_B(T^{k\tilde{h}_n} I_{n,j})| &\leq \sum_{j=kb_n}^{h_n-1} |\lambda_B(T^{k\tilde{h}_n} I_{n,j})| + kb_n\mu(I_n) \\
&\leq \sum_{j=kb_n}^{h_n-1} \left(\int_{I_{n,j}} \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n-1} \chi_B \circ T^{-k\ell} \right| \, d\mu + \frac{k+1}{a_n}\mu(I_{n,j}) \right) + kb_n\mu(I_n) \\
&\leq \int \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n-1} \chi_B \circ T^{-k\ell} \right| \, d\mu + \frac{k+1}{a_n} + \frac{kb_n}{h_n}
\end{aligned}
$$

□

Proposition 4.8. *Let T be a quasi-staircase transformation and $k \in \mathbb{N}$. If T^k is ergodic then $\{k\tilde{h}_n\}$ and $\{kh_n\}$ are rank-one uniform mixing.*

Proof. Since $\frac{b_n}{h_n} \to 0$ and $a_n \to \infty$ there exists N such that for all $n \geq N$ we have $k < a_n$ and $kb_n < h_n$. That $\{k\tilde{h}_n\}$ is rank-one uniform mixing follows from Lemma 4.7 since T^k is ergodic, $b_n \to \infty$, $a_n \to \infty$ and $\frac{b_n}{h_n} \to 0$. Then

$$
\sum_{j=0}^{h_n-1} |\lambda_B(T^{kh_n} I_{n,j})| \leq \sum_{j=kc_n}^{h_n} |\lambda_B(T^{kh_n} I_{n,j})| + \frac{kc_n}{h_n} = \sum_{j=0}^{h_n-kc_n} |\lambda_B(T^{k\tilde{h}_n} I_{n,j})| + \frac{kc_n}{h_n} \to 0
$$

as $\frac{c_n}{h_n} \to 0$, k is fixed and $\{k\tilde{h}_n\}$ is rank-one uniform mixing. □

Lemma 4.9 ([CPR22] Proposition A.13)**.** *Let T be a rank-one transformation and $\{c_n\}$ a sequence such that $\frac{c_n}{h_n} \to 0$. If $k \in \mathbb{N}$ and $\{q(h_n + c_n)\}$ is rank-one uniform mixing for each $q \leq k+1$ and $\{t_n\}$ is a sequence such that $h_n + c_n \leq t_n < (q+1)(h_n + c_n)$ for all n then $\{t_n\}$ is mixing.*

Lemma 4.10 ([CPR22] Proposition A.16)**.** *Let T be a rank-one transformation and $\{c_n\}$ a sequence such that $\frac{c_n}{h_n} \to 0$. If $\{q(h_n + c_n)\}$ is rank-one uniform mixing for each fixed q and $k_n \to \infty$ is such that $\frac{k_n}{n} \leq 1$ then for any measurable set B, $\int |\frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{-jk_n}| \, d\mu \to 0$.*

Proposition 4.11. *Let T be a quasi-staircase transformation and B a measurable set. Then*

$$
\max_{1 \leq k \leq n} \int \left| \frac{1}{n} \sum_{j=0}^{n-1} \chi_B \circ T^{-jk} \right| \, d\mu \to 0
$$

Proof. As T is ergodic, Proposition 4.8 with $k = 1$ gives that $\{\tilde{h}_n\}$ is rank-one uniform mixing, hence mixing, so T is totally ergodic. Then Proposition 4.8 gives that for each fixed k the sequence $\{k\tilde{h}_n\}$ is rank-one uniform mixing so Lemma 4.10 gives the claim. □

Proposition 4.12. *Let T be a quasi-staircase transformation, B a measurable set and $Q > 0$. Then*

$$\max_{h_n+c_n \le t < Q\tilde{h}_n} |\lambda_B(T^t B)| \to 0$$

Proof. As in the proof of Proposition 4.11, for each fixed k the sequence $\{k\tilde{h}_n\}$ is rank-one uniform mixing so Lemma 4.9 gives the claim. □

Lemma 4.13. *Let T be a quasi-staircase transformation. Let $n > 0$ and $0 \le x < b_n$ and $0 \le q < a_n$. If $0 \le \ell < b_n - x$ and $0 \le i < a_n - q$ and $j \ge \frac{1}{2}a_n x(x-1) + qx + ix + \ell(xa_n + q)$ then*

$$T^{(xa_n+q)\tilde{h}_n} I_{n,j}^{[\ell a_n + i]} = I_{n,j-\frac{1}{2}a_n x(x-1)-qx-ix-\ell(xa_n+q)}^{[(\ell+x)a_n+i+q]}$$

Proof. If $x = 0$ then Lemma 4.6 applied with q in place of k gives the claim. So we can write

$$xa_n + q = (a_n - i) + (x-1)a_n + (q+i)$$

and assume all three terms on the right are nonnegative.

Using Lemma 4.6,

$$T^{(a_n-i)\tilde{h}_n} I_{n,j}^{[\ell a_n+i]} = I_{n,j-(a_n-i)\ell}^{[\ell a_n+i+a_n-i]} = I_{n,j-(a_n-i)\ell}^{[(\ell+1)a_n]}$$

Now observe that, by Lemma 4.6 with 0 as i and a_n as k, for any $0 \le v < x$ and any $a_n v \le z < h_n$,

$$T^{a_n\tilde{h}_n} I_{n,z}^{[va_n]} = I_{n,z-a_n v}^{[(v+1)a_n]}$$

so applying that $x - 1$ times for $v = \ell+1, \ell+2, \ldots, \ell+x-1$,

$$T^{(x-1)a_n\tilde{h}_n} I_{n,j-(a_n-i)\ell}^{[(\ell+1)a_n]} = I_{n,j-(a_n-i)\ell-(x-1)\ell a_n-\frac{1}{2}x(x-1)a_n}^{[(\ell+x)a_n]}$$

since $\sum_{v=\ell+1}^{\ell+x-1} v = \frac{1}{2}(\ell+x)(\ell+x-1) - \frac{1}{2}\ell(\ell+1) = (x-1)\ell + \frac{1}{2}x(x-1)$. Then applying Lemma 4.6 one final time with $q+i$ in place of k,

$$\begin{aligned} T^{(q+i)\tilde{h}_n} I_{nj-(a_n-i)\ell-(x-1)\ell a_n-\frac{1}{2}x(x-1)a_n}^{[(\ell+x)a_n]} &= I_{n,j-(a_n-i)\ell-(x-1)\ell a_n-\frac{1}{2}x(x-1)a_n-(x+\ell)(q+i)}^{[(\ell+x)a_n+q+i]} \\ &= I_{n,j-x\ell a_n-\frac{1}{2}x(x-1)a_n-xi-xq-\ell q}^{[(\ell+x)a_n+q+i]} \end{aligned}$$ □

Lemma 4.14. *Let T be a quasi-staircase transformation. Let $n > 0$ and $0 \le x < b_n$ and $0 \le q < a_n$. If $0 \le \ell < b_n - x - 1$ and $a_n - q \le i < a_n$ and $j \ge \frac{1}{2}a_n x(x+1) + q(x+1) + i(x+1) + \ell(xa_n + 1)$ then*

$$T^{(xa_n+q)\tilde{h}_n} I_{n,j}^{[\ell a_n+i]} = I_{n,j-\frac{1}{2}a_n x(x+1)-(q+i-a_n)(x+1)-\ell(xa_n+q)}^{[(\ell+x)a_n+i+q]}$$

Proof. The same proof as Lemma 4.13 except we write $xa_n + q = (a_n - i) + xa_n + (q + i - x)$. □

Lemma 4.15. *Let T be a quasi-staircase transformation. Let B be a union of levels C_N. For $n \ge N$ and $k_n\tilde{h}_n \le t_n < (k_n+1)\tilde{h}_n$,*

$$\sum_{j=0}^{h_n-1} |\lambda_B(T^{t_n} I_{n,j})| \le \sum_{x=0}^{h_n-1} |\lambda_B(T^{k_n\tilde{h}_n} I_{n,x})| + c_n\mu(I_n) + \sum_{x=0}^{h_n-1} |\lambda_B(T^{(k_n+1)\tilde{h}_n} I_{n,x})|$$

Proof. Write $t_n = k_n\tilde{h}_n + z_n$ for $0 \leq z_n < \tilde{h}_n$. Then

$$\begin{aligned}\sum_{j=0}^{h_n-1} |\lambda_B(T^{t_n} I_{n,j})| &\leq \sum_{j=0}^{h_n-z_n-1} |\lambda_B(T^{t_n} I_{n,j})| + c_n\mu(I_n) + \sum_{j=h_n-z_n+c_n}^{h_n-1} |\lambda_B(T^{t_n} I_{n,j})| \\ &\leq \sum_{j=0}^{h_n-z_n-1} |\lambda_B(T^{k_n\tilde{h}_n} I_{n,j+z_n})| + c_n\mu(I_n) + \sum_{j=\tilde{h}_n-z_n}^{h_n-1} |\lambda_B(T^{(k_n+1)\tilde{h}_n} I_{n,j+z_n-\tilde{h}_n})| \\ &\leq \sum_{x=0}^{h_n-1} |\lambda_B(T^{k_n\tilde{h}_n} I_{n,x})| + c_n\mu(I_n) + \sum_{x=0}^{h_n-1} |\lambda_B(T^{(k_n+1)\tilde{h}_n} I_{n,x})| \qquad \square\end{aligned}$$

Proposition 4.16. *Let T be a quasi-staircase transformation such that $\frac{a_n b_n^2}{h_n} \to 0$ and $\frac{b_n}{a_n} \to 0$ and B be a union of levels in some fixed C_N. For $n > N$, set*

$$M_{B,n} := \max_{a_n\tilde{h}_n \leq t < \tilde{h}_{n+1}} \sum_{j=0}^{h_n-1} |\lambda_B(T^t I_{n,j})|$$

Then $\lim_{n\to\infty} M_{B,n} = 0$.

Proof. Let t_n attain the maximum in $M_{B,n}$. If $t_n \geq (r_n - 1)\tilde{h}_n$ then $h_{n+1} + c_{n+1} - t_n \leq c_{n+1} + 2h_n + c_n + \frac{1}{2}a_n b_n(b_n - 1)$ so

$$\begin{aligned}\sum_{j=0}^{h_{n+1}-1} |\lambda_B(T^{t_n} I_{n+1,j})| &\leq \sum_{j=h_{n+1}+c_{n+1}-t_n}^{h_{n+1}-1} |\lambda_B(T^{t_n} I_{n+1,j})| + (h_{n+1} + c_{n+1} - t_n)\mu(I_{n+1}) \\ &\leq \sum_{j=0}^{t_n-c_{n+1}-1} |\lambda_B(T^{\tilde{h}_{n+1}} I_{n+1,j})| + \frac{c_{n+1} + 2h_n + c_n\frac{1}{2}a_n b_n(b_n-1)}{h_{n+1}} \to 0\end{aligned}$$

since $\{\tilde{h}_{n+1}\}$ is rank-one uniform mixing.

So we may assume $t_n < (r_n - 1)\tilde{h}_n$ and therefore write $t_n = k_n\tilde{h}_n + z_n$ for $a_n \leq k_n < r_n - 1$ and $0 \leq z_n < \tilde{h}_n$. By Lemma 4.15,

$$\sum_{j=0}^{h_n-1} |\lambda_B(T^{t_n} I_{n,j})| \leq \sum_{x=0}^{h_n-1} |\lambda_B(T^{k_n\tilde{h}_n} I_{n,x})| + c_n\mu(I_n) + \sum_{x=0}^{h_n-1} |\lambda_B(T^{(k_n+1)\tilde{h}_n} I_{n,x})|$$

We will show the sum on the left tends to zero; the same argument with $k_n + 1$ in place of k_n gives the same for the right sum. As $c_n\mu(I_n) \to 0$, this will complete the proof.

Write $k_n = x_n a_n + q_n$ for $0 \leq q_n < a_n$ and $1 \leq x_n < b_n$. Observe that

$$\begin{aligned}\sum_{j=0}^{h_n-1} |\lambda_B(T^{k_n\tilde{h}_n} I_{n,j})| &\leq \sum_{j=0}^{h_n-1} \left| \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=0}^{a_n-1} \lambda_B(T^{k_n\tilde{h}_n} I_{n,j}^{[\ell a_n+i]}) \right| + 2a_n h_n \mu(I_{n+1}) && (\star) \\ &\quad + \sum_{j=0}^{h_n-1} \left| \sum_{\ell=b_n-x_n+1}^{b_n-1} \sum_{i=0}^{a_n-1} \lambda_B(T^{k_n\tilde{h}_n} I_{n,j}^{[\ell a_n+i]}) \right| + \frac{1}{r_n+1} && (\star\star)\end{aligned}$$

We handle the sum $(\star\star)$ first and return to the sum in $(\star)$ shortly.

For $0 \leq \ell < b_n$ and $0 \leq i < a_n$, we have that

$$I_{n,0}^{[\ell a_n+i]} = T^{(\ell a_n+i)\tilde{h}_n} I_{n,\frac{1}{2}\ell(\ell-1)a_n+i\ell}^{[0]}$$

since $\frac{1}{2}\ell(\ell-1)a_n + i\ell \leq a_n b_n^2 + a_n b_n < h_n$ (as $\frac{a_n b_n^2}{h_n} \to 0$).

For $b_n - x_n + 1 \leq \ell < b_n$ and $0 \leq i < a_n$, since $x + \ell \geq b_n + 1$,

$$\begin{aligned} k_n\tilde{h}_n + (\ell a_n + i)\tilde{h}_n &= (x_n a_n + q_n + \ell a_n + i)(h_n + c_n) \\ &\geq (b_n a_n + a_n)\tilde{h}_n \\ &= (b_n a_n + 1)h_n + b_n a_n c_n + (a_n - 1)h_n + a_n c_n \geq h_{n+1} \end{aligned}$$

since $\frac{1}{2}a_n b_n(b_n - 1) \leq h_n$. Also,

$$\begin{aligned} k_n\tilde{h}_n + (\ell a_n + i)\tilde{h}_n + \frac{1}{2}\ell(\ell-1)a_n + i\ell &= ((x_n + \ell)a_n + q_n + i)(h_n + c_n) + \frac{1}{2}\ell(\ell-1)a_n + i\ell \\ &\leq 2b_n a_n(h_n + c_n) + \frac{1}{2}b_n(b_n - 1)a_n + a_n b_n < 2h_{n+1} \end{aligned}$$

Since a sublevel in I_n is a level in I_{n+1} and $\{h_{n+1}\}$ is rank-one uniform mixing (Proposition 4.8),

$$\sum_{j=0}^{h_n-1}\sum_{\ell=b_n-x_n+1}^{b_n-1}\sum_{i=0}^{a_n-1} |\lambda_B(T^{k_n\tilde{h}_n} I_{n,j}^{[\ell a_n + i]})| \leq \sum_{y=0}^{h_{n+1}-1} |\lambda_B(T^{h_{n+1}} I_{n+1,y})| \to 0$$

As $2a_n h_n \mu(I_{n+1}) \leq \frac{2a_n h_n}{h_{n+1}} \leq \frac{2}{b_n} \to 0$ and $r_n \to \infty$, it remains only to show that the sum in ($\star$) tends to zero. Observe that

$$\begin{aligned} \sum_{\ell=0}^{b_n-x_n-2}\sum_{i=0}^{a_n-1} \lambda_B(T^{k_n\tilde{h}_n} I_{n,j}^{[\ell a_n+i]}) = &\sum_{\ell=0}^{b_n-x_n-2}\sum_{i=0}^{a_n-q_n-1} \lambda_B(T^{k_n\tilde{h}_n} I_{n,j}^{[\ell a_n+i]}) && (\dagger) \\ &+ \sum_{\ell=0}^{b_n-x_n-2}\sum_{i=a_n-q_n}^{a_n-1} \lambda_B(T^{k_n\tilde{h}_n} I_{n,j}^{[\ell a_n+i]}) && (\ddagger) \end{aligned}$$

First, we address ($\dagger$): set $y_n = \frac{1}{2}a_n x_n(x_n - 1) + q_n x_n$. For $i < a_n - q_n$ and $\ell < b_n - x_n - 1$, we have $y_n + ix_n + \ell k_n \leq 3a_n b_n^2$ so for $j \geq 3a_n b_n^2$, by Lemma 4.13 and Lemma 4.5,

$$\begin{aligned} \sum_{\ell=0}^{b_n-x_n-2}\sum_{i=0}^{a_n-q_n-1} \lambda_B(T^{k_n\tilde{h}_n} I_{n,j}^{[\ell a_n+i]}) &= \sum_{\ell=0}^{b_n-x_n-2}\sum_{i=0}^{a_n-q_n-1} \lambda_B(I_{n,j-y_n-ix_n-\ell k_n}^{[(\ell+x_n)a_n+i+q_n]}) \\ = \frac{1}{r_n+1}\sum_{\ell=0}^{b_n-x_n-2}\sum_{i=0}^{a_n-q_n-1} \lambda_B(I_{n,j-y_n-ix_n-\ell k_n}) &= \frac{1}{r_n+1}\sum_{\ell=0}^{b_n-x_n-2}\sum_{i=0}^{a_n-q_n-1} \lambda_B(T^{-\ell k_n - ix_n - y_n} I_{n,j}) \end{aligned}$$

Then, summing over all $3a_n b_n^2 \leq j < h_n$,

$$\begin{aligned} &\sum_{j=3a_nb_n^2}^{h_n-1} \left| \sum_{\ell=0}^{b_n-x_n-2}\sum_{i=0}^{a_n-q_n-1} \lambda_B(T^{k_n\tilde{h}_n} I_{n,j}^{[\ell a_n+i]}) \right| \\ &\quad= \sum_{j=3a_nb_n^2}^{h_n-1} \left| \frac{1}{r_n+1}\sum_{\ell=0}^{b_n-x_n-2}\sum_{i=0}^{a_n-q_n-1} \lambda_B(T^{-\ell k_n - ix_n - y_n} I_{n,j}) \right| \\ &\quad\leq \frac{1}{r_n+1}\sum_{j=0}^{h_n-1}\sum_{\ell=0}^{b_n-x_n-2} \left| \sum_{i=0}^{a_n-q_n-1} \lambda_B(T^{-\ell k_n - ix_n - y_n} I_{n,j}) \right| \\ &\quad\leq \frac{1}{r_n+1}\sum_{\ell=0}^{b_n-x_n-2} \int \left| \sum_{i=0}^{a_n-q_n-1} \chi_B \circ T^{-\ell k_n - ix_n - y_n} \right| \, d\mu \\ &\quad= \frac{(b_n - x_n - 2)(a_n - q_n)}{r_n+1} \int \left| \frac{1}{a_n - q_n}\sum_{i=0}^{a_n-q_n-1} \chi_B \circ T^{-ix_n} \right| \, d\mu \end{aligned}$$

$$\leq \min\left(\frac{a_n-q_n}{a_n}, \int \left| \frac{1}{a_n-q_n} \sum_{i=0}^{a_n-q_n-1} \chi_B \circ T^{-ix_n} \right| d\mu\right)$$

since $\frac{(b_n-2)}{r_n+1} < \frac{1}{a_n}$ and $\int |\chi_B| \, d\mu \leq 1$. For a subsequence along which $x_n \leq a_n - q_n$, Proposition 4.11 implies the integral tends to zero. For n such that $a_n - q_n < x_n < b_n$, the quantity on the left is bounded by $\frac{b_n}{a_n} \to 0$.

For (‡): set $y'_n = \frac{1}{2}a_n x_n(x_n+1) + (q_n - a_n)(x_n+1)$. By Lemma 4.14 and Lemma 4.5, for $j \geq 3a_n b_n^2$,

$$\sum_{\ell=0}^{b_n-x_n-2} \sum_{i=a_n-q_n}^{a_n-1} \lambda_B(T^{k_n\tilde{h}_n} I_{n,j}^{[\ell a_n+i]}) = \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=a_n-q_n}^{a_n-1} \lambda_B(I_{n,j-y'_n-i(x_n+1)-\ell k_n}^{[(\ell+x_n)a_n+i+q_n]})$$

$$= \frac{1}{r_n+1} \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=a_n-q_n}^{a_n-1} \lambda_B(I_{n,j-y'_n-i(x_n+1)-\ell k_n}) = \frac{1}{r_n+1} \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=a_n-q_n}^{a_n-1} \lambda_B(T^{-\ell k_n - i(x_n+1)-y'_n} I_{n,j})$$

Similar to the sum (†), then

$$\sum_{j=3a_nb_n^2}^{h_n-1} \left| \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=a_n-q_n}^{a_n-1} \lambda_B(T^{k_n\tilde{h}_n} I_{n,j}^{[\ell a_n+i]}) \right|$$

$$= \sum_{j=3a_nb_n^2}^{h_n-1} \left| \frac{1}{r_n+1} \sum_{\ell=0}^{b_n-x_n-2} \sum_{i=a_n-q_n}^{a_n-1} \lambda_B(T^{-\ell k_n - i(x_n+1)-y'_n} I_{n,j}) \right|$$

$$\leq \frac{(b_n-x_n-2)q_n}{r_n+1} \int \left| \frac{1}{q_n} \sum_{i=a_n-q_n}^{a_n-1} \chi_B \circ T^{-i(x_n+1)} \right| d\mu$$

$$= \frac{(b_n-x_n-2)q_n}{r_n+1} \int \left| \frac{1}{q_n} \sum_{i'=0}^{q_n-1} \chi_B \circ T^{-i'(x_n+1)} \right| d\mu \leq \min\left(\frac{q_n}{a_n}, \int \left| \frac{1}{q_n} \sum_{i'=0}^{q_n-1} \chi_B \circ T^{-i'(x_n+1)} \right| d\mu\right)$$

and along any subsequence where $x_n + 1 \leq q_n$, this tends to zero by Proposition 4.11, and for $q_n \leq x_n + 1 < b_n + 1$, the quantity on the left is bounded by $\frac{b_n}{a_n} \to 0$, completing the proof. □

Proposition 4.17. *Let T be a quasi-staircase transformation with $\frac{b_n^2}{h_n} \to 0$ and $\frac{b_n}{a_n} \to 0$ and B be a union of levels in some fixed C_N. For $n > N$, set*

$$\widehat{M}_{B,n} := \max_{\tilde{h}_n \leq t < b_n\tilde{h}_n} \sum_{j=0}^{h_n-1} |\lambda_B(T^t I_{n,j})|$$

Then $\lim_{n\to\infty} \widehat{M}_{B,n} = 0$.

Proof. Let t_n attain the maximum in $\widehat{M}_{B,n}$. By Lemma 4.15, writing $t_n = k_n\tilde{h}_n + z_n$ for $1 \leq k_n < b_n$ and $0 \leq z_n < \tilde{h}_n$,

$$\sum_{j=0}^{h_n-1} |\lambda_B(T^{t_n} I_{n,j})| \leq \sum_{x=0}^{h_n-1} |\lambda_B(T^{k_n\tilde{h}_n} I_{n,x})| + c_n\mu(I_n) + \sum_{x=0}^{h_n-1} |\lambda_B(T^{(k_n+1)\tilde{h}_n} I_{n,x})|$$

By Lemma 4.7,

$$\sum_{j=0}^{h_n-1} |\lambda_B(T^{k_n\tilde{h}_n} I_{n,j})| \leq \int \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n-1} \chi_B \circ T^{-k_n\ell} \right| d\mu + \frac{k_n+1}{a_n} + \frac{k_nb_n}{h_n} \to 0$$

since $k_n < b_n$ so Proposition 4.11 implies the integral tends to zero. Similar reasoning for $k_n + 1 \leq b_n$ then completes the proof. □

Lemma 4.18. *Let T be a quasi-staircase transformation, B a union of levels in some C_N, $n > N$, $b_n \leq k < a_n$ and $0 \leq y < \tilde{h}_n$. Let $\epsilon > 0$ such that $\sup_{t \geq b_n} \left(\int \left| \frac{1}{t} \sum_{i=0}^{t-1} \chi_B \circ T^{-i} \right| \, d\mu + \frac{2}{t} \right) < \epsilon$. Then*

$$\sum_{j=a_nb_n+b_{n+1}+c_{n+1}-c_n}^{\tilde{h}_n-y} \left| \lambda_B(T^{k\tilde{h}_n+y} I_{n,j}) - \frac{a_n-k}{r_n+1} \sum_{\ell=0}^{b_n-1} \lambda_B(T^{y-k\ell} I_{n,j}) \right| < \frac{k}{a_n}\epsilon$$

$$\sum_{j=a_nb_n+b_{n+1}+c_{n+1}-c_n+\tilde{h}_n-y}^{\tilde{h}_n} \left| \lambda_B(T^{k\tilde{h}_n+y} I_{n,j}) - \frac{a_n-k-1}{r_n+1} \sum_{\ell=0}^{b_n-1} \lambda_B(T^{y-\tilde{h}_n-(k+1)\ell} I_{n,j}) \right| < \frac{k+1}{a_n}\epsilon$$

Proof. For $a_nb_n + b_{n+1} + c_{n+1} - c_n \leq j < \tilde{h}_n - y$, by Lemmas 4.13 and 4.14,

$$\begin{aligned}
\lambda_B(T^{k\tilde{h}_n+y} I_{n,j}) &= \sum_{i=0}^{a_n-1} \sum_{\ell=0}^{b_n-1} \lambda_B(T^{k\tilde{h}_n+y} I_{n,j}^{[\ell a_n+i]}) + \lambda_B(T^{k\tilde{h}_n+y} I_{n,j}^{[r_n]}) \\
&= \sum_{i=0}^{a_n-k-1} \sum_{\ell=0}^{b_n-1} \lambda_B(T^{-k\ell} I_{n,j+y}^{[\ell a_n+i+k]}) + \sum_{i=a_n-k}^{a_n-1} \sum_{\ell=0}^{b_n-2} \lambda_B(T^{-k\ell-(i+k-a_n)} I_{n,j+y}^{[\ell a_n+i+k+1]}) \\
&\quad + \sum_{i=0}^{k} \lambda_B(T^{k\tilde{h}_n+y} I_{n,j}^{[r_n-i]})
\end{aligned}$$

and since $k\ell \leq a_nb_n$ and $j + y \geq j \geq a_nb_n$, using Lemma 4.5,

$$\sum_{i=0}^{a_n-k-1} \sum_{\ell=0}^{b_n-1} \lambda_B(T^{-k\ell} I_{n,j+y}^{[\ell a_n+i+k]}) = \frac{1}{r_n+1} \sum_{i=0}^{a_n-k-1} \sum_{\ell=0}^{b_n-1} \lambda_B(T^{-k\ell} I_{n,j+y}) = \frac{a_n-k}{r_n+1} \sum_{\ell=0}^{b_n-1} \lambda_B(T^{y-k\ell} I_{n,j})$$

Likewise, since $k\ell + (i + k - a_n) \leq a_nb_n$,

$$\begin{aligned}
\left| \sum_{i=a_n-k}^{a_n-1} \sum_{\ell=0}^{b_n-2} \lambda_B(T^{-k\ell-(i+k-a_n)} I_{n,j+y}^{[\ell a_n+i+k+1]}) \right| &= \left| \frac{1}{r_n+1} \sum_{i=a_n-k}^{a_n-1} \sum_{\ell=0}^{b_n-2} \lambda_B(T^{-k\ell-(i+k-a_n)} I_{n,j+y}) \right| \\
&= \left| \frac{1}{r_n+1} \sum_{i=0}^{k-1} \sum_{\ell=0}^{b_n-2} \lambda_B(T^{y-k\ell-i} I_{n,j}) \right| \leq \frac{k}{r_n+1} \sum_{\ell=0}^{b_n-2} \int_{T^{y-k\ell} I_{n,j}} \left| \frac{1}{k} \sum_{i=0}^{k-1} \chi_B \circ T^{-i} \right| \, d\mu
\end{aligned}$$

and therefore

$$\sum_{j=a_nb_n+b_{n+1}+c_{n+1}-c_n}^{\tilde{h}_n-y} \left| \sum_{i=a_n-k}^{a_n-1} \sum_{\ell=0}^{b_n-2} \lambda_B(T^{-k\ell-(i+k-a_n)} I_{n,j+y}^{[\ell a_n+i+k+1]}) \right| < \frac{k(b_n-1)}{r_n+1} \int \left| \frac{1}{k} \sum_{i=0}^{k-1} \chi_B \circ T^{-i} \right| \, d\mu$$

For $0 \leq i \leq k-1$, using that $j \geq c_{n+1} - c_n + b_{n+1} + a_nb_n$ and that $I_{n,j}^{[0]} = I_{n+1,j}$,

$$\begin{aligned}
T^{k\tilde{h}_n+y} I_{n,j}^{[r_n-i]} &= T^{k\tilde{h}_n+y+h_{n+1}-h_n-i(\tilde{h}_n+b_n-1)} I_{n,j}^{[0]} \\
&= T^{\tilde{h}_{n+1}+(k-i-1)\tilde{h}_n+c_n-c_{n+1}-i(b_n-1)+y} I_{n,j}^{[0]} = T^{\tilde{h}_{n+1}} I_{n+1,j+(k-i-1)\tilde{h}_n+c_n-c_{n+1}-i(b_n-1)+y}
\end{aligned}$$

therefore, since $|\lambda_B(T^{\tilde{h}_{n+1}} I_{n+1,j'})| = |\sum_{t=0}^{b_{n+1}-1} (\sum_{i=0}^{a_{n+1}-2} \lambda_B(T^{-t} I_{n+1,j'}^{[ta_{n+1}+i+1]}) + \lambda_B(T^{-t-1} I_{n+1,j'}^{[(t+1)a_{n+1}]})) + \lambda_B(T^{\tilde{h}_{n+1}} I_{n+1,j'}^{[r_{n+1}]})| \leq \frac{a_{n+1}}{r_{n+1}+1} |\sum_{t=0}^{b_{n+1}-1} \lambda_B(T^{-t} I_{n+1,j'})| + \frac{2\mu(I_{n+1,j'})}{r_{n+1}+1}$ whenever $j' \geq b_{n+1}$,

$$\begin{aligned}
\left| \lambda_B(T^{k\tilde{h}_n+y} I_{n,j}^{[r_n-i]}) \right| &\leq \left| \frac{a_{n+1}}{r_{n+1}+1} \sum_{t=0}^{b_{n+1}-1} \lambda_B \left(T^{-t} I_{n+1,j+(k-i-1)\tilde{h}_n+c_n-c_{n+1}-i(b_n-1)+y} \right) \right| + \frac{2\mu(I_{n+1})}{r_{n+1}+1} \\
&= \left| \frac{a_{n+1}}{r_{n+1}+1} \sum_{t=0}^{b_{n+1}-1} \lambda_B \left(T^{-t} I_{n,j+c_n-c_{n+1}-i(b_n-1)+y}^{[k-i-1]} \right) \right| + \frac{2\mu(I_{n+1})}{r_{n+1}+1}
\end{aligned}$$

$$= \left| \frac{a_{n+1}}{r_{n+1}+1} \frac{1}{r_n+1} \sum_{t=0}^{b_{n+1}-1} \lambda_B \left(T^{-t} I_{n,j+c_n-c_{n+1}-i(b_n-1)+y}\right) \right| + \frac{2\mu(I_{n+1})}{r_{n+1}+1}$$

$$\leq \frac{a_{n+1}b_{n+1}}{(r_{n+1}+1)(r_n+1)} \int_{T^{y+c_n-c_{n+1}-i(b_n-1)}I_{n,j}} \left| \frac{1}{b_{n+1}} \sum_{t=0}^{b_{n+1}-1} \chi_B \circ T^{-t} \right| \, d\mu + \frac{2\mu(I_{n+1,j})}{r_{n+1}+1}$$

and so

$$\sum_{j=a_nb_n+b_{n+1}+c_{n+1}-c_n}^{\tilde{h}_n-y} \sum_{i=0}^{k} \left| \lambda_B(T^{k\tilde{h}_n+y} I_{n,j}^{[r_n-i]}) \right|$$

$$\leq \frac{k}{r_n+1} \int \left| \frac{1}{b_{n+1}} \sum_{t=0}^{b_{n+1}-1} \chi_B \circ T^{-t} \right| \, d\mu + \frac{1}{r_n+1} + \frac{2}{(r_{n+1}+1)(r_n+1)}$$

Therefore, since $\sup_{t \geq b_n} \left(\int \left| \frac{1}{t} \sum_{i=0}^{t-1} \chi_B \circ T^{-i} \right| \, d\mu + \frac{2}{t} \right) < \epsilon$,

$$\sum_{j=a_nb_n+b_{n+1}+c_{n+1}-c_n}^{\tilde{h}_n-y} \left| \lambda_B(T^{k\tilde{h}_n+y} I_{n,j}) - \frac{a_n-k}{r_n+1} \sum_{\ell=0}^{b_n-1} \lambda_B(T^{y-k\ell} I_{n,j}) \right| \leq \frac{kb_n}{r_n+1}\epsilon < \frac{k}{a_n}\epsilon$$

For $a_nb_n + c_{n+1} - c_n + \tilde{h}_n - y \leq j < \tilde{h}_n$,

$$T^{k\tilde{h}_n+y} I_{n,j} = T^{(k+1)\tilde{h}_n+0} I_{n,j-(\tilde{h}_n-y)}$$

and since $a_nb_n + b_{n+1} + c_{n+1} - c_n \leq j - (\tilde{h}_n - y) < \tilde{h}_n - 0$, the claim follows from the above replacing k by $k+1$, j by $j - (\tilde{h}_n - y)$ and y by 0. ☐

Proposition 4.19. *Let T be a quasi-staircase transformation such that $\frac{a_{n+1}b_{n+1}+c_{n+1}+a_nb_n^2}{h_n} \to 0$. Let B be a union of levels in some column C_N. For $n > N$, set*

$$\widetilde{M}_{B,n} = \max_{b_n \leq k < a_n} \max_{a_{n-1}\tilde{h}_{n-1} \leq y \leq \tilde{h}_n - a_{n-1}\tilde{h}_{n-1}} \sum_{j=0}^{\tilde{h}_{n-1}-1} |\lambda_B(T^{k\tilde{h}_n+y} I_{n-1,j})|$$

Then $\lim_{n\to\infty} \widetilde{M}_{B,n} = 0$.

Proof. Let $\epsilon > 0$ such that $\sup_{t \geq b_n} \left(\int \left| \frac{1}{t} \sum_{i=0}^{t-1} \chi_B \circ T^{-i} \right| \, d\mu + \frac{2}{t} \right) < \epsilon$. Write $y = xa_{n-1}\tilde{h}_{n-1} + z\tilde{h}_{n-1} + w$ for $1 \leq x \leq b_n$ and $0 \leq z < a_{n-1}$ and $0 \leq w < \tilde{h}_{n-1}$. Observe that if $0 \leq i < (b_{n-1} - x)a_{n-1}$ then $I_{n-1,j}^{[i]}$ is a level in C_n below $I_{n,\tilde{h}_n-y}$ and that if $(b_{n-1} - x)a_{n-1} < i \leq r_{n-1}$ then $I_{n-1,j}^{[i]}$ is a level in C_n above $I_{n,\tilde{h}_n-y}$. Then by Lemma 4.18, as $\frac{2k+1}{a_n}\epsilon \leq \frac{3k}{a_n}\epsilon$,

$$\sum_{j=0}^{\tilde{h}_{n-1}-1} \Bigg| \lambda_B(T^{k\tilde{h}_n+y} I_{n-1,j}) - \frac{a_n-k}{r_n+1} \sum_{\ell=0}^{b_n-1} \sum_{i=0}^{(b_{n-1}-x)a_{n-1}-1} \lambda_B(T^{y-k\ell} I_{n-1,j}^{[i]})$$
$$- \frac{a_n-k-1}{r_n+1} \sum_{\ell=0}^{b_n-1} \sum_{i=(b_{n-1}-x+1)a_{n-1}}^{r_{n-1}} \lambda_B(T^{y-\tilde{h}_n-(k+1)\ell} I_{n-1,j}^{[i]}) \Bigg| < \frac{3k}{a_n}\epsilon + \frac{a_n}{r_n+1} + \frac{4(a_nb_n+b_{n+1}+c_{n+1})}{\tilde{h}_n}$$

Now observe that, via Lemma 4.15, writing $k' = xa_{n-1} + z$,

$$\sum_{j=0}^{\tilde{h}_{n-1}-1} \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n-1} \sum_{i=0}^{(b_{n-1}-x)a_{n-1}-1} \lambda_B(T^{y-k\ell} I_{n-1,j}^{[i]}) \right| \leq \frac{1}{b_n} \sum_{\ell=0}^{b_n-1} \sum_{j=0}^{\tilde{h}_{n-1}-1} \left| \sum_{i=0}^{(b_{n-1}-x)a_{n-1}-1} \lambda_B(T^{y-k\ell} I_{n-1,j}^{[i]}) \right|$$

$$\leq \frac{c_{n-1}}{\tilde{h}_{n-1}} + \sum_{j=0}^{\tilde{h}_{n-1}-1} \left(\left| \sum_{i=0}^{(b_{n-1}-x)a_{n-1}-1} \lambda_B(T^{k'\tilde{h}_{n-1}} I_{n-1,j}^{[i]}) \right| + \left| \sum_{i=0}^{(b_{n-1}-x)a_{n-1}-1} \lambda_B(T^{(k'+1)\tilde{h}_{n-1}} I_{n-1,j}^{[i]}) \right| \right)$$

which are precisely the sums $(\star)$ in the proof Proposition 4.16 (since $x \geq 1$ so $k' \geq a_{n-1}$). Therefore

$$\sum_{j=0}^{\tilde{h}_{n-1}-1} \left| \frac{a_n - k}{r_n + 1} \sum_{\ell=0}^{b_n-1} \sum_{i=0}^{(b_{n-1}-x)a_{n-1}-1} \lambda_B(T^{y-k\ell} I_{n-1,j}^{[i]}) \right| \to 0$$

Now observe that for $0 \leq i < a_{n-1}$ and $0 \leq q < b_{n-1}$,

$$I_{n-1,j}^{[qa_{n-1}+i]} = T^{qa_{n-1}\tilde{h}_{n-1} + \frac{1}{2}a_{n-1}q(q-1) + i\tilde{h}_{n-1} + iq} I_{n-1,j}^{[0]}$$

so for $0 \leq i < a_{n-1} - 1$, as $(b_{n-1} - x + q)(b_{n-1} - x + q - 1) - q(q-1) = (b_{n-1} - x)(b_{n-1} - x - 1 + 2q)$,

$$I_{n-1,j}^{[(b_{n-1}-x+q)a_{n-1}+i+1]} = T^{(b_{n-1}-x)a_{n-1}\tilde{h}_{n-1} + \frac{1}{2}a_{n-1}(b_{n-1}-x)(b_{n-1}-x-1+2q) + \tilde{h}_{n-1} + q + (i+1)(b_{n-1}-x)} I_{n-1,j}^{[qa_{n-1}+i]}$$

Set $Q = Q_q = -c_n + c_{n-1} - (k+1)\ell + \frac{1}{2}a_{n-1}(b_{n-1}-x)(b_{n-1}-x-1+2q) + q - \frac{1}{2}a_{n-1}b_{n-1}(b_{n-1}-1) + b_{n-1} - x$ and note that $|Q| \leq c_n + a_n b_n + 2a_{n-1}b_{n-1}^2$. Then, since $b_{n-1}a_{n-1}\tilde{h}_{n-1} + \tilde{h}_{n-1} - \tilde{h}_n = -c_n + c_{n-1} - \frac{1}{2}a_{n-1}b_{n-1}(b_{n-1}-1)$,

$$T^{y-(k+1)\ell-\tilde{h}_n} I_{n-1,j}^{[(b_{n-1}-x+q)a_{n-1}+i]} = T^{z\tilde{h}_{n-1} + w + i(b_{n-1}-x) + Q} I_{n-1,j}^{[qa_{n-1}+i]}$$

Consider j such that $0 \leq j + Q - a_{n-1}b_{n-1} < \tilde{h}_{n-1} - w - a_{n-1}b_{n-1}$. If $z + i \geq a_{n-1}$,

$$\begin{aligned} T^{y-(k+1)\ell-\tilde{h}_n} I_{n-1,j}^{[(b_{n-1}-x+q)a_{n-1}+i]} &= T^{z\tilde{h}_{n-1}} I_{n-1,j+Q+w+i(b_{n-1}-x)}^{[qa_{n-1}+i]} = I_{n-1,j+Q+w+i(b_{n-1}-x)-zq-(z+i-a_{n-1})}^{[qa_{n-1}+i+z]} \\ &= T^{i(b_{n-1}-x-1)} I_{n-1,j+Q+w-zq-(z-a_{n-1})}^{[qa_{n-1}+i+z]} \end{aligned}$$

and therefore

$$\lambda_B(T^{y-(k+1)\ell-\tilde{h}_n} I_{n-1,j}^{[qa_{n-1}+i]}) = \frac{1}{r_{n-1}} \lambda_B\left(T^{i(b_{n-1}-x-1)} I_{n-1,j+Q+w-zq-(z-a_{n-1})}\right)$$

Similarly, if $z + i < a_{n-1}$,

$$\begin{aligned} T^{y-(k+1)\ell-\tilde{h}_n} I_{n-1,j}^{[(b_{n-1}-x+q)a_{n-1}+i]} &= T^{z\tilde{h}_{n-1}} I_{n-1,j+Q+w+i(b_{n-1}-x)}^{[qa_{n-1}+i]} = I_{n-1,j+Q+w+i(b_{n-1}-x)-zq}^{[qa_{n-1}+i+z]} \\ &= T^{i(b_{n-1}-x)} I_{n-1,j+Q+w-zq}^{[qa_{n-1}+i+z]} \end{aligned}$$

so

$$\lambda_B(T^{y-(k+1)\ell-\tilde{h}_n} I_{n-1,j}^{[qa_{n-1}+i]}) = \frac{1}{r_{n-1}} \lambda_B\left(T^{i(b_{n-1}-x)} I_{n-1,j+Q+w-zq-(z-a_{n-1})}\right)$$

Therefore, as $\frac{x}{r_{n-1}} \leq \frac{b_{n-1}}{r_{n-1}} < \frac{1}{a_{n-1}}$,

$$\begin{aligned} &\sum_{j=a_{n-1}b_{n-1}-Q}^{\tilde{h}_{n-1}-w-a_{n-1}b_{n-1}} \left| \sum_{i=(b_{n-1}-x+1)a_{n-1}}^{r_{n-1}} \lambda_B(T^{y-(k+1)\ell-\tilde{h}_n} I_{n-1,j}^{[i]}) \right| \\ &\leq \sum_{j=a_{n-1}b_{n-1}-Q}^{\tilde{h}_{n-1}-w-a_{n-1}b_{n-1}} \left| \sum_{q=b_{n-1}-x+1}^{b_{n-1}-1} \sum_{i=0}^{a_{n-1}-2} \lambda_B(T^{y-(k+1)\ell-\tilde{h}_n} I_{n-1,j}^{[qa_{n-1}+i]}) \right| + \frac{x+1}{r_{n-1}} \\ &\leq \frac{1}{r_{n-1}} \sum_{q=0}^{x-1} \int \left| \sum_{i=0}^{a_{n-1}-z-1} \chi_B \circ T^{i(b_{n-1}-x-1)} \right| d\mu + \frac{1}{r_{n-1}} \sum_{q=0}^{x-1} \int \left| \sum_{i=0}^{z-1} \chi_B \circ T^{i(b_{n-1}-x)} \right| d\mu + \frac{x+1}{r_{n-1}} \\ &\leq \int \left| \frac{1}{a_{n-1}} \sum_{i=0}^{a_{n-1}-z-1} \chi_B \circ T^{i(b_{n-1}-x-1)} \right| d\mu + \int \left| \frac{1}{a_{n-1}} \sum_{i=0}^{z-1} \chi_B \circ T^{i(b_{n-1}-x)} \right| d\mu + \frac{x+1}{r_{n-1}} \end{aligned}$$

Now consider j such that $\tilde{h}_{n-1} - w + a_{n-1}b_{n-1} - Q \leq j < \tilde{h}_{n-1} - a_{n-1}b_{n-1}$. Then

$$T^{y-(k+1)\ell-\tilde{h}_n} I_{n-1,j}^{[qa_{n-1}+i]} = T^{(z+1)\tilde{h}_{n-1}} I_{n-1,j+Q+w+i(b_{n-1}-x)-\tilde{h}_{n-1}}$$

so similar reasoning as above shows that

$$\sum_{j=\tilde{h}_{n-1}-w+a_{n-1}b_{n-1}-Q}^{\tilde{h}_{n-1}-a_{n-1}b_{n-1}} \left| \sum_{i=(b_{n-1}-x+1)a_{n-1}}^{r_{n-1}} \lambda_B(T^{y-(k+1)\ell-\tilde{h}_n} I_{n-1,j}^{[i]}) \right|$$
$$\leq \int \left| \frac{1}{a_{n-1}} \sum_{i=0}^{a_{n-1}-z-1} \chi_B \circ T^{i(b_{n-1}-x-1)} \right| d\mu + \int \left| \frac{1}{a_{n-1}} \sum_{i=0}^{z-1} \chi_B \circ T^{i(b_{n-1}-x)} \right| d\mu + \frac{x+1}{r_{n-1}}$$

Note that $y < \tilde{h}_n - a_{n-1}\tilde{h}_{n-1} = (b_{n-1}-1)a_{n-1}\tilde{h}_{n-1} + h_{n-1} + \frac{1}{2}a_{n-1}b_{n-1}(b_{n-1}-1) + c_n < (b_{n-1}-1)a_{n-1}\tilde{h}_{n-1} + 2\tilde{h}_{n-1}$. Therefore $x \leq b_{n-1}-1$ and if $x = b_{n-1}-1$ then $z \leq 1$. When $x \leq b_{n-1}-1$, both $b_{n-1}-x \geq 1$ and $b_{n-1}-x-1 \geq 1$ so both integrals tend to zero by Proposition 4.11. When $x = b_{n-1}-1$, the first integral tends to zero by Proposition 4.11 and the second is bounded by $\frac{z}{a_{n-1}} \to 0$.

Since $\frac{|Q|}{\tilde{h}_{n-1}} \leq \frac{c_n + a_nb_n + a_{n-1}b_{n-1}^2}{\tilde{h_{n-1}}} \to 0$, then $\sum_{j=0}^{\tilde{h}_{n-1}} \left| \sum_{i=(b_{n-1}-x+1)a_{n-1}}^{r_{n-1}} \lambda_B(T^{y-(k+1)\ell-\tilde{h}_n} I_{n-1,j}^{[i]}) \right| \to 0$. □

Notation 4.20. *Define* $\tau_n = \frac{4(a_nb_n + b_{n+1} + c_{n+1})}{\tilde{h}_n}$.

Lemma 4.21. *Let T be a quasi-staircase transformation, B a union of levels in some C_N, $\epsilon > 0$ such that* $\sup_{t \geq b_N} \left(\int \left| \frac{1}{t} \sum_{i=0}^{t-1} \chi_B \circ T^{-i} \right| d\mu + \frac{2}{t} \right) < \frac{\epsilon}{3}$, *$n > N$, $b_n \leq k < a_n$ and $0 \leq |y| < a_{n-1}\tilde{h}_{n-1}$. Then*

$$\left| \lambda_B(T^{k\tilde{h}_n+y}B) - \frac{a_n-k}{r_n+1} \sum_{\ell=0}^{b_n-1} \lambda_B(T^{y-k\ell}B) \right| \leq \frac{k}{a_n}\epsilon + \tau_n + \left(1 - \frac{k}{a_n}\right) \frac{1}{b_n} \sum_{\ell=0}^{b_n-1} \frac{2|y-k\ell|}{\tilde{h}_n}$$

Proof. Consider first when $y \geq 0$. Write $\beta = \{a_nb_n + b_{n+1} + c_{n+1} \leq j < \tilde{h}_n - y : I_{n,j} \subseteq B\}$ and $\beta' = \{a_nb_n + b_{n+1} + c_{n+1} + \tilde{h}_n - y \leq j < \tilde{h}_n : I_{n,j} \subseteq B\}$. By Lemma 4.18,

$$\left| \sum_{j \in \beta \cup \beta'} \lambda_B(T^{k\tilde{h}_n+y} I_{n,j}) - \sum_{\ell=0}^{b_n-1} \left(\frac{a_n-k}{r_n+1} \sum_{j\in\beta} \lambda_B(T^{y-k\ell} I_{n,j}) - \frac{a_n-k-1}{r_n+1} \sum_{j\in\beta'} \lambda_B(T^{y-\tilde{h}_n-(k+1)\ell} I_{n,j}) \right) \right|$$

is bounded by $\frac{k}{a_n}\frac{\epsilon}{3} + \frac{k+1}{a_n}\frac{\epsilon}{3} \leq \frac{k\epsilon}{a_n}$ and therefore

$$\left| \lambda_B(T^{k\tilde{h}_n+y}B) - \frac{a_n-k}{r_n+1} \sum_{\ell=0}^{b_n-1} \sum_{j\in\beta\cup\beta'} \lambda_B(T^{y-k\ell} I_{n,j}) \right|$$
$$\leq \frac{k}{a_n}\epsilon + \frac{\tau_n}{2} + \frac{a_n-k}{r_n+1} \sum_{\ell=0}^{b_n-1} \sum_{j\in\beta'} \left| \lambda_B(T^{y-k\ell} I_{n,j}) - \frac{a_n-k-1}{a_n-k} \lambda_B(T^{y-\tilde{h}_n-(k+1)\ell} I_{n,j}) \right|$$
$$\leq \frac{k}{a_n}\epsilon + \frac{\tau_n}{2} + \frac{a_n-k}{r_n+1} b_n|\beta'|\mu(I_n) \frac{2a_n-2k-1}{a_n-k} < \frac{k}{a_n}\epsilon + \frac{\tau_n}{2} + \left(1 - \frac{k}{a_n}\right) \frac{2|\beta'|}{\tilde{h}_n}$$

so the claim follows for $y \geq 0$ as $|\beta'| = y - a_nb_n - c_{n+1} \leq |y - k\ell|$ for all $0 \leq \ell < b_n$ (and if $y < a_nb_n + b_{n+1} + c_{n+1}$ then $\beta' = \emptyset$) and since $|\lambda_B(T^{k\tilde{h}_n+y}B) - \sum_{j\in\beta\cup\beta'} \lambda_B(T^{k\tilde{h}_n+y} I_{n,j})| \leq \frac{\tau_n}{2}$.

Now consider when $y < 0$. Then $k\tilde{h}_n + y = (k-1)\tilde{h}_n + (\tilde{h}_n + y)$ so, following the same reasoning as above and swapping the roles of β' and β,

$$\left| \lambda_B(T^{k\tilde{h}_n+y}B) - \frac{a_n-(k-1)-1}{r_n+1} \sum_{\ell=0}^{b_n-1} \sum_{j\in\beta\cup\beta'} \lambda_B(T^{(y+\tilde{h}_n)-(k-1+1)\ell} I_{n,j}) \right|$$

$$< \frac{k-1+1}{a_n}\epsilon + \frac{\tau_n}{2} + \left(1 - \frac{k-1+1}{a_n}\right)\frac{2|\beta|}{\tilde{h}_n}$$

so the claim follows as in this case $|\beta| \leq |y - k\ell|$. □

Lemma 4.22. *Let $\epsilon > 0$ and $q, k, p, Q, L \in \mathbb{N}$ and for all $0 \leq \ell < L$, let $0 \leq \delta_\ell \leq 1$. If $\frac{pQ}{L} < \epsilon$ and $\frac{1}{Q} < \epsilon$ and $|\lambda_B(T^{kpt}B)| < \epsilon$ for all $1 \leq t < Q$ then*

$$\left| \frac{1}{L} \sum_{\ell=0}^{L-1} \delta_\ell \lambda_B(T^{q-k\ell}B) \right| < (2\epsilon)^{1/2} + \epsilon$$

Proof. Using that T is measure-preserving and the Cauchy-Schwarz inequality,

$$\left| \frac{1}{L} \sum_{\ell=0}^{L-1} \delta_\ell \lambda_B(T^{q-k\ell}B) \right| = \left| \frac{1}{L} \sum_{\ell=0}^{L-1} \int_B \delta_\ell \chi_B \circ T^{q-k\ell} \, d\mu \right| \leq \int \left| \frac{1}{L} \sum_{\ell=0}^{L-1} \delta_\ell \chi_B \circ T^{q-k\ell} \right| \, d\mu$$

$$\leq \frac{pQ\lfloor\frac{L}{pQ}\rfloor}{L} \frac{1}{\lfloor\frac{L}{pQ}\rfloor} \sum_{j=0}^{\lfloor\frac{L}{pQ}\rfloor-1} \frac{1}{p}\sum_{i=0}^{p-1} \int \left| \frac{1}{Q} \sum_{t=0}^{Q-1} \delta_{jpQ+i+pt}\chi_B \circ T^{-kpt} \right| \circ T^{q-kjpQ-ki} \, d\mu + \frac{pQ}{L}$$

$$< \frac{1}{\lfloor\frac{L}{pQ}\rfloor} \sum_{j=0}^{\lfloor\frac{L}{pQ}\rfloor-1} \frac{1}{p}\sum_{i=0}^{p-1} \int \left| \frac{1}{Q} \sum_{t=0}^{Q-1} \delta_{jpQ+i+pt}\chi_B \circ T^{-kpt} \right| \, d\mu + \epsilon$$

$$\leq \frac{1}{\lfloor\frac{L}{pQ}\rfloor} \sum_{j=0}^{\lfloor\frac{L}{pQ}\rfloor-1} \frac{1}{p}\sum_{i=0}^{p-1} \left(\int \left| \frac{1}{Q} \sum_{t=0}^{Q-1} \delta_{jpQ+i+pt}\chi_B \circ T^{-kpt} \right|^2 \, d\mu \right)^{1/2} + \epsilon$$

$$= \frac{1}{\lfloor\frac{L}{pQ}\rfloor} \sum_{j=0}^{\lfloor\frac{L}{pQ}\rfloor-1} \frac{1}{p}\sum_{i=0}^{p-1} \left(\frac{1}{Q^2} \sum_{t,u=0}^{Q-1} \delta_{jpQ+i+pt}\delta_{jpQ+i+pu}\lambda_B(T^{kp(t-u)}B) \right)^{1/2} + \epsilon$$

$$= \frac{1}{\lfloor\frac{L}{pQ}\rfloor} \sum_{j=0}^{\lfloor\frac{L}{pQ}\rfloor-1} \frac{1}{p}\sum_{i=0}^{p-1} \left(\frac{1}{Q^2} \sum_{t=0}^{Q-1} \delta_{jpQ+i+pt}^2\lambda_B(B) + \frac{1}{Q^2}\sum_{t\neq u} \delta_{jpQ+i+pt}\delta_{jpQ+i+pu}\lambda_B(T^{kp(t-u)}B) \right)^{1/2} + \epsilon$$

$$< \frac{1}{\lfloor\frac{L}{pQ}\rfloor} \sum_{j=0}^{\lfloor\frac{L}{pQ}\rfloor-1} \frac{1}{p}\sum_{i=0}^{p-1} \left(\frac{1}{Q} + \frac{1}{Q^2}\sum_{t\neq u} \delta_{jpQ+i+pt}\delta_{jpQ+i+pu}\epsilon \right)^{1/2} + \epsilon \leq \left(\frac{1}{Q} + \frac{1}{Q^2}\sum_{t\neq u}\epsilon \right)^{1/2} + \epsilon$$ □

Proposition 4.23. *Let T be a quasi-staircase transformation such that $\frac{b_n^2}{\tilde{h}_n} \to 0$, $\frac{a_n b_n}{\tilde{h}_n} \to 0$ and $\frac{b_n}{a_n} \to 0$. Let B be a union of levels in some column C_{N_0}. Then*

$$\lim_{N\to\infty} \max_{0\leq\delta_\ell\leq 1} \max_{1\leq k\leq N} \int \left| \frac{1}{N} \sum_{\ell=0}^{N-1} \delta_\ell \chi_B \circ T^{-\ell k} \right| \, d\mu = 0$$

Proof. Fix $\epsilon > 0$. Let m such that $b_m \geq 2\lceil\epsilon^{-1}\rceil$, $\frac{4(r_m+1)\lceil\epsilon^{-1}\rceil^2}{\tilde{h}_m} < \epsilon$ and $\sup_{n\geq m} \widehat{M}_{B,n} < \epsilon$ (using Proposition 4.17). Take any N such that $\frac{\tilde{h}_m\lceil\epsilon^{-1}\rceil}{N} < \epsilon$. Let k and δ_ℓ attain the maximum for N.

Consider first the case when $k \geq \tilde{h}_m$. Let $n \geq m$ such that $\tilde{h}_n \leq k < \tilde{h}_{n+1}$. Let p such that $(p-1)k < \tilde{h}_{n+1} \leq pk$ so that $pk < \tilde{h}_{n+1} + k < 2\tilde{h}_{n+1}$. Then for every $1 \leq q < \lceil\epsilon^{-1}\rceil$, $\tilde{h}_{n+1} \leq qpk < \lceil\epsilon^{-1}\rceil 2\tilde{h}_{n+1} \leq b_n\tilde{h}_n$ meaning that $|\lambda_B(T^{qpk}B)| \leq \widehat{M}_{B,n} < \epsilon$. Now

$$\frac{p\lceil\epsilon^{-1}\rceil}{N} = \frac{pk\lceil\epsilon^{-1}\rceil}{Nk} < \frac{2\tilde{h}_{n+1}\lceil\epsilon^{-1}\rceil}{N\tilde{h}_n} < \frac{4(r_n+1)\lceil\epsilon^{-1}\rceil}{N} \leq \frac{4(r_n+1)\lceil\epsilon^{-1}\rceil}{k} \leq \frac{4(r_n+1)\lceil\epsilon^{-1}\rceil}{\tilde{h}_n} < \epsilon$$

so Lemma 4.22 implies that $\int \left| \frac{1}{N} \sum_{\ell=0}^{N-1} \delta_\ell \chi_B \circ T^{-\ell k} \right| \, d\mu < (2\epsilon)^{1/2} + \epsilon$.

Consider now when $k < \tilde{h}_m$. Let p such that $(p-1)k < \tilde{h}_m \leq pk$ so that $pk < 2\tilde{h}_m$ and $p \leq \tilde{h}_m$. Then $\tilde{h}_m \leq qpk < \lceil \epsilon^{-1} \rceil 2\tilde{h}_m \leq b_m \tilde{h}_m$ for $1 \leq q < \lceil \epsilon^{-1} \rceil$ so $|\lambda_B(T^{qpk}B)| \leq \widehat{M}_{B,m} < \epsilon$. Since $\frac{p\lceil \epsilon^{-1} \rceil}{N} < \frac{\tilde{h}_m \lceil \epsilon^{-1} \rceil}{N} < \epsilon$, Lemma 4.22 again implies that $\int \left| \frac{1}{N} \sum_{\ell=0}^{N-1} \delta_\ell \chi_B \circ T^{-\ell k} \right| \, d\mu < (2\epsilon)^{1/2} + \epsilon$. ☐

Notation 4.24. *For $t \in \mathbb{Z}$, write $\alpha(t)$ for the unique positive integer such that $\tilde{h}_{\alpha(t)} \leq |t| < \tilde{h}_{\alpha(t)+1}$.*

For $\ell, q, k \in \mathbb{Z}$ and $n > 0$ such that $|q - \ell k| < \tilde{h}_{n+1}$, let d be the unique integer such that $|(q-\ell k) - d\tilde{h}_n| \leq \frac{1}{2}\tilde{h}_n$ and define

$$\gamma_\ell^{n,q,k} = \begin{cases} \frac{a_n - |d|}{a_n} & \text{if } (b_n \leq |d| < a_n \text{ or } d = 0) \text{ and } |(q-\ell k) - d\tilde{h}_n| < a_{n-1}\tilde{h}_{n-1} \\ 0 & \text{otherwise} \end{cases}$$

Lemma 4.25. *Let $\epsilon > 0$ and set $\epsilon_0 = (2\lceil \epsilon^{-1} \rceil^{\lceil \epsilon^{-1} \rceil + 1})^{-1}$. Let $L, k, q \in \mathbb{Z}$ with $L \geq \epsilon_0^{-1}$ and for each $0 \leq \ell < L$, let $0 \leq \delta_\ell \leq 1$.*

Let $\alpha_0 = \max\{\alpha(q - \ell k) : 0 \leq \ell < L\}$. Assume that $\max(M_{B,\alpha_0}, M_{B,\alpha_0 - 1}, \widehat{M}_{B,\alpha_0}, \widetilde{M}_{B,\alpha_0}) < \epsilon$ and $b_{\alpha_0 - 1} > 4\epsilon^{-1}\epsilon_0^{-1}$ and $\sup_{m \geq b_{\alpha_0-1}} \left(\int \left| \frac{1}{m} \sum_{i=0}^{m-1} \chi_B \circ T^{-i} \right| d\mu + \frac{2}{m} \right) < \frac{\epsilon}{3}$.

Then either $\left| \frac{1}{L} \sum_{\ell=0}^{L-1} \delta_\ell \lambda_B(T^{q-\ell k}B) \right| + \frac{1}{L} \sum_{\ell=0}^{L-1} (1-\delta_\ell)\epsilon < 6\epsilon^{1/2}$ or there exists integers $t > 0$, $0 < L' < L/t$, $q_{\ell'}, k'_{\ell'} \in \mathbb{Z}$ such that $\alpha_{\ell'} = \max\{\alpha(q_{\ell'} - k'_{\ell'}) : 0 \leq \ell < L'\} < \alpha_0$ and

$$\begin{aligned} &\left| \frac{1}{L} \sum_{\ell=0}^{L-1} \delta_\ell \lambda_B(T^{q-\ell k}B) \right| + \frac{1}{L} \sum_{\ell=0}^{L-1} (1-\delta_\ell)\epsilon \\ &\quad < \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0}-1} \left(\left| \frac{1}{L} \sum_{\ell=0}^{L-1} \mathbb{1}_{\ell < L'} \delta_{\ell_0 + \ell t} \gamma_{\ell_0 + t\ell}^{\alpha_0,q,k} \lambda_B(T^{q_{\ell'} - k'_{\ell'} \ell}B) \right| + \frac{1}{L} \sum_{\ell=0}^{L-1} \left(1 - \mathbb{1}_{\ell < L'} \delta_{\ell_0 + \ell t} \gamma_{\ell_0 + t\ell}^{\alpha_0,q,k} \right) \epsilon \right) \\ &\qquad + \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0}-1} \frac{1}{L} \sum_{\ell=0}^{L-1} \mathbb{1}_{\ell < L'} \delta_{\ell_0 + t\ell} \gamma_{\ell_0 + t\ell}^{\alpha_0,q,k} \left(1 - \gamma_\ell^{\alpha_{\ell'}, q_{\ell'}, k'_{\ell'}} \right) \epsilon + \tau_{\alpha_0} \end{aligned}$$

Proof. Write $q - \ell k = k_\ell \tilde{h}_{\alpha_0} + y_\ell$ with $|y_\ell| \leq \frac{1}{2}\tilde{h}_{\alpha_0}$. Define

$$\mathcal{L} = \left\{ 0 \leq \ell < L : (k_\ell = 0 \text{ or } b_{\alpha_0} \leq |k_\ell| < a_{\alpha_0}) \text{ and } |y_\ell| < a_{\alpha_0 - 1}\tilde{h}_{\alpha_0 - 1} \right\}$$

Since $\lambda_B(T^{-t}B) = \lambda_B(T^t B)$, then for $\ell \notin \mathcal{L}$, $|\lambda_B(T^{q-\ell k}B)| < \epsilon$ as it is bounded by one of M_{B,α_0}, M_{B,α_0-1}, $\widehat{M}_{B,\alpha_0}$ or $\widetilde{M}_{B,\alpha_0}$. Write $k = z\tilde{h}_{\alpha_0} + y$ for $|y| \leq \frac{1}{2}\tilde{h}_{\alpha_0}$ and $q = x\tilde{h}_{\alpha_0} + r$ for $|r| \leq \frac{1}{2}\tilde{h}_{\alpha_0}$.

Claim: Either $\left| \frac{1}{L} \sum_{\ell=0}^{L-1} \lambda_B(T^{q-\ell k}B) \right| + \frac{1}{L} \sum_{\ell=0}^{L-1} (1-\delta_\ell)\epsilon < 6\epsilon^{1/2}$ or there exists $p \in \mathbb{Z}$, $t > 0$ and $0 \leq \ell_0 < L' \leq L$ such that $\mathcal{L} \subseteq \{\ell_0 + it : 0 \leq i < L'\}$ and $|ity - ip\tilde{h}_{\alpha_0}| < \frac{1}{3}\tilde{h}_{\alpha_0}$ for all $0 \leq i < L'$.

Proof. Let $p \in \mathbb{Z}$ and $0 < t < b_{\alpha_0 - 1}L$ such that $\left| \frac{y}{\tilde{h}_{\alpha_0}} - \frac{p}{t} \right| < \frac{1}{Lb_{\alpha_0-1}}$ and either $(p = 0, t = 1)$ or p, t are relatively prime. Let $u \in \mathbb{Z}$ such that $\left| \frac{r}{\tilde{h}_{\alpha_0}} - \frac{u}{t} \right| \leq \frac{1}{2t}$.

For $\ell \in \mathcal{L}$, there exists $n \in \mathbb{Z}$ such that $|r - \ell y - n\tilde{h}_{\alpha_0}| < a_{\alpha_0-1}\tilde{h}_{\alpha_0-1}$ so $\left| \frac{r - \ell y}{\tilde{h}_{\alpha_0}} - n \right| < \frac{1}{b_{\alpha_0-1}}$. Then $\left| \frac{u - \ell p}{t} - n \right| < \frac{1}{2t} + \frac{\ell}{Lb_{\alpha_0-1}} + \frac{1}{b_{\alpha_0-1}} < \frac{1}{2t} + \frac{2}{b_{\alpha_0-1}}$.

Case: $|p| < \frac{2t}{b_{\alpha_0-1}\epsilon_0 L}$

For $0 \leq \ell < L$, then $|\ell p| < \frac{2t}{b_{\alpha_0-1}\epsilon_0} < \frac{2t}{4\epsilon^{-1}\epsilon_0^{-1}\epsilon_0} = 2\epsilon t$ so $\left| \frac{\ell y}{\tilde{h}_n} \right| \leq \frac{\ell}{b_{\alpha_0-1}L} + \left| \frac{\ell p}{t} \right| < \frac{1}{b_{\alpha_0-1}} + 2\epsilon$ meaning $\mathcal{L}$ is consecutive. Also $|\ell \cdot 1 \cdot y - \ell \cdot 0 \cdot \tilde{h}_{\alpha_0}| = |\ell y| = \tilde{h}_{\alpha_0} \left| \frac{\ell y}{\tilde{h}_{\alpha_0}} \right| < \tilde{h}_{\alpha_0} \left(\frac{1}{b_{\alpha_0-1}} + 2\epsilon \right) < \frac{1}{3}\tilde{h}_{\alpha_0}$. This proves the claim replacing t with 1 and p with 0.

Case: $t \leq \frac{b_{\alpha_0-1}}{4}$ and $|p| > 0$

For $\ell \in \mathcal{L}$, $\left|\frac{u-\ell p}{t} - n\right| < \frac{2}{b_{\alpha_0-1}} + \frac{1}{2t} \leq \frac{1}{2t} + \frac{1}{2t} = \frac{1}{t}$ so $u - \ell p \pmod{t} = 0$. Let ℓ_0 be the minimal element of $\mathcal{L}$. As p, t are relatively prime, every $\ell \in \mathcal{L}$ is of the form $\ell = \ell_0 + ti$. Also $|ty - p\tilde{h}_{\alpha_0}| < \frac{t}{Lb_{\alpha_0-1}}\tilde{h}_{\alpha_0} \leq \frac{1}{4L}\tilde{h}_{\alpha_0}$.

Case: $\frac{b_{\alpha_0-1}}{4} < t \leq L$ and $|p| > 0$

For $\ell \in \mathcal{L}$, $\left|\frac{u-\ell p}{t} - n\right| < \frac{2}{b_{\alpha_0-1}} + \frac{1}{2t} < \frac{4}{b_{\alpha_0-1}}$. Since p and t are relatively prime (so $\ell \mapsto \frac{\ell p}{t}$ is cyclic and onto), at most $\frac{8}{b_{\alpha_0-1}}t\lceil\frac{L}{t}\rceil$ values of $0 \leq \ell < L$ have the property that $u - \ell p \pmod{t} < \frac{4}{b_{\alpha_0-1}}t$ or $> t - \frac{4}{b_{\alpha_0-1}}t$. Then $|\mathcal{L}| \leq \frac{8}{b_{\alpha_0-1}}t\lceil\frac{L}{t}\rceil < \frac{8}{b_{\alpha_0-1}}(L+t) < \frac{16}{b_{\alpha_0-1}}L < 4\epsilon\epsilon_0 L$. Therefore $\left|\frac{1}{L}\sum_{\ell=0}^{L-1}\lambda_B(T^{q-\ell k}B)\right| + \frac{1}{L}\sum_{\ell=0}^{L-1}(1-\delta_\ell)\epsilon < \frac{1}{L}|\mathcal{L}| + \epsilon < 4\epsilon\epsilon_0 + \epsilon$.

Case: $L < t$ and $|p| \geq \frac{2t}{b_{\alpha_0-1}\epsilon_0 L}$

Set $p_0 = p \pmod{t}$ and $p_{j+1} = \lceil\frac{t}{p_j}\rceil p_j \pmod{t}$. Suppose that $\epsilon t \leq p_j$ and $p_{j+1} \leq p_j - \epsilon t$ for $0 \leq j < \lceil\epsilon^{-1}\rceil$. Since $p_j \geq \epsilon t$, $\lceil\frac{t}{p_j}\rceil < \lceil\epsilon^{-1}\rceil$ so $p_j = m_j p \pmod{t}$ for some $m_j < \lceil\epsilon^{-1}\rceil^{\lceil\epsilon^{-1}\rceil-1} < L < t$. As p and t are relatively prime, then $m_j p \pmod{t} \neq 0$ so $p_j > 0$.

Since $p_{j+1} \leq p_j - \epsilon t$, then $0 < p_{\lceil\epsilon^{-1}\rceil} \leq p - \lceil\epsilon^{-1}\rceil\epsilon t < p - t < 0$ is a contradiction. So there exists $0 \leq m < \lceil\epsilon^{-1}\rceil$ such that $0 < p_m$ and either $p_m < \epsilon t$ or $p_m > p_{m-1} - \epsilon t$.

Subcase: $\frac{2t}{b_{\alpha_0-1}} \leq p_m < \epsilon t$

For $1 \leq i < \lceil(2\epsilon)^{-1}\rceil$, then $\frac{2t}{b_{\alpha_0-1}} \leq ip_m < \lceil(2\epsilon)^{-1}\rceil\epsilon t < \frac{1}{2}t + \epsilon t$. So, writing $g = \lceil\frac{t}{p_{m-1}}\rceil\cdots\lceil\frac{t}{p_0}\rceil < \lceil\epsilon^{-1}\rceil^{\lceil\epsilon^{-1}\rceil-1}$, we have $igp \pmod{t} \in [\frac{2}{b_{\alpha_0-1}}t, (\frac{1}{2}+\epsilon)t)$. Then $\left|\frac{igy}{\tilde{h}_{\alpha_0}} - \frac{igp}{t}\right| < \frac{ig}{Lb_{\alpha_0-1}} < \frac{2\lceil\epsilon^{-1}\rceil^{\lceil\epsilon^{-1}\rceil}}{Lb_{\alpha_0-1}} < \frac{1}{b_{\alpha_0-1}}$. So $\left|\frac{igy}{\tilde{h}_{\alpha_0}}\right| > \frac{1}{b_{\alpha_0-1}}$ and $< \frac{1}{2} + \epsilon + \frac{1}{b_{\alpha_0-1}}$. Then $|igy| > a_{\alpha_0-1}h_{\tilde{\alpha_0-1}}$ and $< \tilde{h}_{\alpha_0} - a_{\alpha_0-1}h_{\tilde{\alpha_0-1}}$. Since $igk = igz\tilde{h}_{\alpha_0} + igy$, then $|\lambda_B(T^{igk}B)| \leq \widetilde{M}_{B,\alpha_0} < \epsilon$ (or $< M_{B,\alpha_0-1}$ if $z = 0$). Since $\frac{g\lceil(2\epsilon)^{-1}\rceil}{L} < \frac{2\lceil\epsilon^{-1}\rceil^{\lceil\epsilon^{-1}\rceil}}{L} < \lceil\epsilon^{-1}\rceil^{-1} < \epsilon$, by Lemma 4.22 then $\left|\frac{1}{L}\sum_{\ell=0}^{L-1}\lambda_B(T^{q-\ell k}B)\right| + \frac{1}{L}\sum_{\ell=0}^{L-1}(1-\delta_\ell)\epsilon < (2\epsilon)^{1/2} + \epsilon + \epsilon < 4\epsilon^{1/2}$.

Subcase: $\frac{2t}{b_{\alpha_0-1}\epsilon_0 L} \leq p_m < \frac{2t}{b_{\alpha_0-1}}$

Let $g \in \mathbb{N}$ minimal such that $gp_m \geq \frac{2t}{b_{\alpha_0-1}}$. Then $gp_m < \frac{4t}{b_{\alpha_0-1}}$ so $g < 2\epsilon_0 L$. For $1 \leq i < \lceil\epsilon^{-1}\rceil$, then $\frac{2t}{b_{\alpha_0-1}} \leq igp_m < 4\lceil\epsilon^{-1}\rceil\frac{t}{b_{\alpha_0-1}} < \frac{1}{2}t$. Then $zg\lceil\frac{t}{p_{m-1}}\rceil\cdots\lceil\frac{t}{p_0}\rceil p \pmod{t} \in [\frac{2t}{b_{\alpha_0-1}}, \frac{t}{2})$ so, since $\lceil\epsilon^{-1}\rceil g\lceil\frac{t}{p_{m-1}}\rceil\cdots\lceil\frac{t}{p_0}\rceil < \lceil\epsilon^{-1}\rceil^{\lceil\epsilon^{-1}\rceil}\epsilon_0 L < \epsilon L$, as above Lemma 4.22 implies $\left|\frac{1}{L}\sum_{\ell=0}^{L-1}\lambda_B(T^{q-\ell k}B)\right| + \frac{1}{L}\sum_{\ell=0}^{L-1}(1-\delta_\ell)\epsilon < 4\epsilon^{1/2}$.

Subcase: $0 < p_m < \frac{2t}{b_{\alpha_0-1}\epsilon_0 L}$ and $|p| \geq \frac{2t}{b_{\alpha_0-1}\epsilon_0 L}$ and $L < t < \epsilon_0 b_{\alpha_0-1}L$

Set $g = \lceil\frac{2t}{b_{\alpha_0-1}p_m}\rceil$ so $g < \frac{2t}{b_{\alpha_0-1}} + 1 < \frac{2\epsilon_0 b_{\alpha_0-1}L}{b_{\alpha_0}} + 1 = 2\epsilon_0 L + 1$. For $1 \leq i < \lceil\epsilon^{-1}\rceil$, then $\frac{2t}{b_{\alpha_0-1}} \leq zgp_m < \lceil\epsilon^{-1}\rceil(2\epsilon_0 L + 1)\frac{2t}{b_{\alpha_0-1}\epsilon_0 L} < \frac{t}{2}$. Since $g\lceil\frac{t}{p_{m-1}}\rceil\cdots\lceil\frac{t}{p_0}\rceil < (2\epsilon_0 L + 1)\lceil\epsilon^{-1}\rceil^{\lceil\epsilon^{-1}\rceil-1} < \lceil\epsilon^{-1}\rceil^{-2}L$, then, as above, we can apply Lemma 4.22 to obtain $\left|\frac{1}{L}\sum_{\ell=0}^{L-1}\lambda_B(T^{q-\ell k}B)\right| + \frac{1}{L}\sum_{\ell=0}^{L-1}(1-\delta_\ell)\epsilon < 4\epsilon^{1/2}$.

Subcase: $0 < p_m < \frac{2t}{b_{\alpha_0-1}\epsilon_0 L}$ and $|p| \geq \frac{2t}{b_{\alpha_0-1}\epsilon_0 L}$ and $t \geq \epsilon_0 b_{\alpha_0-1}L$

Set $g = \lceil\frac{t}{p_{m-1}}\rceil\cdots\lceil\frac{t}{p_0}\rceil < \lceil\epsilon^{-1}\rceil^{\lceil\epsilon^{-1}\rceil-1}$. Since $|p| \geq \frac{2t}{b_{\alpha_0-1}\epsilon_0 L}$, $m > 0$ and therefore there exists an integer $v \neq 0$ such that $vt \leq gp < vt + \frac{2t}{b_{\alpha_0-1}\epsilon_0 L}$ and we may assume v and g are relatively prime. For $\ell \in \mathcal{L}$, there exists n such that $|u - \ell p - nt| < \frac{1}{2} + \frac{2t}{b_{\alpha_0-1}}$. Therefore $|nvt - ngp| < \frac{2|n|t}{b_{\alpha_0-1}\epsilon_0 L}$ and $|nvt - v(u-\ell p)| < \frac{|v|}{2} + \frac{2t|v|}{b_{\alpha_0-1}}$.

Since $|n| \leq \frac{L|p|}{t}$ and $|v| \leq \frac{g|p|}{t}$ and $t \leq b_{\alpha_0-1}L$ and $L \leq \frac{t}{\epsilon_0 b_{\alpha_0-1}}$,

$$|ngp - v(u-\ell p)| < \frac{2|n|t}{b_{\alpha_0-1}\epsilon_0 L} + \frac{|v|}{2} + \frac{2t|v|}{b_{\alpha_0-1}} < \frac{2|p|}{b_{\alpha_0-1}\epsilon_0} + \frac{g|p|}{2t} + \frac{2g|p|}{b_{\alpha_0-1}}$$

$$< \left(\frac{2}{b_{\alpha_0-1}\epsilon_0} + \frac{g}{2\epsilon_0 b_{\alpha_0-1}L} + \frac{2g}{b_{\alpha_0-1}} \right) |p| < \frac{1}{2}|p|$$

as $g < \epsilon^2\epsilon_0 < \epsilon^2 b_{\alpha_0-1}$. Write $vu = cp + d$ for $c \in \mathbb{Z}$ and $|d| \leq \frac{|p|}{2}$. Then $|ngp - cp - d + v\ell p| < \frac{|p|}{2}$ so $|ngp - cp + v\ell p| < \frac{|p|}{2} + |d| \leq |p|$ meaning that $np_0 - c + v\ell = 0$ for every $\ell \in \mathcal{L}$.

Let ℓ_0 be the minimal element of $\mathcal{L}$. As g and v are relatively prime, every $\ell \in \mathcal{L}$ is then of the form $\ell = \ell_0 + ig$ for some $i \geq 0$. Also $|igy - iv\tilde{h}_{\alpha_0}| = i\tilde{h}_{\alpha_0}|\frac{gy}{\tilde{h}_{\alpha_0}} - v| \leq i\tilde{h}_{\alpha_0}|\frac{gp}{t} - v| + \frac{i\tilde{h}_{\alpha_0}g}{b_{\alpha_0-1}L} < i\tilde{h}_{\alpha_0}\frac{2}{b_{\alpha_0-1}\epsilon_0 L} + i\tilde{h}_{\alpha_0}|\frac{vt}{t} - v| + \frac{\lceil\epsilon^{-1}\rceil^{\lceil\epsilon^{-1}\rceil-1}\tilde{h}_{\alpha_0}}{b_{\alpha_0-1}} < \frac{2\tilde{h}_{\alpha_0}}{b_{\alpha_0-1}\epsilon_0} + \frac{\lceil\epsilon^{-1}\rceil^{\lceil\epsilon^{-1}\rceil-1}\tilde{h}_{\alpha_0}}{b_{\alpha_0-1}} < \frac{1}{3}\tilde{h}_{\alpha_0}$ so the claim holds with g for t and v for p.

Subcase: $p_m > p_{m-1} - \epsilon t$

Set $p^\star = \lfloor\frac{t}{p_{m-1}}\rfloor p_{m-1} = \lceil\frac{t}{p_{m-1}}\rceil p_{m-1} - p_{m-1} = p_m + t - p_{m-1} > t - \epsilon t$.

Subsubcase: $t - \epsilon t < p^\star \leq t - \frac{2t}{b_{\alpha_0-1}}$

For $1 \leq i < \lceil(2\epsilon)^{-1}\rceil$, $\frac{1}{2}t - \epsilon t < ip^\star \pmod{t} \leq t - \frac{2t}{b_{\alpha_0-1}}$. Then $i\lfloor\frac{t}{p_{m-1}}\rfloor\lceil\frac{t}{p_{m-2}}\rceil\cdots\lceil\frac{t}{p_0}\rceil p \pmod{t}$ is nonzero and at least $\frac{2t}{b_{\alpha_0-1}}$ away from every multiple of t. Since $\lceil\epsilon^{-1}\rceil\lfloor\frac{t}{p_{m-1}}\rfloor\cdots\lceil\frac{t}{p_0}\rceil < \lceil\epsilon^{-1}\rceil^{\lceil\epsilon^{-1}\rceil}\epsilon_0 L < \epsilon L$, as above Lemma 4.22 implies $\left|\frac{1}{L}\sum_{\ell=0}^{L-1}\lambda_B(T^{q-\ell k}B)\right| + \frac{1}{L}\sum_{\ell=0}^{L-1}(1-\delta_\ell)\epsilon < 6\epsilon^{1/2}$.

Subsubcase: $t - \frac{2t}{b_{\alpha_0-1}} < p^\star \leq t - \frac{2t}{b_{\alpha_0-1}\epsilon_0 L}$

Let $g \in \mathbb{N}$ minimal such that $gp^\star \pmod{t} \leq t - \frac{2t}{b_{\alpha_0-1}}$. As in the subcase where $\frac{2t}{b_{\alpha_0-1}\epsilon_0 L} \leq p_m < \frac{2t}{b_{\alpha_0-1}}$, $g < 2\epsilon_0 L$ and then similar reasoning as there using Lemma 4.22 gives $\left|\frac{1}{L}\sum_{\ell=0}^{L-1}\lambda_B(T^{q-\ell k}B)\right| + \frac{1}{L}\sum_{\ell=0}^{L-1}(1-\delta_\ell)\epsilon < 4\epsilon^{1/2}$.

Subsubcase: $t - \frac{2t}{b_{\alpha_0-1}\epsilon_0 L} < p^\star < t$ and $L < t \leq \epsilon_0 b_{\alpha_0} L$

Set $g = \lceil\frac{2t}{b_{\alpha_0-1}(t-p^\star)}\rceil < 2\epsilon_0 L + 1$. Then $igp^\star \pmod{t} < t - \frac{2t}{b_{\alpha_0-1}(t-p^\star)}(t - p^\star) = t - \frac{2t}{b_{\alpha_0-1}}$ and $igp^\star \pmod{t} > t - \lceil\epsilon^{-1}\rceil(2\epsilon_0 L + 1)\frac{2}{b_{\alpha_0-1}\epsilon_0 L}$ so again similar reasoning gives $\left|\frac{1}{L}\sum_{\ell=0}^{L-1}\lambda_B(T^{q-\ell k}B)\right| + \frac{1}{L}\sum_{\ell=0}^{L-1}(1-\delta_\ell)\epsilon < 4\epsilon^{1/2}$.

Subsubcase: $t - \frac{2t}{b_{\alpha_0-1}\epsilon_0 L} < p^\star < t$ and $t \geq \epsilon_0 b_{\alpha_0-1} L$ and $|p| \geq \frac{2t}{b_{\alpha_0-1}\epsilon_0 L}$

Set $g = \lfloor\frac{t}{p_{m-1}}\rfloor\lceil\frac{t}{p_{m-2}}\rceil\cdots\lceil\frac{t}{p_0}\rceil < \lceil\epsilon^{-1}\rceil^{\lceil\epsilon^{-1}\rceil-1}$. Then $p^\star = gp \pmod{t}$. Here, as above, $m \neq 0$ so there exists $v \neq 0$ such that $vt - \frac{2t}{b_{\alpha_0-1}\epsilon_0 L} < gp < vt$ and the same argument as in the $0 < p_m < \frac{2t}{b_{\alpha_0-1}\epsilon_0}$ subcase shows that the claim holds. Therefore the claim is proved as all cases have been covered. □

For $\ell_0 + ti \in \mathcal{L}$, since $r - \ell_0 y = a\tilde{h}_{\alpha_0} + y_{\ell_0}$ for some $|a| \leq 1$

$$q - (\ell_0 + ti)k = (x - (\ell_0 + ti)z)\tilde{h}_{\alpha_0} + r - (\ell_0 + ti)y = (x - \ell_0 z - tiz - ip + a)\tilde{h}_{\alpha_0} + y_{\ell_0} + ip\tilde{h}_{\alpha_0} - tiy$$

Since $|y_{\ell_0}| < a_{\alpha_0-1}\tilde{h}_{\alpha_0-1}$ as $\ell_0 \in \mathcal{L}$, then $|y_{\ell_0} + ip\tilde{h}_{\alpha_0} - tiy| < a_{\alpha_0-1}\tilde{h}_{\alpha_0-1} + \frac{1}{3}\tilde{h}_{\alpha_0} < \frac{1}{2}\tilde{h}_{\alpha_0}$ meaning that $y_{\ell_0+ti} = y_{\ell_0} + ip\tilde{h}_{\alpha_0} - tiy$ and $k_{\ell_0+ti} = x - \ell_0 z - tiz - ip + a$.

Then $y_{\ell_0+ti} - k_{\ell_0+ti}\ell' = y_{\ell_0} + ip\tilde{h}_{\alpha_0} - tiy - (x - \ell_0 z - tiz - ip + a)\ell' = (y_{\ell_0} - x\ell' + \ell_0 z\ell' - a\ell') - i(-p\tilde{h}_{\alpha_0} + ty - tz\ell' - p\ell')$ so define $q_{\ell'} = y_{\ell_0} - x\ell' + \ell_0 z\ell' - a\ell'$ and $k'_{\ell'} = -p\tilde{h}_{\alpha_0} + ty - tz\ell' - p\ell'$ so that

$$y_{\ell_0+ti} - k_{\ell_0+ti}\ell' = q_{\ell'} - k'_{\ell'}i$$

and observe that $|y_{\ell_0+ti} - k_{\ell_0+ti}\ell'| < \frac{1}{2}\tilde{h}_{\alpha_0} + a_{\alpha_0}b_{\alpha_0}$ so $\alpha(y_{\ell_0+ti} - k_{\ell_0+ti}\ell') < \alpha_0$ for all ℓ' and i.

For $\ell_0 + ti \in \mathcal{L}$ such that $k_{\ell_0+ti} \neq 0$, by Lemma 4.21,

$$\left|\lambda_B(T^{q-(\ell_0+ti)k}B) - \frac{a_{\alpha_0} - |k_{\ell_0+ti}|}{r_{\alpha_0}+1}\sum_{\ell'=0}^{b_{\alpha_0}-1}\lambda_B(T^{y_{\ell_0+ti}-k_{\ell_0+ti}\ell'}B)\right|$$

$$\leq \frac{a_{\alpha_0} - |k_{\ell_0+ti}|}{a_{\alpha_0}} \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0}-1} \frac{2|y_{\ell_0+ti} - k_{\ell_0+ti}\ell'|}{\tilde{h}_{\alpha_0}} + \frac{|k_{\ell_0+ti}|}{a_{\alpha_0}}\epsilon + \tau_{\alpha_0}$$

For i, ℓ' such that $\alpha(q_{\ell'} - k'_{\ell'}i) = \alpha_{\ell'}$, if $d_{\ell',i}$ is the unique integer such that $|q_{\ell'} - k'_{\ell'}i - d_{\ell',i}\tilde{h}_{\alpha_{\ell'}}| \leq \frac{1}{2}\tilde{h}_{\alpha_{\ell'}}$ then

$$\frac{|y_{\ell_0+ti} - k_{\ell_0+ti}\ell'|}{\tilde{h}_{\alpha_0}} = \frac{|q_{\ell'} - k'_{\ell'}i|}{h_{\alpha_0}} < \frac{(|d_{\ell',i}| + 1)\tilde{h}_{\alpha_{\ell'}}}{\tilde{h}_{\alpha_0}} < \frac{|d_{\ell',i}| + 1}{a_{\alpha_{\ell'}} b_{\alpha_0-1}} < \left(1 - \gamma_i^{\alpha_{\ell'}, q_{\ell'}, k'_{\ell'}}\right) \frac{2}{b_{\alpha_0-1}}$$

and for i, ℓ' such that $\alpha(q_{\ell'} - k'_{\ell'}i) < \alpha_{\ell'}$, as $|q_{\ell'} - k'_{\ell'}i| < \tilde{h}_{\alpha_{\ell'}} \leq \tilde{h}_{\alpha_0-1}$ and $\gamma_i^{\alpha_{\ell'}, q_{\ell'}, k'_{\ell'}} = 0$,

$$\frac{|q_{\ell'} - k'_{\ell'}i|}{\tilde{h}_{\alpha_0}} < \frac{\tilde{h}_{\alpha_0-1}}{\tilde{h}_{\alpha_0}} < \frac{1}{a_{\alpha_0-1}b_{\alpha_0-1}} < \frac{2}{b_{\alpha_0-1}} = \left(1 - \gamma_i^{\alpha_{\ell'}, q_{\ell'}, k'_{\ell'}}\right) \frac{2}{b_{\alpha_0-1}}$$

Then for $\ell_0 + ti \in \mathcal{L}$ such that $k_{\ell_0+ti} \neq 0$, as $\frac{a_{\alpha_0} - |k_{\ell_0+ti}|}{r_{\alpha_0}+1} = \frac{a_{\alpha_0}}{r_{\alpha_0}+1}\gamma_{\ell_0+ti}^{\alpha_0,q,k}$,

$$\left| \lambda_B(T^{q-(\ell_0+ti)k}B) - \frac{r_{\alpha_0}}{r_{\alpha_0}+1}\gamma_{\ell_0+ti}^{\alpha_0,q,k} \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0}-1} \lambda_B(T^{y_{\ell_0+ti} - k_{\ell_0+ti}\ell'}B) \right|$$

$$\leq (1 - \gamma_{\ell_0+ti}^{\alpha_0,q,k})\epsilon + \tau_{\alpha_0} + \gamma_{\ell_0+ti}^{\alpha_0,q,k} \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0-1}} \left(1 - \gamma_i^{\alpha_{\ell'}, q_{\ell'}, k'_{\ell'}}\right) \frac{4}{b_{\alpha_0-1}}$$

For $\ell_0 + ti \in \mathcal{L}$ such that $k_{\ell_0+ti} = 0$, we have $\lambda_B(T^{q-(\ell_0+ti)k}B) = \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0}-1} \lambda_B(T^{y_{\ell_0+ti} - k_{\ell_0+ti}\ell'}B)$ and $\gamma_{\ell_0+ti}^{\alpha_0,q,k} = 1$.

For $\ell_0 + ti \notin \mathcal{L}$, $\gamma_{\ell_0+ti}^{\alpha_0,q,k} = 0$ by definition and $|\lambda_B(T^{q-(\ell_0+ti)k}B)| < \epsilon$ so $|\delta_{\ell_0+ti}\lambda_B(T^{q-(\ell_0+ti)k}B)| + (1 - \delta_{\ell_0+ti})\epsilon < \epsilon = \gamma_{\ell_0+ti}^{\alpha_0,q,k}\lambda_B(T^{q-(\ell_0+ti)k}B) + (1 - \gamma_{\ell_0+ti}^{\alpha_0,q,k})\epsilon$.

Therefore, as $\frac{r_{\alpha_0}}{r_{\alpha_0}+1} < 1$ and $\frac{4}{b_{\alpha_0-1}} < \epsilon$,

$$\begin{aligned}
&\left| \sum_{i=0}^{L'-1} \delta_{\ell_0+ti}\lambda_B(T^{q-(\ell_0+ti)k}B) \right| + \sum_{i=0}^{L'-1} (1 - \delta_{\ell_0+ti})\,\epsilon \\
&\quad < \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0}-1} \left| \sum_{i=0}^{L'-1} \delta_{\ell_0+ti}\gamma_{\ell_0+ti}^{\alpha_0,q,k}\lambda_B(T^{q_{\ell'} - k'_{\ell'}i}B) \right| + \tau_{\alpha_0} + \sum_{i=0}^{L'-1} (1 - \delta_{\ell_0+ti})\,\epsilon \\
&\qquad + \sum_{i=0}^{L'-1} \delta_{\ell_0+ti} \left(\left(1 - \gamma_{\ell_0+ti}^{\alpha_0,q,k}\right)\epsilon + \gamma_{\ell_0+ti}^{\alpha_0,q,k} \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0-1}} \left(1 - \gamma_i^{\alpha_{\ell'}, q_{\ell'}, k'_{\ell'}}\right) \frac{4}{b_{\alpha_0-1}} \right) \\
&\quad = \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0}-1} \left(\left| \sum_{i=0}^{L'-1} \delta_{\ell_0+ti}\gamma_{\ell_0+ti}^{\alpha_0,q,k}\lambda_B(T^{q_{\ell'} - k'_{\ell'}i}B) \right| + \sum_{i=0}^{L'-1} \left((1 - \delta_{\ell_0+ti}) + \delta_{\ell_0+ti}\left(1 - \gamma_{\ell_0+ti}^{\alpha_0,q,k}\right) \right)\epsilon \right) \\
&\qquad + \tau_{\alpha_0} + \sum_{i=0}^{L'-1} \delta_{\ell_0+ti}\gamma_{\ell_0+ti}^{\alpha_0,q,k} \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0-1}} \left(1 - \gamma_i^{\alpha_{\ell'}, q_{\ell'}, k'_{\ell'}}\right)\epsilon \\
&\quad = \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0}-1} \left(\left| \sum_{i=0}^{L-1} \mathbb{1}_{\ell<L'}\delta_{\ell_0+ti}\gamma_{\ell_0+ti}^{\alpha_0,q,k}\lambda_B(T^{q_{\ell'} - k'_{\ell'}i}B) \right| + \sum_{i=0}^{L-1} \mathbb{1}_{\ell<L'}\left(1 - \delta_{\ell_0+ti}\gamma_{\ell_0+ti}^{\alpha_0,q,k}\right)\epsilon \right) \\
&\qquad + \tau_{\alpha_0} + \sum_{i=0}^{L-1} \mathbb{1}_{\ell<L'}\delta_{\ell_0+ti}\gamma_{\ell_0+ti}^{\alpha_0,q,k} \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0-1}} \left(1 - \gamma_i^{\alpha_{\ell'}, q_{\ell'}, k'_{\ell'}}\right)\epsilon
\end{aligned}$$

Since $|\lambda_B(T^{q-\ell k}B)| < \epsilon$ for $\ell \notin \mathcal{L}$ and $|\{\ell : \ell \neq \ell_0 + ti\}| = L - L'$,

$$\left| \sum_{\ell \neq \ell_0 + ti} \delta_\ell \lambda_B(T^{q-k\ell}B) \right| + \sum_{\ell \neq \ell_0 + ti} (1-\delta_\ell)\,\epsilon < (L-L')\epsilon$$

and therefore

$$\begin{aligned}
&\left| \frac{1}{L} \sum_{\ell=0}^{L-1} \delta_\ell \lambda_B(T^{q-\ell k}B) \right| + \frac{1}{L} \sum_{\ell=0}^{L-1} (1-\delta_\ell)\epsilon \\
&\quad \leq \frac{1}{L} \sum_{\ell=0}^{L-1} (1 - \mathbb{1}_{\ell < L'})\,\epsilon + \left| \frac{1}{L} \sum_{\ell=\ell_0+ti} \delta_\ell \lambda_B(T^{q-\ell k}B) \right| + \frac{1}{L} \sum_{\ell=\ell_0+ti} (1-\delta_\ell)\epsilon \\
&\quad < \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0}-1} \left(\left| \sum_{i=0}^{L-1} \mathbb{1}_{\ell<L'} \delta_{\ell_0+ti} \gamma_{\ell_0+ti}^{\alpha_0,q,k} \lambda_B(T^{q_{\ell'} - k'_{\ell'} i}B) \right| + \sum_{i=0}^{L-1} \mathbb{1}_{\ell<L'} \left(1 - \delta_{\ell_0+ti}\gamma_{\ell_0+ti}^{\alpha_0,q,k}\right)\epsilon \right) \\
&\qquad\qquad + \tau_{\alpha_0} + \sum_{i=0}^{L-1} \mathbb{1}_{\ell<L'} \delta_{\ell_0+ti}\gamma_{\ell_0+ti}^{\alpha_0,q,k} \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0-1}} \left(1 - \gamma_i^{\alpha_{\ell'},q_{\ell'},k'_{\ell'}}\right)\epsilon + \frac{1}{L}\sum_{\ell=0}^{L-1} (1 - \mathbb{1}_{\ell<L'})\,\epsilon \\
&\quad = \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0}-1} \left(\left| \sum_{i=0}^{L-1} \mathbb{1}_{\ell<L'} \delta_{\ell_0+ti} \gamma_{\ell_0+ti}^{\alpha_0,q,k} \lambda_B(T^{q_{\ell'} - k'_{\ell'} i}B) \right| + \sum_{i=0}^{L-1} \left(1 - \mathbb{1}_{\ell<L'}\delta_{\ell_0+ti}\gamma_{\ell_0+ti}^{\alpha_0,q,k}\right)\epsilon \right) \\
&\qquad\qquad + \tau_{\alpha_0} + \sum_{i=0}^{L-1} \mathbb{1}_{\ell<L'} \delta_{\ell_0+ti}\gamma_{\ell_0+ti}^{\alpha_0,q,k} \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0-1}} \left(1 - \gamma_i^{\alpha_{\ell'},q_{\ell'},k'_{\ell'}}\right)\epsilon
\end{aligned}$$

□

Proposition 4.26. *Let T be a quasi-staircase transformation such that $\sum \frac{a_n b_n + b_{n+1} + c_{n+1}}{h_n} < \infty$ and $\frac{a_n b_n^2}{h_n} \to 0$ and $\frac{a_{n+1} b_{n+1}}{h_n} \to 0$. Let B be a union of levels in some fixed C_N. Then*

$$\lim_{n\to\infty} \max_{b_n \leq k < a_n} \max_{|q| < a_{n-1}\tilde{h}_{n-1}} \left| \lambda_B(T^{k\tilde{h}_n + q}B) \right| = 0.$$

Proof. Fix $\epsilon > 0$ and set $\epsilon_0 = (2\lceil \epsilon^{-1} \rceil^{\lceil \epsilon^{-1} \rceil + 1})^{-1}$. Using Propositions 4.16, 4.17, 4.19 and 4.23 and that $\sum_n \tau_n < \infty$, there exists N such that $b_N > 4\epsilon^{-1}\epsilon_0^{-1}$, $\sup_{m \geq N-1} M_{B,m} < \epsilon$, $\sup_{m\geq N} \widehat{M}_{B,m} < \epsilon$, $\sup_{m \geq N} \widetilde{M}_{B,m} < \epsilon$, $\sum_{n=N}^\infty \tau_n < \epsilon$ and $\sup_{m \geq b_N - 1} \sup_{k \leq m} \left(\int \left| \frac{1}{m} \sum_{j=0}^{m-1} \chi_B \circ T^{-jk} \right| \, d\mu + \frac{2}{m} \right) < \frac{\epsilon}{3}$.

Take any n such that $b_n > \tilde{h}_{N+1}$. For $b_n \leq k < a_n$ and $|q| < a_{n-1}\tilde{h}_{n-1}$, by Lemma 4.21,

$$\left| \lambda_B(T^{k\tilde{h}_n + q}B) \right| < \frac{a_n - k}{a_n} \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n - 1} \lambda_B(T^{q-k\ell}B) \right| + \frac{k}{a_n}\epsilon + \tau_n < \left| \frac{1}{b_n} \sum_{\ell=0}^{b_n-1} \lambda_B(T^{q-k\ell}B) \right| + 2\epsilon$$

Set $L = b_n$. By Lemma 4.25, $\left| \frac{1}{L} \sum_{\ell=0}^{L-1} \lambda_B(T^{q-k\ell}B) \right| < 6\epsilon^{1/2}$ or there exists $q_{\ell'}, k'_{\ell'}, L', \ell_0, t$ such that

$$\begin{aligned}
\left| \frac{1}{L} \sum_{\ell=0}^{L-1} \lambda_B(T^{q-k\ell}B) \right| &< \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0}-1} \left(\left| \frac{1}{L} \sum_{\ell=0}^{L-1} \mathbb{1}_{\ell<L'} \gamma_{\ell_0+t\ell}^{\alpha_0,q,k} \lambda_B(T^{q_{\ell'} - k'_{\ell'}\ell}B) \right| + \frac{1}{L} \sum_{\ell=0}^{L-1} \left(1 - \mathbb{1}_{\ell<L'}\gamma_{\ell_0+t\ell}^{\alpha_0,q,k}\right)\epsilon \right) \\
&\quad + \frac{1}{b_{\alpha_0}} \sum_{\ell'=0}^{b_{\alpha_0}-1} \frac{1}{L} \sum_{\ell=0}^{L-1} \mathbb{1}_{\ell<L'} \gamma_{\ell_0+t\ell}^{\alpha_0,q,k} \left(1 - \gamma_\ell^{\alpha_{\ell'},q_{\ell'},k'_{\ell'}}\right)\epsilon + \tau_{\alpha_0}
\end{aligned}$$

Let $\mathcal{L}' = \{0 \leq \ell' < b_{\alpha_0} : \alpha_{\ell'} > N$ and Lemma 4.25 does not bound the ℓ' weighted average by $6\epsilon^{1/2}\}$.

Since N is large enough that Proposition 4.23 implies if $k'_{\ell'} < \tilde{h}_{N+1} \leq L$ then $|\frac{1}{L}\sum_{\ell=0}^{L-1}\delta_\ell\lambda_B(T^{-k\ell}B)| < \epsilon$,

$$\frac{1}{b_{\alpha_0}}\sum_{\ell'=0}^{b_{\alpha_0}-1}\left(\left|\frac{1}{L}\sum_{\ell=0}^{L-1}\mathbb{1}_{\ell<L'}\gamma_{\ell_0+t\ell}^{\alpha_0,q,k}\lambda_B(T^{q_{\ell'}-k'_{\ell'}\ell}B)\right| + \frac{1}{L}\sum_{\ell=0}^{L-1}\left(1-\mathbb{1}_{\ell<L'}\gamma_{\ell_0+t\ell}^{\alpha_0,q,k}\right)\epsilon\right) < \left(1-\frac{|\mathcal{L}'|}{b_{\alpha_0}}\right)6\epsilon^{1/2}$$
$$+\frac{|\mathcal{L}'|}{b_{\alpha_0}}\frac{1}{|\mathcal{L}'|}\sum_{\ell'\in\mathcal{L}'}\left(\left|\frac{1}{L}\sum_{\ell=0}^{L-1}\mathbb{1}_{\ell<L'}\gamma_{\ell_0+t\ell}^{\alpha_0,q,k}\lambda_B(T^{q_{\ell'}-k'_{\ell'}\ell}B)\right| + \frac{1}{L}\sum_{\ell=0}^{L-1}\left(1-\mathbb{1}_{\ell<L'}\gamma_{\ell_0+t\ell}^{\alpha_0,q,k}\right)\epsilon\right)$$

Therefore, applying Lemma 4.25 to each ℓ' weighted average, since $\alpha_{\ell'} \leq \alpha_0 - 1$ (and suppressing the explicit dependence on ℓ' of L'', ℓ'_0, t' for clarity),

$$\left|\frac{1}{L}\sum_{\ell=0}^{L-1}\lambda_B(T^{q-k\ell}B)\right| < \left(1-\frac{|\mathcal{L}'|}{b_{\alpha_0}}\right)6\epsilon^{1/2} + \tau_{\alpha_0} + \tau_{\alpha_0-1}$$
$$+\frac{|\mathcal{L}'|}{b_{\alpha_0}}\frac{1}{|\mathcal{L}'|}\sum_{\ell'\in\mathcal{L}'}\frac{1}{b_{\alpha_{\ell'}}}\sum_{\ell''=0}^{b_{\alpha_{\ell'}}-1}\left(\left|\frac{1}{L}\sum_{\ell=0}^{L-1}\mathbb{1}_{\ell<L''}\mathbb{1}_{\ell'_0+t'\ell<L'}\gamma_{\ell_0+t(\ell'_0+t'\ell)}^{\alpha_0,q,k}\gamma_{\ell'_0+t'\ell}^{\alpha_{\ell'},q_{\ell'},k'_{\ell'}}\lambda_B(T^{q_{\ell',\ell''}-k_{\ell',\ell''}\ell}B)\right|\right.$$
$$\left.+\frac{1}{L}\sum_{\ell=0}^{L-1}\left(1-\mathbb{1}_{\ell<L''}\mathbb{1}_{\ell'_0+t'\ell<L'}\gamma_{\ell_0+t(\ell'_0+t'\ell)}^{\alpha_0,q,k}\gamma_{\ell'_0+t'\ell}^{\alpha_{\ell'},q_{\ell'},k'_{\ell'}}\right)\epsilon\right)$$
$$+\frac{|\mathcal{L}'|}{b_{\alpha_0}}\frac{1}{|\mathcal{L}'|}\sum_{\ell'\in\mathcal{L}'}\frac{1}{b_{\alpha_{\ell'}}}\sum_{\ell''=0}^{b_{\alpha_{\ell'}}-1}\frac{1}{L}\sum_{\ell=0}^{L-1}\left(\mathbb{1}_{\ell<L'}\gamma_{\ell_0+t\ell}^{\alpha_0,q,k}\left(1-\gamma_{\ell}^{\alpha_{\ell'},q_{\ell'},k'_{\ell'}}\right)\right.$$
$$\left.+\mathbb{1}_{\ell<L''}\mathbb{1}_{\ell'_0+t'\ell<L'}\gamma_{\ell_0+t(\ell'_0+t'\ell)}^{\alpha_0,q,k})\gamma_{\ell'_0+t'\ell}^{\alpha_{\ell'},q_{\ell'},k'_{\ell'}}\left(1-\gamma_{\ell}^{\alpha_{\ell''},q_{\ell',\ell''},k_{\ell',\ell''}}\right)\right)\epsilon$$

Now observe that

$$\frac{1}{L}\sum_{\ell=0}^{L-1}\left(\mathbb{1}_{\ell<L'}\gamma_{\ell_0+t\ell}^{\alpha_0,q,k}\left(1-\gamma_{\ell}^{\alpha_{\ell'},q_{\ell'},k'_{\ell'}}\right) + \mathbb{1}_{\ell<L''}\mathbb{1}_{\ell'_0+t'\ell<L'}\gamma_{\ell_0+t(\ell'_0+t'\ell)}^{\alpha_0,q,k})\gamma_{\ell'_0+t'\ell}^{\alpha_{\ell'},q_{\ell'},k'_{\ell'}}\left(1-\gamma_{\ell}^{\alpha_{\ell''},q_{\ell',\ell''},k_{\ell',\ell''}}\right)\right)$$
$$=\frac{1}{L}\sum_{\substack{\ell\neq\ell'_0+t'i\\ \ell<L'}}\gamma_{\ell_0+t\ell}^{\alpha_0,q,k}\left(1-\gamma_{\ell}^{\alpha_{\ell'},q_{\ell'},k'_{\ell'}}\right) + \frac{1}{L}\sum_{\ell=0}^{L''-1}\mathbb{1}_{\ell'_0+t'\ell<L'}\gamma_{\ell_0+t(\ell'_0+t'\ell)}^{\alpha_0,q,k}\left(1-\gamma_{\ell'_0+t'\ell}^{\alpha_{\ell'},q_{\ell'},k'_{\ell'}}\gamma_{\ell}^{\alpha_{\ell''},q_{\ell',\ell''},k_{\ell',\ell''}}\right)$$

and that the sets of the original $0 \leq \ell < L$ the two sums range over are disjoint.

Continue iteratively applying Lemma 4.25 until all terms are bounded by $6\epsilon^{1/2}$ or have $k_{\ell',\ell'',\ldots} \leq L$, which must occur as α decrements at each application of the lemma (and the hypotheses of the lemma hold as long as $\alpha_{\ell''\cdots} > N$). Then $\left|\frac{1}{L}\sum_{\ell=0}^{L-1}\lambda_B(T^{q-k\ell}B)\right|$ is bounded by a convex combination of terms less than $6\epsilon^{1/2}$ plus a sum of τ's bounded by $\sum_{n=N}^{\infty}\tau_n < \epsilon$ plus an average over $0 \leq \ell < L'$ of terms of the form

$$\gamma^{\alpha_0}\left(1-\gamma^{\alpha_{\ell'}}\gamma^{\alpha_{\ell''}}\cdots\gamma^{\alpha_{\ell^{(m)}}}\right)\epsilon$$

which are all bounded by ϵ as $0 \leq \gamma \leq 1$. Therefore

$$\left|\frac{1}{L}\sum_{\ell=0}^{L-1}\lambda_B(T^{q-k\ell}B)\right| < 6\epsilon^{1/2}+\epsilon+\epsilon \qquad \text{meaning that} \qquad \left|\lambda_B(T^{k\tilde{h}_n+q}B)\right| < 6\epsilon^{1/2}+4\epsilon \qquad \square$$

Theorem 4.27. *Let T be a quasi-staircase transformation such that $\sum\frac{a_nb_n+b_{n+1}+c_{n+1}}{h_n} < \infty$ and $\frac{a_nb_n^2}{h_n} \to 0$ and $\frac{a_{n+1}b_{n+1}}{h_n} \to 0$ and $\frac{b_n}{a_n} \to 0$. Then T is mixing.*

Proof. By Propositions 4.16, 4.17, 4.19 and 4.26, for any B which is a union of levels in some C_N, $\lim_{n\to\infty}\max_{\tilde{h}_n\leq t<\tilde{h}_{n+1}}|\lambda_B(T^tB)| = 0$. As unions of levels generate the measure algebra, T is Renyi mixing hence mixing. $\square$

5 Non-superlinear word complexity implies partial rigidity

Theorem 5.1. *Let X be a subshift with word complexity p such that $\liminf \frac{p(q)}{q} < \infty$. Then there exists a constant $\delta_X > 0$ such that every ergodic probability measure μ on X is at least δ_X-partially rigid.*

5.1 Word combinatorics

Notation 5.2. *For x a finite or infinite word and $-\infty \leq i < j \leq \infty$,*

$$x_{[i,j)} = \text{ the subword of } x \text{ from position } i \text{ through position } j-1$$

Notation 5.3. $[w] = \{x \in X : x_{[0,\|w\|)} = w\}$ *for finite words w.*

Notation 5.4. *For a word v and $0 \leq q < \|v\|$, let $v^{q/\|v\|}$ be the suffix of v of length q.*
Let $v^{n+q/\|v\|} = v^{q/\|v\|}v^n$ for $n \in \mathbb{N}$.

Definition 5.5. Let $w \in \mathcal{L}(X)$ be a word in the language of a subshift. A word $v \in \mathcal{L}(X)$ is a **root** of w if $wv \in \mathcal{L}(X)$ and $\|v\| \leq \|w\|$ and w is a suffix of v^∞, i.e. there exists $q = p/\|v\|$ with $p \geq \|v\|$ such that $w = v^q$. The **minimal root** of w is the shortest v which is a root of w.

Every word has a unique minimal root as it is a root of itself.

Lemma 5.6. *If $uw = wv$ and $\|v\| \leq \|w\|$ then v is a root of w.*

Proof. As w has v as a suffix, $w = w'v$. Then $uw'v = uw = wv = w'vv$ so $uw' = w'v$. If $\|w'\| \geq \|v\|$, repeat this process until it terminates at $w = w''v^n$ with $\|w''\| < \|v\|$. Then $uw'' = w''v$ so w'' is a suffix of v. □

Lemma 5.7. *If $uv = vu$ then $u = v_0^t$ and $v = v_0^s$ for some word v_0 and $t, s \in \mathbb{N}$.*

Proof. If $\|u\| = \|v\|$ then $uv = vu$ immediately implies $u = v$. Let

$$V = \{(u,v) : uv = vu, \|v\| < \|u\|, \text{ there is no word } v_0 \text{ with } u = v_0^t \text{ and } v = v_0^s \text{ for } s,t \in \mathbb{N}\}$$

and suppose $V \neq \emptyset$. Let $(u,v) \in V$ such that $\|u\|$ is minimal. As $\|u\| > \|v\|$, $uv = vu$ implies $u = vu' = u''v$ for some nonempty words u', u''. Then $vu'v = uv = vu = vu''v$ so $u' = u''$ and $vu' = u'v$. If $\|u'\| = \|v\|$ then $u' = v$ so $u = v^2$ contradicting that $(u,v) \in V$.

Consider when $\|u'\| < \|v\|$. Since $\|u'\| < \|u\|$ and $\|v\| < \|u\|$, the minimality of $\|u\|$ implies that $(v,u') \notin V$. Then $v = v_0^n$ and $u' = v_0^m$ for some word v_0 and $n, m \in \mathbb{N}$. So $u = v_0^{n+m}$ meaning $(u,v) \notin V$. When $\|v\| < \|u'\|$, we have $(u',v) \notin V$ so $u' = v_0^n$ and $u = v_0^{n+m}$. So $V = \emptyset$. □

Lemma 5.8. *If u and v are both roots of a word w and uu is a suffix of w and $\|v\| < \|u\|$ then there exists a suffix v_0 of v such that $u = v_0^n$ and $v = v_0^m$ for some $n, m \in \mathbb{N}$.*

In particular, if v is the minimal root of w and u is a root of w and uu is a suffix of w then u is a multiple of v, i.e. there exists $n \in \mathbb{N}$ such that $u = v^n$.

Proof. Writing u' and v' for the appropriate suffixes of u and v, we have $w = u'u^t = v'v^q$ for some $t, q \in \mathbb{N}$. Then $u = u_0v^a$ for some proper (possibly empty) suffix u_0 of v and $1 \leq a \leq q$. So $u'(u_0v^a)^t = v'v^q$ meaning that $u'(u_0v^a)^{t-1}u_0 = v'v^{q-a}$. As $t \geq 2$, $\|v'v^{q-a}\| = \|u'(u_0v^a)^{t-1}u_0\| \geq \|u_0v^au_0\| \geq \|vu_0\|$ so, as u_0 is a suffix of v, then $v'v^{q-a}$ has u_0v as a suffix. This means $vu_0 = u_0v$ so Lemma 5.7 gives v_0 such that $v = v_0^n$ and $u_0 = v_0^m$ so $u = v_0^{m+an}$. If v is the minimal root then $v = v_0$ since v_0 is a root of w. □

Lemma 5.9. *Let w be a word with minimal root v. If $0 \leq i \leq \frac{1}{2}\|w\|$ and $T^i[w] \cap [w] \neq \emptyset$ then i is a multiple of $\|v\|$.*

Proof. Let u be the prefix of B of length i and v_0 be the suffix of B of length i. For $x \in T^i[w] \cap [w]$, then $x_{[-i,\|w\|)} = uw = wv_0$. By Lemma 5.6, then v_0 is a root of w. As $\|v_0\| = i \leq \frac{1}{2}\|w\|$, w has v_0v_0 as a suffix. By Lemma 5.8, since v is the minimal root then v_0 is a multiple of v. □

5.2 Language analysis

Proposition 5.10. *There exists $C, k > 0$, depending only on X, and $\ell_n \to \infty$ and, for each n, at most C words $B_{n,j}$ so that $X_0 = \{x \in X :$ every finite subword of x is a subword of a concatenation of the $B_{n,j}\}$ has measure one.*

Let $h_{n,j} = \|B_{n,j}\|$. Then $\max_j h_{n,j} \leq k\ell_n$ and $\min_j h_{n,j} \to \infty$. Let

$$W_{B_{n,j}} = W_{n,j} = \{x \in X_0 : x \text{ can be written as a concatenation such that } x_{[0,h_{n,j})} = B_{n,j}\} \subseteq [B_{n,j}]$$

There exists $c_{n,j} \leq k\ell_n$ such that the sets $T^iW_{n,j}$ are disjoint over $0 \leq i < c_{n,j}$.

For j such that $h_{n,j} > \frac{1}{2}\ell_n$, $c_{n,j} \geq \frac{1}{2}\ell_n$.

For j such that $h_{n,j} \leq \frac{1}{2}\ell_n$, $c_{n,j} = h_{n,j}$. For such j, also $W_{n,j} = T^{\ell_n}[B_{n,j}^{\ell_n/h_{n,j}}B_{n,j}]$ and $B_{n,j}$ is the minimal root of $B_{n,j}^{\ell_n/h_{n,j}}B_{n,j}$.

If $x \in T^{h_{n,j}}W_{n,j} \cap W_{n,j'}$ for $j \neq j'$ and $h_{n,j'} \leq \frac{1}{2}\ell_n$ then $x_{(-\infty,0)}$ has $B_{n,j'}^{\ell_n/h_{n,j'}}$ as a suffix and does not have $B_{n,j'}^{\ell_n/h_{n,j'}}B_{n,j'}$ as a suffix.

Proof. Since $\liminf \frac{p(q)}{q} < \infty$, [Bos85] Theorem 2.2 gives a constant k and $\ell_n \to \infty$ such that $p(\ell_n + 1) - p(\ell_n) \leq k$ and $p(\ell_n) \leq k\ell_n$. We perform an analysis similar to Ferenczi [Fer96] Proposition 4.

Let G_q be the Rauzy graphs: the vertices are the words of length q in $\mathcal{L}(X)$ and the directed edges are from words w to w' such that $wa = bw' \in \mathcal{L}(X)$ for some letters a and b and we label the edge with the letter a. As μ is ergodic, exactly one strongly connected component has measure one and the rest have measure zero so we may assume G_q is strongly connected.

Let V_q^{RS} be the set of all vertices with more than one outgoing edge, i.e. the right-special vertices. Let $\mathcal{B}_q$ be the set of all paths from some $v \in V_q^{RS}$ to some $v' \in V_q^{RS}$ that do not pass through any $v'' \in V_q^{RS}$. Then every $v \in V_q$ is necessarily along such a path. Given any word w in $\mathcal{L}(X)$, there exists $x \in X$ such that $x_{[0,\|w\|)} = w$ so w is the label of the path from the vertex corresponding to $x_{[-q,0)}$ to the vertex corresponding to $x_{[\|w\|-q,\|w\|)}$ hence is a subword of some concatenation of labels of paths in $\mathcal{B}_q$.

The labels of the paths between right-special vertices are nested: $\mathcal{B}_{q+1}$ is necessarily a concatenation of paths in $\mathcal{B}_q$ since words corresponding to elements of V_{q+1}^{RS} necessarily have right-special suffixes. There are therefore recursion formulas defining $\mathcal{B}_{q+1}$ in terms of $\mathcal{B}_q$ though we do not make use of this fact.

Writing outdeg(v) for the number of outgoing edges of a vertex, $\sum_{v \in V_{\ell_n}^{RS}}(\text{outdeg}(v) - 1) = p(\ell_n + 1) - p(\ell_n) \leq k$ meaning that $|V_{\ell_n}^{RS}| \leq k$ and therefore $\sum_{v \in V_{\ell_n}^{RS}} \text{outdeg}(v) \leq 2k$. Therefore $|\mathcal{B}_{\ell_n}| \leq 2k$. No path in $\mathcal{B}_{\ell_n}$ properly contains a cycle so $\|B\| \leq p(\ell_n) \leq k\ell_n$ for any label B of a path in $\mathcal{B}_{\ell_n}$.

Let $\mathcal{B}_n^g$ be the set of all concatenations of paths in $\mathcal{B}_{\ell_n}$ of total length at least $\frac{3}{2}\ell_n$ and at most $k\ell_n$ not properly containing any cycles. As such a path contains no cycle properly, it has at most $|\mathcal{B}_{\ell_n}| \leq 2k$ segments from some vertex in $V_{\ell_n}^{RS}$ to another, so there are at most $K = \sum_{t=1}^{2k}(2k)^t$ such paths.

Let $\mathcal{B}_n^c$ be the set of all concatenations of paths in $\mathcal{B}_{\ell_n}$ of total length less than $\frac{3}{2}\ell_n$ which are simple cycles. Then $|\mathcal{B}_n^c| \leq K$ as each path has at most $2k$ segments and at most $2k$ choices for each segment. Every biinfinite concatenation of paths in $\mathcal{B}_{\ell_n}$ is necessarily a concatenation of paths in $\mathcal{B}_n^g \cup \mathcal{B}_n^c$.

Let B be the label of a path in $\mathcal{B}_n^g$ and let v be its minimal root. Suppose that $\|v\| < \frac{1}{2}\ell_n$. Then the vertex at which the path corresponding to B ends is the word $v^{\ell_n/\|v\|}$ as it must be a suffix of B. Let B' such that $B = B'v$. Then $\|B'\| = \|B\| - \|v\| \geq \frac{3}{2}\ell_n - \|v\| > \ell_n$. Then the path corresponding to B reaches its final vertex twice as B' has suffix $v^{\ell_n/\|v\|}$ corresponding to that vertex. This means the path properly contains a cycle which is a contradiction. So all labels of paths in $\mathcal{B}_n^g$ have minimal root of length at least $\frac{1}{2}\ell_n$. By Lemma 5.9, then $T^i W_{n,j} \cap W_{n,j} \neq \emptyset$ for $0 < i \leq \frac{1}{2}\|B\|$ only when i is a multiple of $\|v\|$. Set $c_{n,j} = \min(\|v\|, \frac{1}{2}\|B\|) \geq \frac{1}{2}\ell_n$ and then $T^i W_{n,j}$ are disjoint over $0 \leq i < c_{n,j}$.

Let B be the label of a simple cycle beginning and ending at the word w. Since B is the label of a path beginning at w, every appearance of B as a label in $x \in X$ is preceded by w, i.e. $W_B \subseteq T^{\ell_n}[wB]$. Since B either has w as a suffix or B is a root of w by Lemma 5.6, B is a root of wB. Let v be the minimal root of wB and write $B = B'v$. Then wB' has v as a root and $\|wB'\| = \ell_n + \|B'\|$ so wB' has suffix $v^{\ell_n/\|v\|}$. If B' is nonempty then the path corresponding to B passes through its final vertex before the path ends, contradicting that it is a simple cycle. So $B = v$ is the minimal root of wB.

Then Lemma 5.9 implies that $T^i W_B \cap W_B \neq \emptyset$ for $0 < i \leq \frac{1}{2}\|wB\|$ only when i is a multiple of $\|B\|$. So if $\|B\| > \frac{1}{2}\ell_n$ then set $c_{n,j} = \min(\|B\|, \frac{1}{2}\|wB\|) > \frac{1}{2}\ell_n$. If $\|B\| \leq \frac{1}{2}\ell_n$, set $c_{n,j} = \|B\|$. For such B, since $W_B \subseteq T^{\ell_n}[wB]$, we have that every occurrence of B as a label of a path is preceded by $w = B^{\ell_n/\|B\|}$. Moreover, if $x_{[-\ell_n,\|B\|)} = wB$ then $x_{[0,\|B\|)}$ is the label of a path beginning at the vertex w and ending at w so $x \in W_B$.

For $x \in W_B$, if $x_{(-\infty,0)}$ has $B^{\ell_n/\|B\|}B$ as a suffix then the path reaches w prior to the final B in that suffix. As no word B' appearing in the concatenation is the label of a path properly containing a cycle, this means the word preceding $x_{[0,\|B\|)} = B$ in x must be B, i.e. $x \in T^{\ell_n+\|B\|}[B^{\ell_n/\|B\|}B]$ so $x \in T^{\|B\|}W_B \cap W_B$ and $x \notin T^{\|B'\|}W_{B'} \cap W_B$ for every $B' \neq B$ as the path for B' does not properly contain a cycle.

Let $\mathcal{B}_n^* = \mathcal{B}_n^g \cup \mathcal{B}_n^c$. Then $|\mathcal{B}_n^*| \leq 2K = C$ for all n and every word in $\mathcal{L}(X)$ is a subword of some concatenation of labels of paths in $\mathcal{B}_n^*$. Let $\mathcal{R}_n$ be the set of all labels of paths in $\mathcal{B}_n^*$.

Let $\mathcal{D}_M = \{B : \|B\| \leq M \text{ and } B \in \mathcal{R}_n \text{ infinitely often}\}$. Then $|D_M| < \infty$ as there only finitely many words of length at most M (as non-superlinear complexity implies finite alphabet rank [DDMP21]). Let X_M be the set of $x \in X$ such that for infinitely many n, x cannot be written as a concatenation of labels in $\mathcal{B}_n^*$ without using at least one label in $\mathcal{D}_M$.

For $x \in X_M$, there exist infinitely many t such that x has $B_t^{r_t}$ as a subword for some $B_t \in \mathcal{D}_M$ and $r_t \to \infty$ (since the label B_t is preceded by the word $B_t^{\lfloor \ell_n/(\|B_t\|) \rfloor}$). As $|\mathcal{D}_M| < \infty$, there exists B such that $B_t = B$ infinitely often. Then B^{r_t} is a subword of x for $r_t \to \infty$ meaning x is periodic. Therefore $\bigcup_M X_M \subseteq \{\text{periodic words}\}$ so $\mu(\bigcup_M X_M) = 0$ as μ is ergodic hence nonatomic and a periodic word of positive measure would be an atom (there are at most countably many periodic words).

Define $\{B_{n,j}\}$ to be the set of all labels of paths in $\mathcal{B}_n^*$ which are in $\mathcal{R}_n \setminus \bigcup_M \mathcal{D}_M$. If $\liminf_n \min_j \|B_{n,j}\| < \infty$ then $B_{n,j} = B$ for some fixed B infinitely often (as there are only finitely many words of up to some fixed length). But then $B \in \mathcal{D}_{\|B\|}$, a contradiction, so $\lim_n \min_j \|B_{n,j}\| = \infty$. As $X_0 = X \setminus \bigcup_M X_M$, we have $\mu(X_0) = 1$. $\square$

5.3 Measure-theoretic analysis

Definition 5.11. Let $C_{n,j} = \displaystyle\bigcup_{i=0}^{h_{n,j}-1} T^i W_{n,j}$.

Definition 5.12. For j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, let

$$\begin{aligned} Z_{n,j} &= [B_{n,j}^{\ell_n/h_{n,j}} B_{n,j}] \setminus T^{h_{n,j}}[B_{n,j}^{\ell_n/h_{n,j}} B_{n,j}] \\ &= \{x \in X : x_{[0,\ell_n+h_{n,j})} = B_{n,j}^{\ell_n/h_{n,j}} B_{n,j} \text{ and } x_{[-h_{n,j},\ell_n)} \neq B_{n,j}^{\ell_n/h_{n,j}} B_{n,j}\} \end{aligned}$$

Proposition 5.13. *For j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, the sets $T^{ah_{n,j}} Z_{n,j}$ are disjoint over $0 \leq a \leq \left\lfloor \frac{\ell_n}{h_{n,j}} \right\rfloor$.*

Proof. For $0 \leq a < b \leq \left\lfloor \frac{\ell_n}{h_{n,j}} \right\rfloor$ and $x \in T^{ah_{n,j}} Z_{n,j} \cap T^{bh_{n,j}} Z_{n,j}$, writing $z = \ell_n - \left\lfloor \frac{\ell_n}{h_{n,j}} \right\rfloor h_{n,j}$, we would have $x_{[z-(a+1)h_{n,j}, z-ah_{n,j})} \neq B_{n,j}$ but $x_{[z-bh_{n,j},z)} = B_{n,j}^b$ which is impossible. $\square$

Proposition 5.14. *For j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, the sets $T^i Z_{n,j}$ are disjoint over $0 \leq i < c_{n,j}$.*

Proof. Lemma 5.9 as $B_{n,j}$ is the minimal root of $B_{n,j}^{\ell_n/h_{n,j}} B_{n,j}$ and $c_{n,j} \leq \frac{1}{2}\ell_n < \frac{1}{2}\|B_{n,j}^{\ell_n/h_{n,j}} B_{n,j}\|$. □

Definition 5.15. For j such that $\|B_{n,j}\| > \frac{1}{2}\ell_n$, let, for $0 \leq i < c_{n,j}$,

$$I_{n,j,i} = T^i W_{n,j}$$

and for j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, let, for $0 \leq i < c_{n,j}$,

$$I_{n,j,i} = T^i \Big(\bigsqcup_{a=0}^{\lfloor \frac{\ell_n}{h_{n,j}} \rfloor} T^{a h_{n,j}} Z_{n,j} \Big)$$

As T is measure-preserving, $\mu(I_{n,j,i}) = \mu(I_{n,j,0})$ for all n, j and $0 \leq i < c_{n,j}$.

Definition 5.16. Let $\tilde{C}_{n,j} = \bigsqcup_{i=0}^{c_{n,j}-1} I_{n,j,i}$. For j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, let $\widehat{C}_{n,j} = \bigsqcup_{i=0}^{h_{n,j}-1} T^i W_{n,j}$.

Proposition 5.17. *For j such that $\|B_{n,j}\| > \frac{1}{2}\ell_n$, we have $\mu(\tilde{C}_{n,j}) \geq \frac{1}{2k}\mu(C_{n,j})$.*

Proof. $\mu(C_{n,j}) \leq h_{n,j}\mu(W_{n,j}) = h_{n,j}\mu(I_{n,j,0}) = \dfrac{h_{n,j}}{c_{n,j}}\mu(\tilde{C}_{n,j}) \leq \dfrac{k\ell_n}{\frac{1}{2}\ell_n}\mu(\tilde{C}_{n,j}) = 2k\mu(\tilde{C}_{n,j})$. □

Proposition 5.18. $\lim_n \max_j \{\mu(I_{n,j,0})\} = 0$.

Proof. For j such that $\|B_{n,j}\| > \frac{1}{2}\ell_n$, we have $1 \geq \mu(\tilde{C}_{n,j}) = c_{n,j}\mu(I_{n,j,0}) \geq \frac{1}{2}\ell_n\mu(I_{n,j,0})$ and $\ell_n \to \infty$. For j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, we have $1 \geq \mu(\tilde{C}_{n,j}) = h_{n,j}\mu(I_{n,j,0})$ and $\min_j h_{n,j} \to \infty$. □

Proposition 5.19. *$T^{h_{n,j}} W_{n,j} \subseteq \bigcup_{j'} W_{n,j'}$ and $X_0 = \bigcup_j C_{n,j}$.*

Proof. Every $x \in X_0$ is a concatenation of words of the form $B_{n,j}$ so every occurrence of $B_{n,j}$ is followed immediately by some $B_{n,j'}$ and $x_{[0,\infty)} = uB_1B_2\cdots$ for some u a suffix of some $B_{n,j}$ and $B_\ell \in \{B_{n,j}\}$. □

Proposition 5.20. *Let $E \subseteq W_{n,j}$. Then there exists j' such that $\mu(T^{h_{n,j}}E \cap W_{n,j'}) \geq \frac{1}{C}\mu(E)$.*

Proof. $T^{h_n}E = T^{h_n}E \cap T^{h_{n,j}}W_{n,j} \subseteq T^{h_n}E \cap \bigcup_{j'} W_{n,j'}$ and there are at most C choices of j'. □

Lemma 5.21. $\mu(W_{n,j}) \geq \frac{1}{k\ell_n}\mu(\tilde{C}_{n,j})$.

Proof. For j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, by Proposition 5.10, $T^{-\ell_n}W_{n,j} = [B_{n,j}^{\ell_n/h_{n,j}}B_{n,j}] \supseteq Z_{n,j}$ so

$$\mu(W_{n,j}) \geq \mu(Z_{n,j}) = \frac{1}{\lfloor \frac{\ell_n}{h_{n,j}} \rfloor + 1}\mu(I_{n,j,0}) \geq \frac{1}{\frac{\ell_n}{h_{n,j}}}\frac{1}{h_{n,j}}\mu(\tilde{C}_{n,j}) = \frac{1}{\ell_n}\mu(\tilde{C}_{n,j})$$

and for j such that $\|B_{n,j}\| > \frac{1}{2}\ell_n$, we have $\mu(W_{n,j}) = \frac{1}{c_{n,j}}\mu(\tilde{C}_{n,j}) \geq \frac{1}{k\ell_n}\mu(\tilde{C}_{n,j})$ since $c_{n,j} \leq k\ell_n$. □

Proposition 5.22. *If $\mu(T^{h_{n,j}}W_{n,j} \cap W_{n,j'}) \geq \delta\mu(W_{n,j''})$ for $j \neq j'$ then $\mu(\tilde{C}_{n,j'}) \geq \frac{1}{2k}\delta\mu(\tilde{C}_{n,j''})$.*

Proof. For j' such that $h_{n,j'} < \frac{1}{2}\ell_n$, Proposition 5.10 states that, as $j \neq j'$, for $x \in T^{h_{n,j}}W_{n,j} \cap W_{n,j'}$, the word $x_{(-\infty,0)}$ has $B_{n,j'}^{\ell_n/h_{n,j'}}$ as a suffix but does not have $B_{n,j'}^{\ell_n/h_{n,j'}}B_{n,j'}$ as a suffix. Therefore $T^{-\ell_n}(T^{h_{n,j}}W_{n,j} \cap W_{n,j'}) \subseteq [B_{n,j'}^{\ell_n/h_{n,j'}}B_{n,j'}] \setminus T^{h_{n,j'}}[B_{n,j'}^{\ell_n/h_{n,j'}}B_{n,j'}] = Z_{n,j'}$. This means that $\mu(Z_{n,j'}) \geq \mu(T^{h_{n,j}}W_{n,j} \cap W_{n,j'}) \geq \delta\mu(W_{n,j''})$ so

$$\begin{aligned}\mu(\tilde{C}_{n,j'}) = h_{n,j'}\mu(I_{n,j',0}) = h_{n,j'}\Big(\Big\lfloor \frac{\ell_n}{h_{n,j'}}\Big\rfloor + 1\Big)\mu(Z_{n,j'}) &\geq h_{n,j'}\frac{\ell_n}{h_{n,j'}}\delta\mu(W_{n,j''}) \\ &\geq \ell_n\delta\frac{1}{c_{n,j''}}\mu(\tilde{C}_{n,j''}) \geq \ell_n\delta\frac{1}{k\ell_n}\mu(\tilde{C}_{n,j''}) = \delta\frac{1}{k}\mu(\tilde{C}_{n,j''})\end{aligned}$$

For j' such that $h_{n,j'} > \frac{1}{2}\ell_n$, using Lemma 5.21 and that $\mu(W_{n,j'}) \geq \delta\mu(W_{n,j''})$,

$$\mu(\tilde{C}_{n,j'}) = c_{n,j'}\mu(W_{n,j'}) \geq c_{n,j'}\delta\mu(W_{n,j''}) \geq c_{n,j'}\delta\frac{1}{k\ell_n}\mu(\tilde{C}_{n,j''}) \geq \frac{\ell_n}{2}\delta\frac{1}{k\ell_n}\mu(\tilde{C}_{n,j''}) = \frac{1}{2k}\delta\mu(\tilde{C}_{n,j''}) \quad \square$$

Proposition 5.23. *For j such that $\|B_{n,j}\| \leq \frac{1}{2}\ell_n$, we have $\mu(T^{h_{n,j}}I_{n,j,0} \cap I_{n,j,0}) \geq \frac{1}{2}\mu(I_{n,j,0})$.*

Proof.

$$\mu(T^{h_{n,j}}I_{n,j,0} \cap I_{n,j,0}) \geq \mu\Big(\bigsqcup_{a=1}^{\lfloor \frac{\ell_n}{h_{n,j}} \rfloor} T^{ah_{n,j}}Z_{n,j}\Big) = \Big\lfloor \frac{\ell_n}{h_{n,j}}\Big\rfloor\mu(Z_{n,j}) = \frac{\big\lfloor \frac{\ell_n}{h_{n,j}}\big\rfloor}{\big\lfloor \frac{\ell_n}{h_{n,j}}\big\rfloor + 1}\mu(I_{n,j,0}) \geq \frac{1}{2}\mu(I_{n,j,0}) \quad \square$$

5.4 Partial rigidity

We employ ideas similar to Danilenko's [Dan16] proof that exact finite rank implies partial rigidity:

Proposition 5.24. *If there exists $\delta > 0$ and j_n and $t_n \to \infty$ with $\mu(\tilde{C}_{n,j_n}) \geq \delta$ (or $\mu(\widehat{C}_{n,j_n}) \geq \delta$ when applicable) and $\mu(T^{t_n}I_{n,j_n} \cap I_{n,j_n}) \geq \delta\mu(I_{n,j_n})$ then (X,μ) is $\frac{1}{2}\delta^2$-partially rigid.*

Proof. Let $A = W_{N,J}$ for some fixed N and J. Define $\alpha_n = \{0 \leq i < c_{n,j_n} - h_{N,J} : I_{n,j_n,i} \subseteq A\}$.

For j_n such that $h_{n,j_n} > \frac{1}{2}\ell_n$, if $x \in I_{n,j_n,i} \cap W_{N,J}$ then $x_{[-i,-i+h_{n,j_n})} = B_{n,j_n}$ and $x_{[0,h_{N,J})} = B_{N,J}$ meaning that $(B_{n,j_n})_{[i,i+h_{N,J})} = B_{N,J}$. This implies that $T^iW_{n,j_n} \subseteq W_{N,J}$ provided $i < h_{n,j_n} - h_{N,J}$.

For j_n such that $h_{n,j_n} \leq \frac{1}{2}\ell_n$, if $x \in I_{n,j_n,i} \cap W_{N,J}$ then $x_{[-i,-i+\ell_n/h_{n,j_n})} = B_{n,j_n}^{\ell_n/h_{n,j_n}}$ and $x_{[0,h_{N,J})} = B_{N,J}$ so $(B_{n,j_n}^{\ell_n/h_{n,j_n}})_{[i,i+h_{N,J})} = B_{N,J}$ which implies $I_{n,j_n,i} \subseteq W_{N,J}$ provided $i < h_{n,j_n} - h_{N,J}$.

Therefore $(|\alpha_n| + h_{N,J})\mu(I_{n,j_n,0}) \geq \mu(A \cap \tilde{C}_{n,j_n}) \geq |\alpha_n|\mu(I_{n,j_n,0})$. Likewise, if $\|B_{n,j_n}\| \leq \frac{1}{2}\ell_n$ then $(|\alpha_n| + h_{N,J})\mu(W_{n,j_n}) \geq \mu(A \cap \widehat{C}_{n,j_n}) \geq |\alpha_n|\mu(W_{n,j_n})$ using $\alpha_n = \{0 \leq i < h_{n,j_n} - h_{N,J} : T^iW_{n,j_n} \subseteq A\}$.

For $m < c_{n,j_n}$, $\mu(T^m\tilde{C}_{n,j_n} \triangle \tilde{C}_{n,j_n}) \leq 2m\mu(I_{n,j_n,0})$, (and likewise $\mu(T^m\widehat{C}_{n,j} \triangle \widehat{C}_{n,j}) \leq 2m\mu(W_{n,j})$ when applicable) therefore

$$\int \big|\mathbb{1}_{\tilde{C}_{n,j_n}} \circ T^{-m} - \mathbb{1}_{\tilde{C}_{n,j_n}}\big|^2 \, d\mu = 2\mu(\tilde{C}_{n,j_n}) - 2\mu(T^m\tilde{C}_{n,j_n} \cap \tilde{C}_{n,j_n}) \leq 2m\mu(I_{n,j_n,0})$$

Therefore for $M < c_{n,j_n}$,

$$\begin{aligned}\Big|\frac{1}{M}\sum_{m=1}^{M}\mu(T^{-m}A \cap \tilde{C}_{n,j_n}) - \mu(A \cap \tilde{C}_{n,j_n})\Big| &= \Big|\frac{1}{M}\sum_{m=1}^{M}\mu(A \cap T^m\tilde{C}_{n,j_n}) - \mu(A \cap \tilde{C}_{n,j_n})\Big| \\ &\leq \frac{1}{M}\sum_{m=1}^{M}\big|\mu(A \cap T^m\tilde{C}_{n,j_n}) - \mu(A \cap \tilde{C}_{n,j_n})\big| \leq \frac{1}{M}\sum_{m=1}^{M}\int_A \big|\mathbb{1}_{\tilde{C}_{n,j_n}} \circ T^{-m} - \mathbb{1}_{\tilde{C}_{n,j_n}}\big| \, d\mu\end{aligned}$$

$$\leq \frac{1}{M}\sum_{m=1}^{M}\Big(\int \big|\mathbb{1}_{\tilde{C}_{n,j_n}}\circ T^{-m} - \mathbb{1}_{\tilde{C}_{n,j_n}}\big|^2\,d\mu\Big)^{1/2} \leq \frac{1}{M}\sum_{m=1}^{M}\sqrt{2m\mu(I_{n,j_n,0})} \leq \sqrt{2M\mu(I_{n,j_n,0})}$$

The mean ergodic theorem gives M such that $\int \big|\frac{1}{M}\sum_{m=1}^{M}\mathbb{1}_A\circ T^m - \mu(A)\big|^2 d\mu < (\frac{1}{4}\delta\mu(A))^2$ so

$$\Big|\frac{1}{M}\sum_{m=1}^{M}\mu(T^{-m}A\cap\tilde{C}_{n,j_n}) - \mu(A)\mu(\tilde{C}_{n,j_n})\Big| = \Big|\int_{\tilde{C}_{n,j_n}}\frac{1}{M}\sum_{m=1}^{M}\mathbb{1}_A\circ T^m - \mu(A)\,d\mu\Big|$$
$$\leq \int_{\tilde{C}_{n,j_n}}\Big|\frac{1}{M}\sum_{m=1}^{M}\mathbb{1}_A\circ T^m - \mu(A)\Big|\,d\mu \leq \Big(\int\Big|\frac{1}{M}\sum_{m=1}^{M}\mathbb{1}_A\circ T^m - \mu(A)\Big|^2\,d\mu\Big)^{1/2} < \frac{1}{4}\delta\mu(A)$$

For n large enough that $c_{n,j_n} > M$ and $\sqrt{2M\mu(I_{n,j_n,0})} < \frac{1}{4}\delta\mu(A)$ (Proposition 5.18 states $\mu(I_{n,j_n,0})\to 0$) then $|\mu(A\cap\tilde{C}_{n,j_n}) - \mu(A)\mu(\tilde{C}_{n,j_n})| < \frac{1}{2}\delta\mu(A)$. Then

$$\begin{aligned}\mu(T^{t_n}A\cap A) &\geq \mu(T^{t_n}(A\cap\tilde{C}_{n,j_n})\cap(A\cap\tilde{C}_{n,j_n})) \geq \sum_{i\in\alpha_n}\mu(T^{t_n}T^iI_{n,j_n,0}\cap T^iI_{n,j_n,0})\\ &= |\alpha_n|\mu(T^{t_n}I_{n,j_n,0}\cap I_{n,j_n,0}) \geq |\alpha_n|\delta\mu(I_{n,j_n,0}) \geq \delta(\mu(A\cap\tilde{C}_{n,j_n}) - h_{N,J}\mu(I_{n,j_n,0}))\\ &> \delta\Big(\mu(A)\mu(\tilde{C}_{n,j_n}) - \frac{1}{2}\delta\mu(A)\Big) - \delta h_{N,J}\mu(I_{n,j_n,0})\\ &\geq \delta\Big(\mu(A)\delta - \frac{1}{2}\delta\mu(A)\Big) - \delta h_{N,J}\mu(I_{n,j_n,0}) = \frac{1}{2}\delta^2\mu(A) - \delta h_{N,J}\mu(I_{n,j_n,0})\end{aligned}$$

with the same applying to $\hat{C}_{n,j_n}$ when applicable. Therefore for fixed N and J and $0\leq i < h_{N,J}$,

$$\liminf \mu(T^{t_n}T^iW_{N,J}\cap T^iW_{N,J}) = \liminf \mu(T^{t_n}W_{N,J}\cap W_{N,J}) \geq \frac{1}{2}\delta^2\mu(W_{N,J}) = \frac{1}{2}\delta^2\mu(T^iW_{N,J})$$

and since the sets $T^iW_{N,J}$ generate the Borel algebra, μ is $\frac{1}{2}\delta^2$-partially rigid. □

Proof of Theorem 5.1. We aim to apply Proposition 5.24. Set $\delta = \frac{1}{4k^2C^{C+1}}$ which depends only on X.

There exists a_0 such that $\mu(C_{n,a_0}) \geq \frac{1}{C}$ since $X_0 = \bigcup_j C_{n,j}$. If $\|B_{n,a_0}\| \leq \frac{1}{2}\ell_n$ then $\mu(\widehat{C}_{n,a_0}) = \mu(C_{n,a_0}) \geq \frac{1}{C}$ and Proposition 5.23 implies $\mu(T^{h_{n,a_0}}I_{n,a_0,0}\cap I_{n,a_0,0}) \geq \frac{1}{2}\mu(I_{n,a_0,0})$ so take $t_n = h_{n,a_0}$ and $j_n = a_0$.

Now consider when $\|B_{n,a_0}\| > \frac{1}{2}\ell_n$ so Proposition 5.17 implies $\mu(\tilde{C}_{n,a_0}) \geq \frac{1}{2k}\mu(C_{n,a_0}) \geq \frac{1}{2kC}$.

By Proposition 5.20, there exists a_1 such that $\mu(T^{h_{n,a_0}}W_{n,a_0}\cap W_{n,a_1}) \geq \frac{1}{C}\mu(W_{n,a_0})$. If $a_1 = a_0$ then $\mu(\tilde{C}_{n,a_1}) = \mu(\tilde{C}_{n,a_0}) \geq \frac{1}{2kC}$ and if $a_1 \neq a_0$ then Proposition 5.22 implies $\mu(\tilde{C}_{n,a_1}) \geq \frac{1}{2k}\mu(\tilde{C}_{n,a_0}) \geq \frac{1}{4k^2C}$.

Proposition 5.20 then says there exists a_2 such that

$$\mu(T^{h_{n,a_1}}(T^{h_{n,a_0}}W_{n,a_0}\cap W_{n,a_1})\cap W_{n,a_2}) \geq \frac{1}{C}\mu(T^{h_{n,a_0}}W_{n,a_0}\cap W_{n,a_1}) \geq \frac{1}{C^2}\mu(W_{n,a_0})$$

and then Proposition 5.22 gives $\mu(\tilde{C}_{n,a_2}) \geq \frac{1}{C^2}\frac{1}{2k}\mu(\tilde{C}_{n,a_0}) \geq \frac{1}{4k^2C^3}$.

Repeating this process, we obtain a_ℓ for $0\leq \ell\leq C$ such that $\mu(\tilde{C}_{n,a_\ell}) \geq \frac{1}{4k^2C^{\ell+1}} \geq \frac{1}{4k^2C^{C+1}}$ and

$$\mu(W_{n,a_C}\cap\bigcap_{\ell=0}^{C-1}T^{\sum_{z=\ell}^{C-1}h_{n,a_z}}W_{n,a_\ell}) \geq \frac{1}{C^C}\mu(W_{n,a_0})$$

If any of the a_ℓ are such that $h_{n,a_\ell} \leq \frac{1}{2}\ell_n$ then Proposition 5.23 implies $\mu(T^{h_{n,a_\ell}}I_{n,a_\ell,0}\cap I_{n,a_\ell,0}) \geq \frac{1}{2}\mu(I_{n,a_\ell,0})$ so take $t_n = h_{n,a_\ell}$ and $j_n = a_\ell$.

If $h_{n,a_\ell} > \frac{1}{2}\ell_n$ for all $0\leq\ell\leq C$ then, since there are at most C choices of j, for some $q < s$ we must have $a_q = a_s$ so setting $j_n = a_q$ and $t_n = \sum_{z=q}^{s-1} h_{n,a_z}$,

$$\mu(T^{t_n} I_{n,j_n,0} \cap I_{n,j_n,0}) = \mu(T^{\sum_{z=q}^{s-1} h_{n,a_z}} W_{n,a_q} \cap W_{n,a_s}) \geq \mu(W_{n,a_C} \cap \bigcap_{\ell=0}^{C-1} T^{\sum_{z=\ell}^{C-1} h_{n,a_z}} W_{n,a_\ell}) \geq \frac{1}{C^C} \mu(W_{n,a_0})$$

As
$$\begin{aligned}\mu(W_{n,a_0}) = \mu(I_{n,a_0,0}) &= \frac{1}{c_{n,a_0}} \mu(\tilde{C}_{n,a_0}) \geq \frac{1}{h_{n,a_0}} \frac{1}{2kC} \geq \frac{1}{k\ell_n} \frac{1}{2kC} \geq \frac{1}{k\ell_n} \frac{1}{2kC} \mu(\tilde{C}_{n,j_n}) \\ &= \frac{1}{k\ell_n} \frac{1}{2kC} c_{n,j_n} \mu(I_{n,j_n,0}) \geq \frac{1}{k\ell_n} \frac{1}{2kC} \frac{\ell_n}{2} \mu(I_{n,j_n,0}) = \frac{1}{4k^2C} \mu(I_{n,j_n,0})\end{aligned}$$

we then have $\mu(T^{t_n} I_{n,j_n,0} \cap I_{n,j_n,0}) \geq \frac{1}{4k^2C^{C+1}} \mu(I_{n,j_n,0})$.

In all cases, by Proposition 5.24, we have that (X, μ, T) is $\frac{1}{2}\delta^2$-partially rigid. □

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